# Signal and information processing in time

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February 27, 2018

# Continuous time, discrete time, discrete signals

- We have studied continuous time, discrete time, and discrete signals
- Complex exponentials (CE), discrete time CE, and discrete CE
- And also the Fourier transform (FT), the DTFT, and the DFT
- For which we respectively studied the iFT, iDTFT and the iDFT

- Different versions of related concepts
  $\Rightarrow$ Let's take time to summarize
  $\Rightarrow$ And to emphasize analogies and differences

# Signals

- Continuous time (CT) $t \in \mathbb{R}$ $\Rightarrow$ Continuous time signals

$$x : \mathbb{R} \to \mathbb{C}$$

- Discrete time (DT) $n \in \mathbb{Z}$ $\Rightarrow$ Discrete time signals

$$x : \mathbb{Z} \to \mathbb{C}$$

- Discrete and finite $n \in [0, N-1]$ $\Rightarrow$ Discrete signals

$$x : [0, N-1] \to \mathbb{C}$$

- From discrete signals we go to ...
  ... infinity $\Rightarrow$ discrete time signals (extend borders)
  ... and beyond $\Rightarrow$ continuos time signal (fill in spaces, dense)

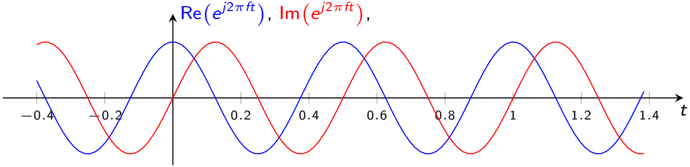
# Inner products and energy

- Inner product in continuous time $\Rightarrow \langle x, y\rangle := \displaystyle\int_{-\infty}^{\infty} x(t)y^*(t)dt$
- Inner product in discrete time $\Rightarrow \langle x, y\rangle := \displaystyle\sum_{n=-\infty}^{\infty} x(n)y^*(n)$
- Inner product of discrete signals $\Rightarrow \langle x, y\rangle := \displaystyle\sum_{n=0}^{N-1} x(n)y^*(n)$
- How much signals $x$ and $y$ are like each other
- Unrelated signals = orthogonality $\Rightarrow \langle x, y\rangle = 0$
- Energy, same definition works for all $\Rightarrow \|x\|^2 = \langle x, x\rangle$
- Inner product may not exist and energy may be infinite (CT and DT)

# Continuous time complex exponentials

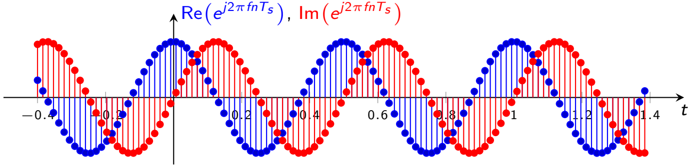
- Continuous time complex exponential $e_f \;\Rightarrow e_f(t) = e^{j2\pi ft}$

  $\Rightarrow$ Signal is dense and extend to plus and minus infinity




- Frequency $f = 2$Hz shown. Time $t$ in seconds

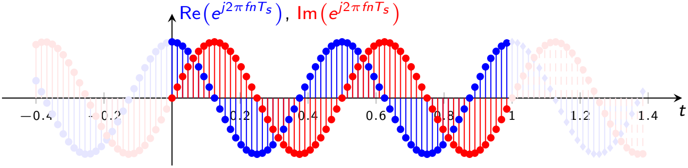
- Discrete time complex exponential $e_{fT_s} \Rightarrow e_{fT_s}(n) = e^{j2\pi fnT_s}$
  - $\Rightarrow$ Sample continuous time CE with sampling frequency $f_s = 1/T_s$
  - $\Rightarrow$ Signal extend to plus and minus infinity but is not dense




- Frequency $f = 2\text{Hz}$. Sampling freq. $f_s = 64\text{Hz}$. Time $t$ in seconds.

# Discrete complex exponentials

- Discrete complex exponential $\Rightarrow \sqrt{N}e_{kN}(n) = e^{j2\pi kn/N} = e^{j2\pi fnT_s}$
  - $\Rightarrow$ Discrete time CE observed during $N$ samples $= NT_s$ time units
  - $\Rightarrow$ Defined for frequencies of the form $f = (k/N)f_s$ only
  - $\Rightarrow$ Exactly $k$ oscillations during observation period $N \Leftrightarrow T$




- Frequency $f = 2$Hz. Sampling freq. $f_s = 64$Hz. Time $t$ in seconds
- Observation time $T = 1$s $\Rightarrow$ number samples $N = Tf_s = 64$.
- Discrete frequency $k = N(f/f_s) = 2$

# Orthogonality of complex exponentials

- Discrete complex exponentials are a set of $N$ orthonormal signals

$$\langle e_{kN}, e_{lN} \rangle = \delta(k-l)$$

- We restrict $k$ and $l$ to interval of length $N$. E.g., $[-N/2+1, N/2]$
- CE with freqs. $N$ apart are equivalent. Opposites are conjugates

- Discrete time complex exponentials are (sort of) orthogonal

$$\langle e_{fT_s}, e_{gT_s} \rangle = \delta(f-g)$$

- Continuous time delta $\Rightarrow$ Involves a limit. Generalized function

- Same is true in continuous time $\Rightarrow \langle e_f, e_g \rangle = \delta(f-g)$

# Fourier transforms

# Fourier transforms

- Fourier transform (FT) of continuous time signal $x$ is the function

$$X(f) := \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\, dt$$

- The discrete time (DT)FT of discrete time signal $x$ is the function

$$X(f) := T_s \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s}$$

- The discrete (D)FT of discrete signal $x$ is the function

$$X(k) := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi k n/N} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi f n T_s}$$

- Discrete frequency $k$ equivalent to real $f = k/NT_s = kf_s/N$
- DFT is undefined for frequencies that are not $f = kf_s/N$ for some $k$

# Fourier transforms as inner products

- Recall definitions of inner products and complex exponentials

- Write the FT of $x$ as $\Rightarrow X(f) = \langle x, e_f \rangle = \int_{-\infty}^{\infty} x(t) e_f^*(t)\, dt$

- Write DTFT of $x$ as $\Rightarrow X(f) = \langle x, e_{fT_s} \rangle = T_s \sum_{n=-\infty}^{\infty} x(n) e_{fT_s}^*(n)$

- Write the DFT of $x$ as $\Rightarrow X(k) = \langle x, e_{kN} \rangle = \sum_{n=0}^{N-1} x(n) e_{kN}^*(n)$

- All three transforms written as inner products in respective spaces

# Different formalizations of the same concept

- Inner products with frequency $f$ $(f = kf_s/N)$ complex exponentials
- It follows that they are different formalizations of the same concept
  - $\Rightarrow$ They are projections of $x$ onto oscillations of frequency $f$
  - $\Rightarrow$ They measure how much $x$ resembles oscillation of frequency $f$
- Integrals, indefinite sums, sums $\Rightarrow$ Inherent differences in signals
- FT and DTFT are analysis tools. DFT is a computational tool

# Input and output spaces

- Input and output spaces for FTs are continuous
- For DTFTs, discrete inputs, continuous and periodic outputs (odd)
- For DFTs, input and outputs are discrete and periodic or finite

| | **Input space** | **Output space** |
|---|---|---|
| Fourier transform | Continuous | |
| | | Continuous |
| DTFT | Discrete | Periodic |
| | | Continuous |
| DFT | Discrete | Periodic |
| | Periodic | Discrete |

- Observe the duality between sampling and periodicity or finiteness

- Filter $\Rightarrow$ multiply in frequency by $H$ $\Rightarrow$ convolve in time with $h$

$$X_f = HX \quad \Longleftrightarrow \quad x_f = x * h$$

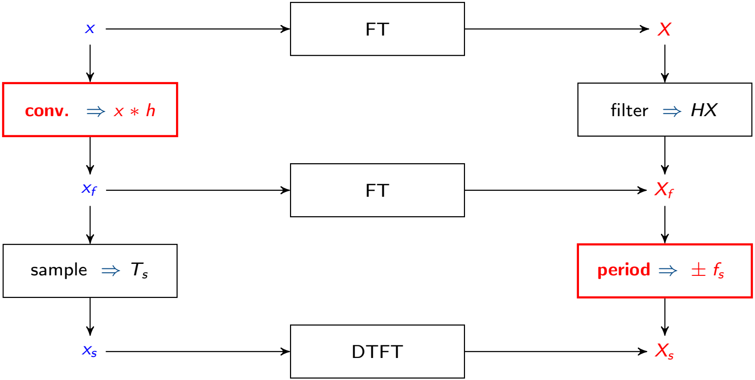
- Sample filtered signal $X_f$ $\Rightarrow$ Periodize filtered spectrum $X_f$

$$x_s(n) = x_f(nT_s) \quad \Longleftrightarrow \quad X_s(f) = \sum_{k=-\infty}^{\infty} X_f(f - kf_s)$$

- Distortion (information loss) occurs during filtering step
  - $\Rightarrow$ Frequency $\Rightarrow$ Loss above $f_s/2$ + some distortion if $H$ not perfect
  - $\Rightarrow$ Time $\Rightarrow$ Convolution with $h$

# The DTFT as proxy for the FT (2 of 3)

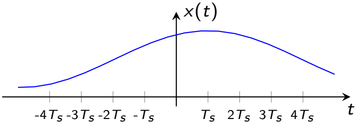
- Filtering (chop) induces convolution. Sampling induces periodization

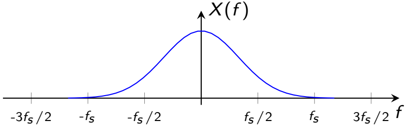



- Small distortion $\Rightarrow$ Make $f_s$ so that $X(f) \approx 0$ for $f \notin [-f_s/2, f_s/2]$

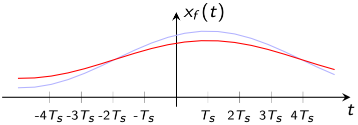
# The DTFT as proxy for the FT (3 of 3)

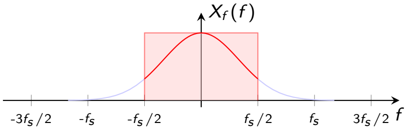
- Continuous time signal $x$ with FT $X$ $\Rightarrow$ Not necessarily bandlimited



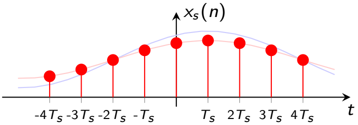



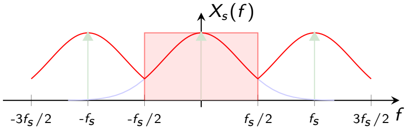

- Continuous time filtered signal $x_f$ $\Rightarrow$ filtering smoothes (distorts) $x$




- Sampled signal $x_s$ obtained from filtered $x_f$ $\Rightarrow$ No further distortion



# The DFT as proxy for the DTFT (1 of 3)

- Filter $\Rightarrow$ multiply by window $w_N$ $\Rightarrow$ convolve in frequency with $W_N$

$$x_w(n) = x(n) \times w_N(n) \quad \Longleftrightarrow \quad X_w(f) = X_s(f) * W_N(f)$$

- Sample windowed spectrum $X_w$ $\Rightarrow$ Periodize windowed signal $x_w$

$$x_d(n) = \sum_{k=-\infty}^{\infty} x_w(n - kN) \quad \Longleftrightarrow \quad X_d\left(\frac{kf_s}{N}\right) = T_s\sqrt{N}\, X_w(k)$$

- Distortion (information loss) occurs during windowing step
  $\Rightarrow$ Frequency sampling is with no loss of information

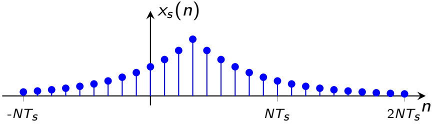
- Window (chop) induces convolution. Sampling induces periodization

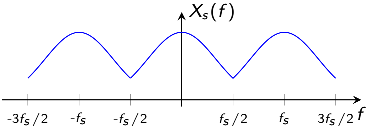
$$\begin{tikzcd}
x_s \arrow[r] \arrow[d] & \boxed{\text{DTFT}} \arrow[r] & X_s \arrow[d] \\
\boxed{\text{window} \Rightarrow xw_N} \arrow[d] & & \boxed{\textbf{conv} \Rightarrow X * W} \arrow[d] \\
x_w \arrow[r] \arrow[d] & \boxed{\text{DTFT}} \arrow[r] & X_w \arrow[d] \\
\boxed{\textbf{periodize} \Rightarrow N} \arrow[d] & & \boxed{\text{sample} \Rightarrow f_s/N} \arrow[d] \\
x_d \arrow[r] & \boxed{\text{DFT}} \arrow[r] & X_d
\end{tikzcd}$$

- Small distortion $\Rightarrow$ Make $N$ so that $x(n) \approx 0$ for $n \notin [0, N-1]$

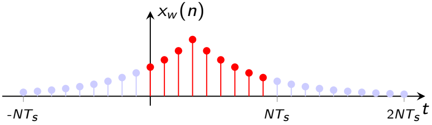
# The DFT as proxy for the DTFT (3 of 3)

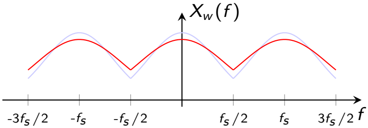
- Discrete time signal $x_s$ with DTFT $X_s$ $\Rightarrow$ Not necessarily finite



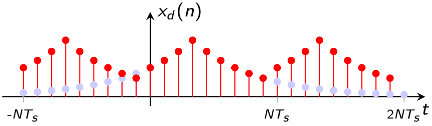



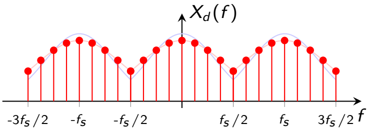

- Discrete time windowed signal $x_w$ $\Rightarrow$ windowing smoothes (distorts) $X_s$




- Discrete DFT $X_D$ samples windowed DTFT $X_w$ $\Rightarrow$ No further distortion



# Bandlimited and finite (periodic) signals

- If signal is bandlimited and sampled at frequency $f_s \geq W$
  $\Rightarrow$ The DTFT and the FT coincide in the interval $[-f_s/2, f_s/2]$
- If signal is finite, and windowed with $N$ larger than its length
  $\Rightarrow$ DFT and DTFT coincide at the sampled frequencies $f = kf_s/N$
- What happens when signal is bandlimited and finite?
  $\Rightarrow$ Doesn't matter. These signals don't exist. Uncertainty principle

# Inverse Fourier transforms

# Inverse Fourier transforms

- Given a transform $X$, the inverse Fourier transform is defined as

$$x(t) := \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} \, df$$

- The iDTFT $x$ of DTFT $X$, is the discrete time signal with elements

$$x(n) = \int_{-f_s/2}^{f_s/2} X(f) e^{j2\pi f n T_s} \, df = \int_{0}^{f_s} X(f) e^{j2\pi f n T_s} \, df$$

- Given a Fourier transform $X$, the inverse (i)DFT is defined as

$$x(n) := \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} = \frac{1}{\sqrt{N}} \sum_{k=-N/2+1}^{N/2} X(k) e^{j2\pi kn/N}$$

- Same as direct transform but for sign in the exponent $\Rightarrow$ duality

# The inverses are inverses indeed

**Theorem**

*The inverse FT (or inverse DTFT or inverse DFT) $\tilde{x}$ of the FT (respectively, DTFT or DFT) $X$ of a given signal $x$ is the given signal $x$*

$$\tilde{x} = \mathcal{F}^{-1}(X) = \mathcal{F}^{-1}[\mathcal{F}(x)] = x$$

- We can recover signal from transform $\Rightarrow$ equivalent representation
  - $\Rightarrow$ Neither less, nor more information. Just different interpretability
- Implies that we can write signal as a sum of complex exponentials
  - $\Rightarrow$ Literally for iDFT, conceptually for iDTFT and iFT

# Inverse DFT as sum of complex exponentials

- Signal as sum of exponentials $\Rightarrow x(n) = \dfrac{1}{\sqrt{N}} \displaystyle\sum_{k=-N/2+1}^{N/2} X(k) e^{j2\pi kn/N}$

- Expand the sum inside out from $k = 0$ to $k = \pm 1$, to $k = \pm 2$, ...

$$
\begin{array}{lllllll}
x(n) = & X(0) & e^{j2\pi 0n/N} & & & & \text{constant} \\
& + X(1) & e^{j2\pi 1n/N} & + X(-1) & e^{-j2\pi 1n/N} & & \text{single oscillation} \\
& + X(2) & e^{j2\pi 2n/N} & + X(-2) & e^{-j2\pi 2n/N} & & \text{double oscillation} \\
& \vdots & \vdots & \vdots & \vdots & & \vdots \\
& + X\left(\frac{N}{2}-1\right) & e^{j2\pi\left(\frac{N}{2}-1\right)n/N} & + X\left(-\frac{N}{2}+1\right) & e^{-j2\pi\left(\frac{N}{2}-1\right)n/N} & & \left(\frac{N}{2}-1\right) - \text{oscillation} \\
& + X\left(\frac{N}{2}\right) & e^{j2\pi\left(\frac{N}{2}\right)n/N} & & & & \frac{N}{2} - \text{oscillation}
\end{array}
$$

- Start with slow variations and progress on to add faster variations

# Properties of Fourier transforms

# Linearity and conjugate symmetry

**Theorem**

*The FT, DTFT, and DFT of linear combinations of signals are linear combinations of the respective transforms of the individual signals,*

$$\mathcal{F}(ax + by) = a\mathcal{F}(x) + b\mathcal{F}(y).$$

- Useful to compute transforms when considering sums of signals

**Theorem**

*The FT, DTFT, and DFT $X = \mathcal{F}(x)$ of a real signal $x$ (one with $Im(x) \equiv 0$) are conjugate symmetric*

$$X(-f) = X^*(f)$$

- Only the positive half of the spectrum carries information

# Energy conservation (Parseval's Theorem)

### Theorem (Parseval)

*The energy of a signal $x$ and its FT, DTFT, or DFT $X = \mathcal{F}(x)$ are the same, i.e.,*

$$\|x\|^2 = \|X\|^2$$

- Energy definitions are different for different signal spaces
- For the FT $\Rightarrow \int_{-\infty}^{\infty} |x(t)|^2 dt = \|x\|^2 = \|X\|^2 = \int_{-\infty}^{\infty} |X(f)|^2 df$
- For the DTFT $\Rightarrow \sum_{n=-\infty}^{\infty} |x(n)|^2 = \|x\|^2 = \|X\|^2 = \int_{-f_s/2}^{f_s/2} |X(f)|^2 df$
- For the DFT $\Rightarrow \sum_{n=0}^{N-1} |x(n)|^2 = \|x\|^2 = \|X\|^2 = \sum_{k=-N/2+1}^{N/2} |X(k)|^2$

# Shift and modulation

Theorem

*A time shift of $\tau$ units in the time domain is equivalent to multiplication by a complex exponential of frequency $-\tau$ in the frequency domain*

$$x_\tau = x(t - \tau) \qquad \Longleftrightarrow \qquad X_\tau(f) = e^{-j2\pi f\tau} X(f)$$

Theorem

*A multiplication by a complex exponential of frequency $g$ in the time domain is equivalent to a shift of $g$ units in the frequency domain*

$$x_g = e^{j2\pi gt} x(t) \qquad \Longleftrightarrow \qquad X_g(f) = X(f - g)$$

- Theorems are duals of each other. True for FT and DTFT
- For DFT we need to define circular shifts. Not covered in this course

# Convolutions in continuous and discrete time

- Let $x$ and $h$ be continuous time signals
- Convolution of $x$ with $h$ is the signal $y = x * h$ with values

$$[x * h](t) = y(t) = \int_{-\infty}^{\infty} x(u)h(t-u)\, du$$

- Let $x$ and $h$ be discrete time signals
- Convolution of $x$ with $h$ is the signal $y = x * h$ with values

$$[x * h](n) = y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

# Multiplication and convolution

- Convolution in time domain $\equiv$ to multiplication in frequency domain

**Theorem (Convolution theorem)**

*Given signals $x$ and $y$ with transforms $X = \mathcal{F}(x)$ and $Y = \mathcal{F}(y)$. The FT $Z = \mathcal{F}(z)$ of the convolved signal $z = x * y$ is the product $Z = XY$*

$$z = x * y \qquad \Longleftrightarrow \qquad Z = XY$$

- True for FT and DTFT. For DFT need to define circular convolution
- The dual is also true
- Convolution in frequency domain $\equiv$ to multiplication in time domain

# Sampling and reconstruction

# Sampling

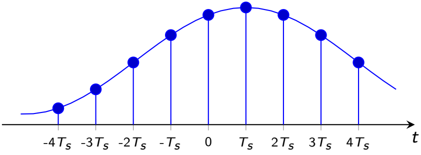
- The sampled signal $x_s$ is a discrete time signal with values

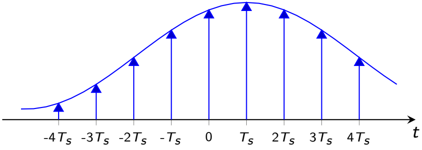
$$x_s(n) = x(nT_s)$$

- Creates discrete time signal $x_s$ from continuous time signal $x$
- Equivalently, we represent sampling as multiplication by a Dirac train

$$x_\delta(t) = x(t) \times T_s \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

- Dirac train lives in continuous time. Compare FT of $x_\delta$ to FT of $x$



# Spectral effect of sampling

- Multiplication $\Leftrightarrow$ Convolution . Thus spectrum $X_\delta = \mathcal{F}(x_\delta)$ is

$$X_\delta = X * \mathcal{F}\left[T_s \sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right]$$

- Fourier transform of the Dirac train ($T_s$) is another Dirac train ($f_s$)

$$X_\delta = X * T_s \sum_{n=-\infty}^{\infty} \delta(f - kf_s) = \sum_{n=-\infty}^{\infty} X * \delta(f - kf_s)$$

Theorem

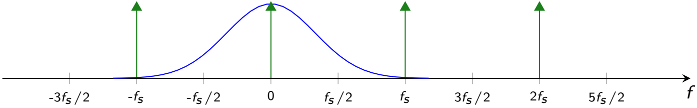
*Sampled signal spectrum is a sum of shifted versions of original spectrum*

$$X_s(f) = X_\delta(f) = \sum_{k=-\infty}^{\infty} X(f - kf_s)$$

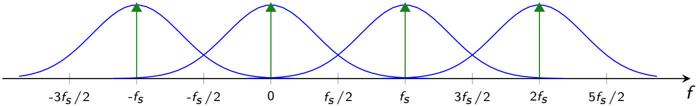
- We say the spectrum of $X$ is periodized when the signal is sampled

# Spectrum periodization

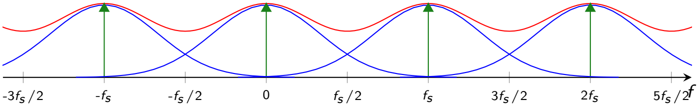
- Start with the spectrum $X$ of $x$ and the Dirac train in frequency




- First convolution step is to duplicate and shift spectrum to $kf_s$



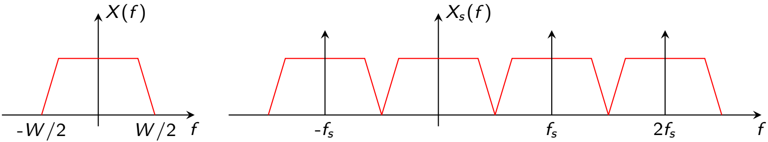

- Second convolution step is to sum all shifted copies




- Loose all info. above $f_s/2$. And some below to aliasing distortion

# Sampling of bandlimited signals

- Signal with bandwidth $W \Rightarrow X(f) = 0$ for all $f \notin [-W/2, W/2]$
- Upon sampling, spectrum is periodized but not aliased



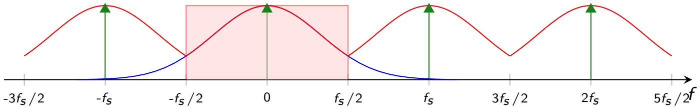

- This means that sampling entails no loss of information

# Prefiltering

- To avoid aliasing preprocess $x$ into $x_{f_s}$ with a low pass filter

$$X_{f_s}(f) = X(f)\sqcap_{f_s}(f)$$

- The signal $x_{f_s}$ has bandwidth $f_s$ and can be sampled without aliasing
  $\Rightarrow$ Frequency components below $f_s/2$ retained with no distortion




- Prefiltering can be implemented as convolution in the time domain

$$x_{f_s} = x * h, \qquad h(t) = f_s \text{sinc}(\pi f_s t)$$

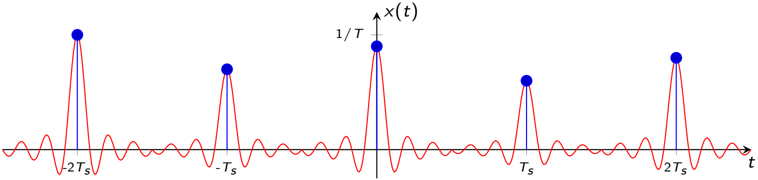
- iFT of low pass filter with cutoff $f_s/2$ is the sinc pulse with freq. $f_s$

# Reconstruction

- In principle, we can recover $x$ from $x_\delta$ with a low pass filter
- Since Dirac train can't be generated, we modulate train of pulses

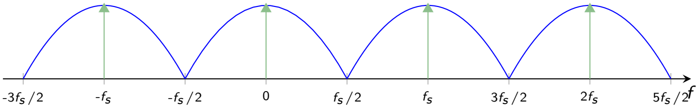
$$x_p(t) = T_s \sum_{n=-\infty}^{\infty} x_s(n) p(t - nT_s)$$

- For narrow pulses, pulse and Dirac modulation are close, i.e, $x_p \approx x_\delta$

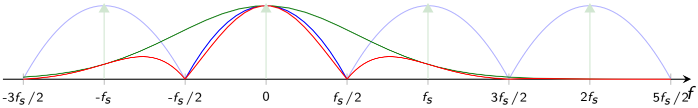


# The spectrum of the reconstructed signal

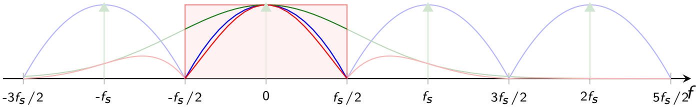
- Spectrum $X_s$ of sampled signal $\Rightarrow X_s(f) = \sum_{k=-\infty}^{\infty} X(f - kf_s)$




- Spectrum $X_p$ of pulse train $\Rightarrow X_p(f) = P(f) \times \sum_{k=-\infty}^{\infty} X(f - kf_s)$




- Reconstructed spectrum $X_r$ $\Rightarrow X_r(f) = \sqcap_{f_s}(f) P(f) X(f - kf_s)$




- Good pulse for recovery $\Rightarrow X(f) = 1$ for $f \in [-f_s/2, f_s/2]$

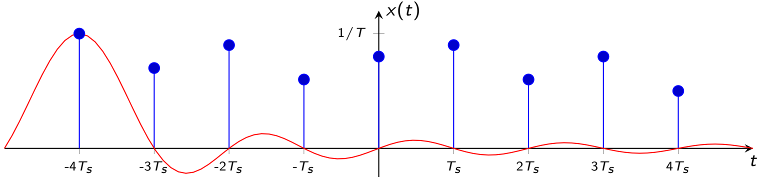
# Modulation of a sinc train

- The sinc pulse $f_s \text{sinc}(\pi f_s t)$ has a flat spectrum for $f \in [-f_s/2, f_s/2]$
- Don't even need to use low pass filter $\Rightarrow$ sinc pulse already lowpass

Theorem

*A signal of bandwidth $W \leq f_s$ can be recovered from samples $x(nT_s)$ as*

$$x(t) = f_s T_s \sum_{n=-\infty}^{\infty} x(nT_s)\text{sinc}\big(\pi f_s(t - nT_s)\big)$$



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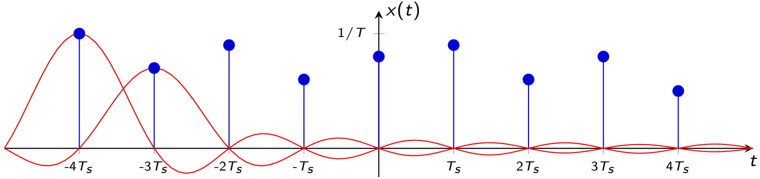

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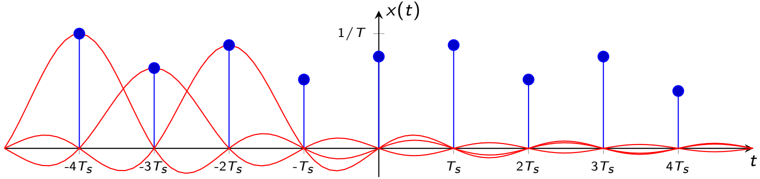

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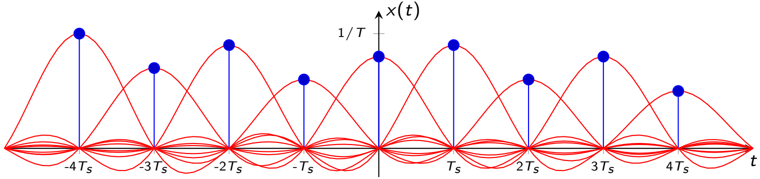


# Philosophical digression

- Sampling is a straightforward operation, but its effects are obscure
  - $\Rightarrow$ Or not. If we look at the signal in frequency effects are also clear
- Loss of information contained at frequencies $f > f_s/2$
- Aliasing distortion for frequencies $f \leq f_s/2$
- Perfect recovery of bandlimited signals
- Avoid aliasing with prefiltering
- Reconstruction distortion when modulating a train of pulses
- If we had a sixth sense for frequencies, all of this would be obvious
  - $\Rightarrow$ But we do have that sense, or rather have grown that sense

# Linear time invariant systems

Signals and information

Fourier transforms

Inverse Fourier transforms

Properties of Fourier transforms

Sampling and reconstruction

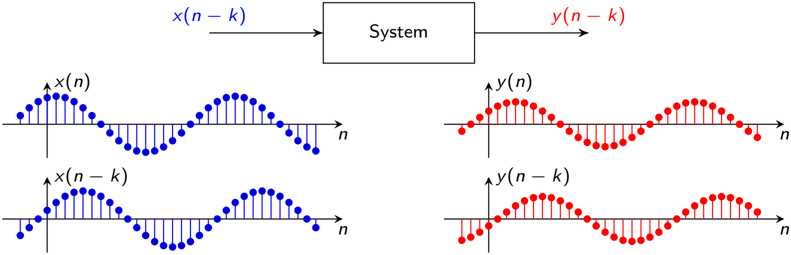
Linear time invariant systems

Applications

Signal representation

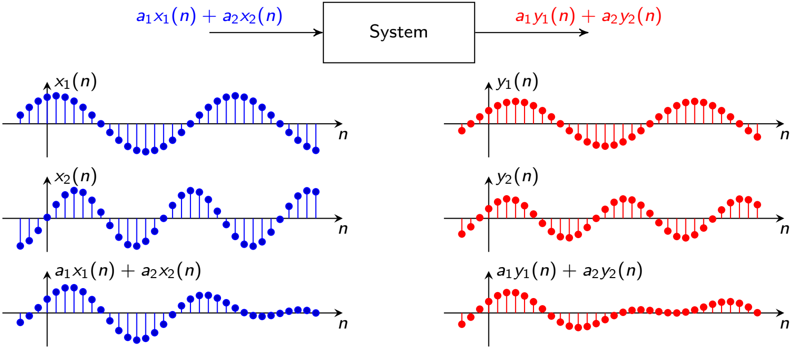
# Time invariant systems

- Systems are characterized by input-output ($x \to y$) relationships
- A system is time invariant if a delayed input yields a delayed output
- If input $x(n)$ yields output $y(n)$ then input $x(n-k)$ yields $y(n-k)$



# Linear systems

- In a linear system $\Rightarrow$ input a linear combination of inputs
  $\Rightarrow$ Output the same linear combination of the respective outputs
- I.e., if input $x_1(n)$ yields output $y_1(n)$ and $x_2(n)$ yields $y_2(n)$
  $\Rightarrow$ Input $a_1x_1(n) + a_2x_2(n)$ yields output $a_1y_1(n) + a_2y_2(n)$



# Output of a linear time invariant system

- linear time invariant system (LTI) $\Rightarrow$ Linear + time invariant

**Theorem**

*A linear time invariant system is completely determined by its impulse response h. In particular, the response to input x is the signal $y = x * h$.*

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow (x * h)(n) = \sum_{-\infty}^{\infty} x(k)h(t-k)$$

- Theorem true for discrete time and continuous time signals

  $\Rightarrow$ Convolutions are defined differently

- For discrete signals we need to use circular convolutions

# Linear time invariant system frequency response

- Frequency response $\Rightarrow$ impulse response transform $\Rightarrow H = \mathcal{F}(h)$

**Corollary**

*A linear time invariant system is completely determined by its frequency response $H$. In particular, the response to input $X$ is the signal $Y = HX$.*

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f) = H(f)X(f)$$

- What a LTI system does to a signal is obscure
  - $\Rightarrow$ Or not. If we look at the signal in frequency the effects are clear
- If we had a sixth sense for frequencies. Oh wait, we do
- It is obvious what LTI filters do $\Rightarrow$ They alter frequency components
- But they don't mix frequency components. Each of them is separate

# Applications

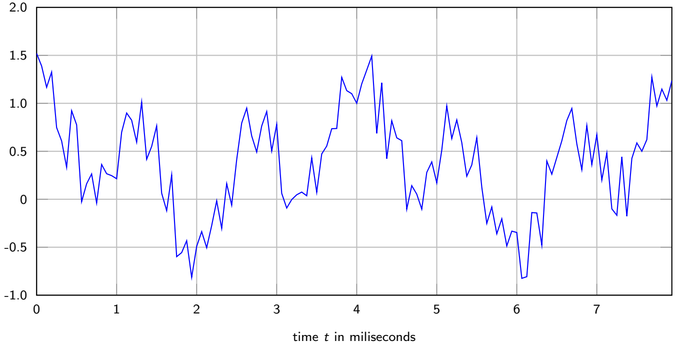
# Applications

- Practical applications of frequency analysis are very common
- Here are a few applications that we have covered
  - ⇒ Noise removal,
  - ⇒ Music synthesis,
  - ⇒ Compression,
  - ⇒ Modulation,
  - ⇒ Signal detection (voice recognition)
- There are many more we have not covered
  - ⇒ E.g., equalization, high-pass filtering, band-pass filtering
- In all of these applications understanding time is complicated
  - ⇒ But understanding frequency is straightforward

# Noise removal

- There is signal and noise, but what is signal and what is noise?
- We already know answer $\Rightarrow$ Signal discernible in frequency domain

Original signal $x(t)$. It moves randomly, but not that much

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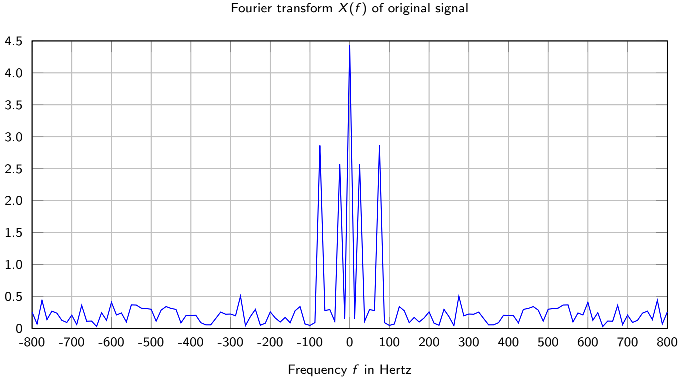
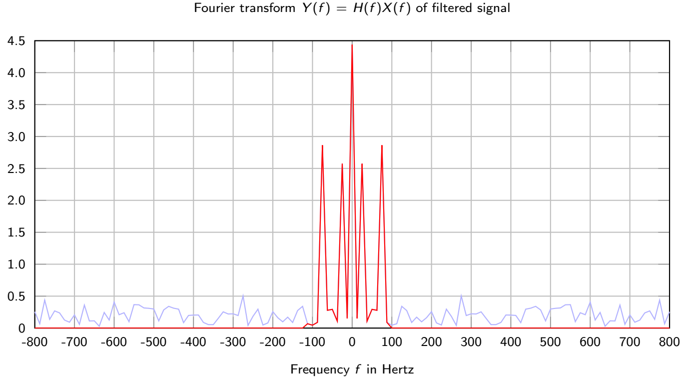



- Filter out all frequencies above 100Hz (and below -100Hz)

# Low pass filter design

- Multiply spectrum with low pass filter $H(f) = \sqcap_W(f)$ with $W = 200\text{Hz}$
  $\Rightarrow$ Only frequencies between $\pm W/2 = \pm 100\text{Hz}$ are retained




- This spectral operation does separate signal from noise

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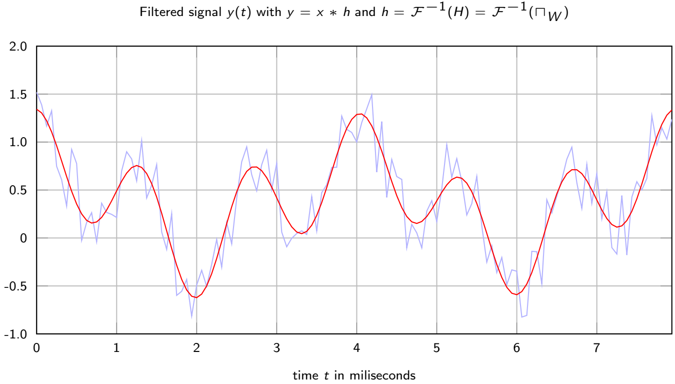



- This spectral operation does separate signal from noise

# Low pass filter implementation

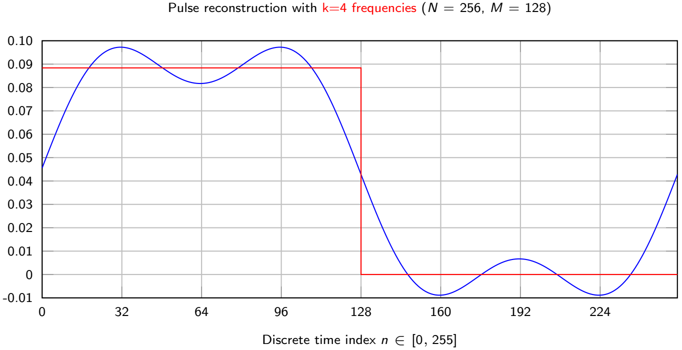
- We can implement filtering in the frequency domain
  $\Rightarrow$ Sample $\Rightarrow$ DFT $\Rightarrow$ Multiply by $H(f) = \sqcap_W(f)$ $\Rightarrow$ iDFT

$$x \longrightarrow \boxed{h(t) = W\text{sinc}(\pi W t)} \longrightarrow y = x * h$$

- We can also implement filtering in the time domain
  $\Rightarrow$ Inverse transform of $\sqcap_W(f)$ is $h(t) = W\text{sinc}(\pi W t)$
- How is it that convolving with a sinc removes noise? $\Rightarrow$ obscure
- But is is very clear if we use our frequency sense
- Signal occupies some frequencies but noise occupies all frequencies

# Signal compression

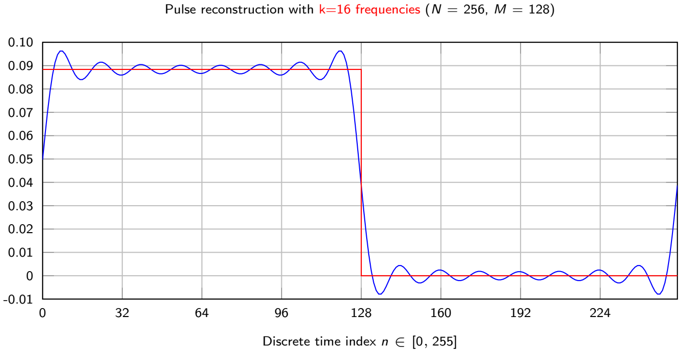
- Consider square pulse of duration $N = 256$ and length $M = 128$
- Reconstruct with 9 frequency components ($k \in [-4, 4]$)




- Compression $\Rightarrow$ Store 9 DFT values instead of $N = 128$ samples

# Signal compression

- Consider square pulse of duration $N = 256$ and length $M = 128$
- Reconstruct with $k = 16$ frequency components




- Can tradeoff less compression for better signal accuracy

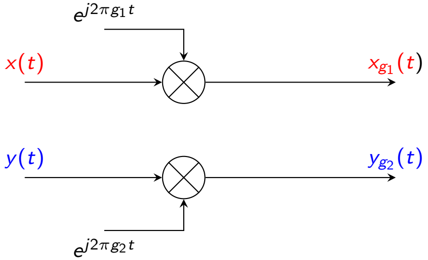
# Signal compression

- Generic compression $\Rightarrow$ Keep largest DFT coefficients
  - $\Rightarrow$ Not necessarily the lowest frequencies
- The approximation error energy is that of the coefficients dropped
- What's the advantage of comprising in frequency domain?
- Well, how would you compress in time domain
- Keep largest coefficients?
  - $\Rightarrow$ No. Close values are redundant. Small values also important
- Keep values at certain spacing?
  - $\Rightarrow$ Maybe. Actually that's sampling. Better think in freq. domain
- Compression is obscure but becomes clear if we use frequency sense

# Modulation of multiple bandlimited signals

- Transmit multiple bandlimited signals ($W$) in a common support
  - $\Rightarrow$ Wireless, optical fiber, coaxial cable, twisted pair
- Modulate (multiply by complex exponentials) with freqs. $g_1$ and $g_2$

$$z(t) = e^{j2\pi g_1 t} x(t) + e^{j2\pi g_2 t} y(t)$$

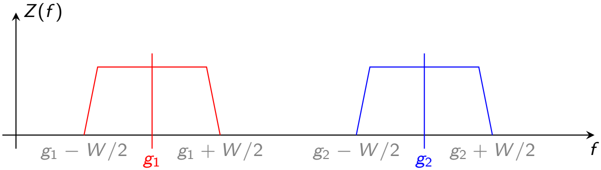



$$z(t) = x_{g_1}(t) + y_{g_2}(t)$$

- Spectrum of $x$ recentered at $g_1$. Spectrum of $y$ recentered at $g_2$

# Spectrum of multiple modulated signals

- No spectral mixing if modulating frequencies satisfy $g_2 - g_1 > W$




- To recover $x$ multiply by conjugate frequency $e^{-j2\pi g_1 t}$
- And eliminate all frequencies outside the interval $[-W/2, W/2]$

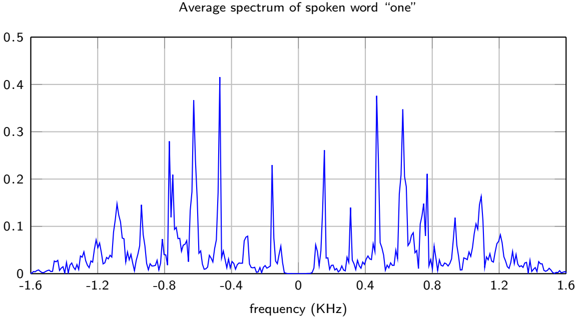
- To recover $y$ multiply by conjugate frequency $e^{-j2\pi g_2 t}$
- And eliminate all frequencies outside the interval $[-W/2, W/2]$

# Modulation analysis and design

- Can we understand modulation in time?
  - $\Rightarrow$ Actually, yes. Use orthogonality of complex exponentials
- But still, spectral analysis is clearer. Simplifies design
- Modulation is not entirely obscure
  - $\Rightarrow$ But it becomes clearer if we use frequency sense

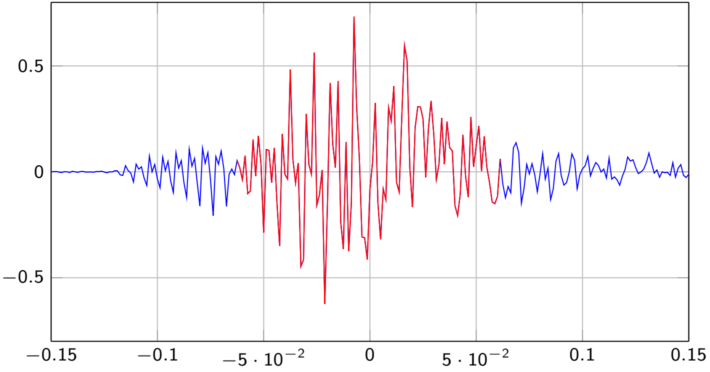
# Signal detection (voice recognition)

- For a given word to be recognized we compare the spectra $\bar{X}$ and $X$
  - $\Rightarrow \bar{X} \Rightarrow$ Average spectrum magnitude of word to be recognized
  - $\Rightarrow X \Rightarrow$ Recorded spectrum during execution time




- Energy $\sum_{k=-N/2+1}^{N/2} (X_i \bar{X}_i)^2 \Rightarrow$ Filter $X$ with $\bar{X}$, i.e., $Y(f) = H(f)X(f)$ with $H(f) = \bar{X}$

- Determine impulse response $h(n)$ as inverse DFT of spectrum $\bar{X}$
- Window $h(n)$ to keep, say, $N = 1,000$ largest consecutive taps



# Signal detection analysis and design

- Can we understand signal detection in time?
  $\Rightarrow$ Actually, yes. It's called a matched filter
- But, as in modulation, spectral analysis is clearer. Simplifies design
- Signal detection is not entirely obscure
  $\Rightarrow$ But it becomes clearer if we use frequency sense

# It's all oh so simple

- Once and again, things are invisible or obscure in time domain
  - $\Rightarrow$ But they become, visible and clear in the frequency domain
- Even when clear in time, they are easier to understand in frequency
- Literally a new sense to view things that are otherwise invisible

*"On ne voit bien qu'avec le coeur.*
*L'essentiel est invisible pour les yeux."*

*The Little Prince*

- One sees clearly only with the frequency
  The essential is invisible to the eyes

# Signal representation

- Why a new sense? $\Rightarrow$ We can write signals as sums of shifted deltas

$$x(n) = \sum_{k=1}^{N} x(k)\delta(k-n)$$

- Conceptually, the same as writing signals as sums of oscillations

$$x(n) = \sum_{k=1}^{N} X(k)e^{j2\pi kn/N}$$

- Only difference is that we sense time but we don't sense frequency
- We say we change the signal representation or we change the basis
- It all hinges in our ability to represent the signal in a different domain

# Moving forward

- If something is obscure in time but also obscure in frequency
  $\Rightarrow$ Change the representation $\equiv$ Change the basis
- Images $\Rightarrow$ multidimensional DFT, Discrete cosine transform (DCT)
- Stochastic processes $\Rightarrow$ Principal component analysis (PCA)
  $\Rightarrow$ Eigenvectors of the correlation matrix
- Signals defined on graphs $\Rightarrow$ Graph signal processing
  $\Rightarrow$ Eigenvalues of the graph Laplacian