Signal and Information Processing

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Contents

Chapter 1

Discrete sines, cosines, and complex exponentials

1.1 Discrete Signals

A signal is a collection of all possible values that can be mapped from a set of indices to their value in real time. Signals can represent various types of information, and thus it is important to gain some experience and intuition on how these signals look like and behave. The signals we consider here are discrete because they are indexed by a finite and integer time index $n = 0, 1, \dots, N-1$. The constant N is referred to as the length of the signal. This discrete signal x is a function mapping its time index n to the real value $x(n)$ that the signal takes, represented as

$$x : [0, N-1] \to \mathbb{R} \tag{1.1}$$

Signals can also be complex, with the mapping

$$x : [0, N-1] \to \mathbb{C}, \tag{1.2}$$

, and such are represented as $x(t) = x_R(t) + j\, x_I(t)$. Therefore, the space of signals is the space of N-dimensional vectors in $\mathbb{R}^N$ or $\mathbb{C}^N$. An example of a discrete signal is a delta function $\delta(n)$ that spikes at initial time $n = 0$ seen in the figure 1.1, and shown by the equation

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{else} \end{cases} \tag{1.3}$$

It is important to note that in this case, and as for the case with signals, the first index is always $n = 0$ and so the last time index of a signal will be $n = N - 1$. For example, in Figure 1.1, the signal is of length 8, but the last time index is $n = 7$. Figure 1.2 is a shifted version of this function, $\delta(n - n_0)$, that spikes at time $n = n_0$, shown by the equation

$$\delta(n - n_0) = \begin{cases} 1 & \text{if } n = n_0 \\ 0 & \text{else} \end{cases} \tag{1.4}$$

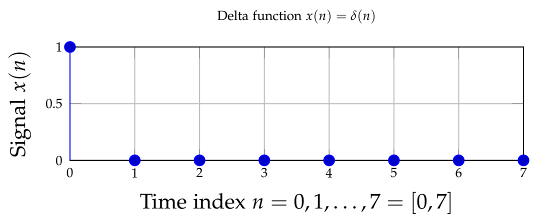

Figure 1.1. Discrete delta function $\delta(n)$ that has value $\delta(n) = 1$ at initial time $n = 0$ and value $\delta(n) = 0$ elsewhere.

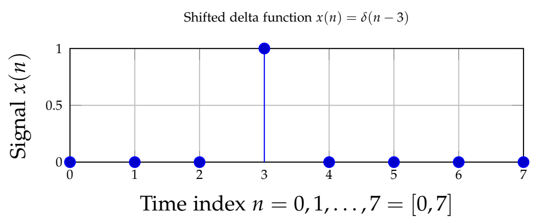

Figure 1.2. Shifted delta function $\delta(n - n_0)$ that has value 1 at time $n = n_0$, and value 0 elsewhere.

This is simply a time shift of the original delta function. Some examples of common signals include a constant function $x(n)$, that has the same value c for all n, shown in figure 1.3, with the function

$$x(n) = c, \quad \text{for all } n \tag{1.5}$$

Also to note is a square pulse signal, $\sqcap_M(n)$, which is of width M, and equals one for the first M values, shown in figure 1.4, denoted by

$$\sqcap_M(n) = \begin{cases} 1 & \text{if } 0 \leq n < M \\ 0 & \text{if } M \leq n \end{cases} \tag{1.6}$$

When dealing with sampling of a signal, some units to be aware of are the sampling time T_s, which is the time elapsed between indexes n and $n+1$ and sampling frequency $f_s := 1/T_s$. Time index n represents the actual time $t = nT_s$. Therefore, the signal duration $T = NT_s$ is the total length in real time of the signal. Figure 1.5 is an example of a square pulse plotted in actual time t, which was taken at sampling rate $T_s = 125ms$ for duration $T = 2s$.

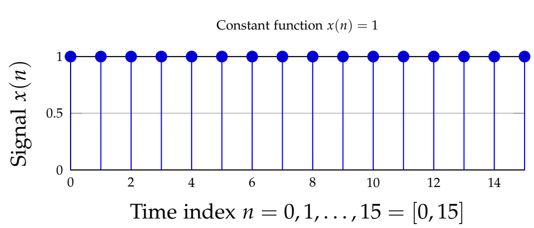

Figure 1.3. Example of a constant function, $x(n) = 1$.

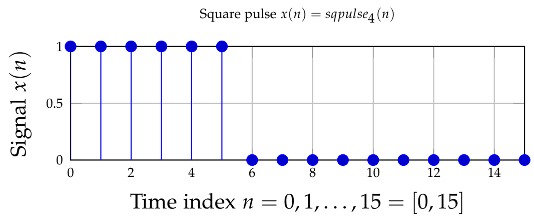

Figure 1.4. Example of a square pulse, $x(n) = \sqcap_6(n)$

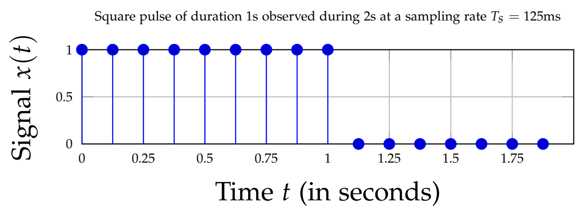

Figure 1.5. Example of a square pulse plotted in real time for the signal $x(t)$.

1.2 Discrete Cosines and Sines

Discrete cosines and sines are important signals to understand for the basis of signal and information processing. For a signal of duration N (assume N is even), there exists the discrete cosine $x(n)$ of frequency k, where

$$x(n) = \cos(2\pi kn/N), \tag{1.7}$$

and the discrete sine $x(n)$ of frequency k, where

$$x(n) = \sin(2\pi kn/N). \tag{1.8}$$

Frequency k is discrete, with an integer number of complete oscillations, (i.e. $k = 0, 1, 2, \ldots$). Figure 1.6 illustrates a cosine and sine signal of length $N = 32$ and frequency $k = 2$. Other examples of cosines of different frequencies can be seen in figure 1.7. When dealing with discete cosines and sines, it is necessary to note certain properties about these signals. Since we are in discrete time, for a cosine or sine signal of length N, you can only physically denote up to $N/2$ times that the signal can oscillate, i.e go from $1 \to -1 \to 1, \to -1, \ldots$, as seen in figure 1.8. After this, in order to capture frequencies of higher order ($k > N/2$), one would need to go into continuous time, where points between, for example, 0 and 1 can be referenced. When solely dealing with discrete cosines, we observe that for frequencies k and $N - k$, the same cosine is represented, i.e for discrete cosines of frequency k and l where $k + l = \dot{N}$, cosines of frequencies k and l are equivalent. Note, though, that this is not true for sines, where the signals have opposite signs. Examples of equivalent discrete cosines are shown in Figure 1.9.

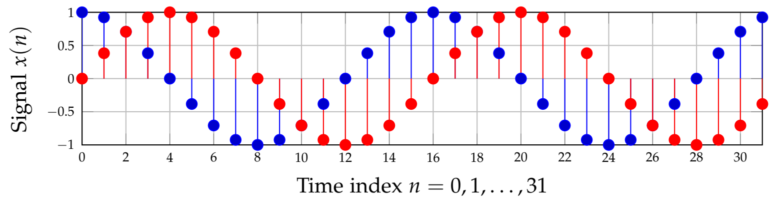

Figure 1.6. Cosine $x(n) = \cos(2\pi k n/N)$ and sine $x(n) = \sin(2\pi k n/N)$. Frequency $k = 2$ and number of samples $N = 32$.

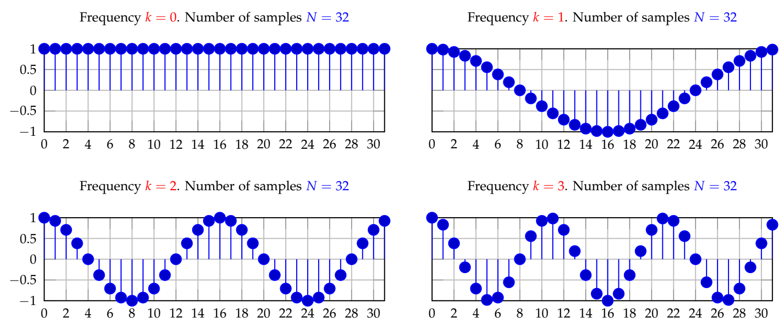

Figure 1.7. Discrete cosines of different discrete frequencies. The discrete frequency k dictates the number of complete oscillations during the duration of the signal. When $k = 1$ there is one complete oscillation (top right), when $k = 2$ there are two complete oscillations (bottom left), there are then three complete oscillations for $k = 3$ (bottom right), and so on. When $k = 0$ there are no oscillations (top left). The cosine is just the constant signal with $x(n) = 1$ for al n.

1.2.1 Physical meaning of discrete frequencies

Discrete cosines are digital representations of real continuous time cosines. Given that a cosine is characterized by its frequency f_0 we ask the question of what is the frequency f_0 that is associated with a cosine of duration N and discrete frequency k. The answer to this question depends on the sampling time T_s.

To see how k, N, and T_s determine f_0, start by recalling that the duration of the signal x when measured in continuous time is $T = NT_s$; see Figure 1.10. Since we also know that the discrete frequency k represents the number of complete oscillations during the duration of the signal, the *period* of the cosine must be $T/k = NT_s/k$. Further observing that the frequency of a cosine is the inverse of its period, we conclude that the continuous time frequency f_0 must be given by

$$f_0 = \frac{k}{T} = \frac{k}{NT_s} = \frac{k}{N} f_s, \tag{1.9}$$

where we have used the definition of the sampling frequency $f_s = 1/T_s$ to write the last equality.

In the example in Figure 1.10, we have a discrete cosine of discrete frequency $k = 4$ with $N = 64$ samples and a sample period of $T_s = 0.5$s. Given the length of the signal

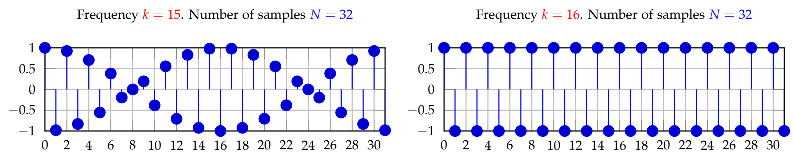

Figure 1.8. For discrete cosines of large discrete frequencies ($k = 15$ for sample size $N = 32$), the oscillations before dofficult to see (left). For a signal of sample size $N = 32$ the last discrete frequency with physical meaning is $k = 16$ (right).

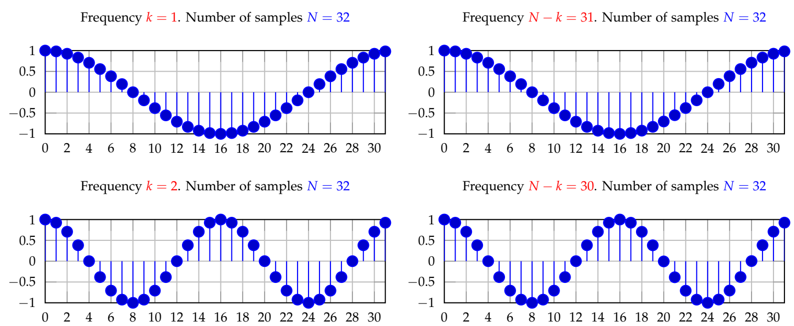

Figure 1.9. Discrete cosines of length $N = 32$, where for frequencies $k + l = N$, the same signal is represented. Cosines of frequencies $k = 1$ and $l = 31$ (top) and frequencies $k = 2$ and $k = 30$ (bottom) are equivalent.

$N = 64$ and the sampling time $T_s = 0.5$s, the duration of the signal is $T = NT_s = 32$s. Considering now that the discrete frequency is $k = 4$ we have 4 complete oscillations during the duration of the signal. This implies that the period of the cosine, i.e., the time it takes for a complete oscillation, is $NT_s/k = 8$s. The frequency of the cosine is the inverse of its period leading to $f_0 = 1/8 = 0.125$Hz. This is, of course, the same result that follows from direct application of (1.9).

It is important to observe here that since only discrete frequencies $k \leq N/2$ have meaning, only frequencies $f_0 \leq f_s/2$ have physical meaning. This means that if we use a sampling frequency f_s we can represent cosines that oscillate with a frequency smaller than $f_s/2$.

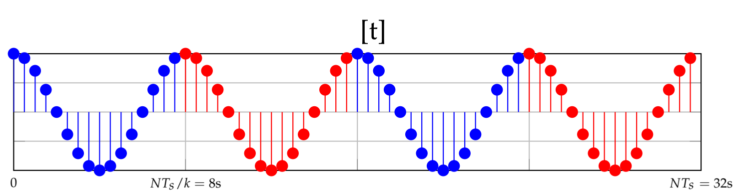

Figure 1.10. Transforming discrete frequencies to continuous time frequencies. A discrete cosine of discrete frequency $k = 4$ with $N = 64$ samples and a sample period of $T_s = 0.5$s is shown. The duration of the signal is $T = NT_s = 32$s and the period of the cosine is $NT_s/k = 8$s. The frequency of the cosine is the inverse of its period, namely $f_0 = 1/8 = 0.125$Hz [cf. (1.9)].

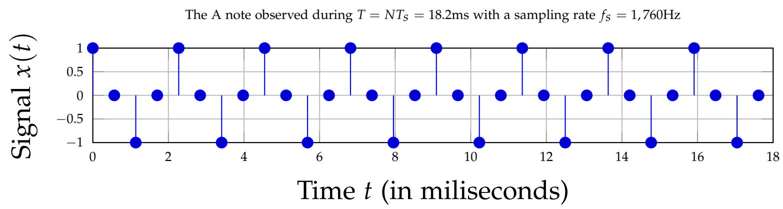

Figure 1.11. Creating an A note observed for $T = N * T_s = 18.2$ms using a sampling rate of $f_s = 1760$Hz. Note there are $k = 8$ oscillations.

Further observe that from the definition of the discrete cosine in (1.8) we can write the value of the signal at continuous time $t = nT_s$ as

$$x(t) = x(nTs) = \cos(2\pi kn/N). \tag{1.10}$$

If we use the expression in (1.9) to convert the discrete frequency k into the continuous time frequency $f_0 = (k/N)T_s$ we can rewrite (1.10) as

$$x(t) = x(nTs) = \cos(2\pi f_0 nT_s) = \cos(2\pi f_0 t). \tag{1.11}$$

We can reinterpret (1.11) as a way of extending the signal $x(nTs)$ to times t other than those available. If we are given the discrete cosine x with signal values $x(n)$, we can transform it into its continuous time representation $x(nTs) = \cos(2\pi f_0 nT_s)$, which is identical to the representation $x(t) = \cos(2\pi f_0 t)$ for all times $t = nT_s$. However, we can also think of (1.11) as defining intermediate values for all times t, not just those times having the form $t = nT_s$ for some n between 0 and $N - 1$. Conversely, we can think that the signal $x(t) = \cos(2\pi f_0 t)$ exists for all times t and that the discrete signal x with values $x(n) = x(nTs) = \cos(2\pi f_0 nT_s)$ is a sampling of the continuous time signal $x(t)$. These matters are related to sampling and reconstruction of continuous time signals, which we will study later.

Example 1 *Generate and signal of an A note with $N = 32$ samples and with sampling frequency $f_s = 1,760$Hz.*

Solution: Fist consider the real frequency of an A, $f_0 = 440$Hz. Therefore, the discrete frequency must be

$$k = \frac{f_0}{f_s} N = \frac{440\text{Hz}}{1{,}760\text{Hz}} 32 = 8$$

Reference Figure 1.11 for the generated signal. Instead of converting to discrete frequency to calculate the discrete cosine, $x(n) = \cos\left[2\pi kn/N\right]$, we can directly substiture the real frequency and sampling frequency into the equation $x(n) = \cos\left[2\pi(f_0/f_s)Nn/N\right]$, which simplifies to $x(n) = \cos\left[2\pi(f_0/f_s)n\right] = \cos\left[2\pi f_0(nT_s)\right]$ ■

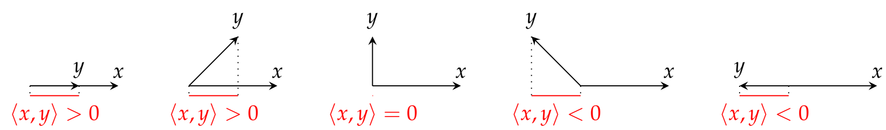

Figure 1.12. (left) The two signals x and y that point in same direction will have a large and positive inner product. (middle) For two signals that are completely perpendicular, they have orthogonal signals and their inner product is 0. This signifies that knowing x tells you very little about y. (right) The two signals x and y that point in completely opposite directions have an inner product that is large and negative. This signifies that the signals are similar, but point in opposite directions.

1.3 Inner Product and Energy

An important mathematical operation used in signal processing is the inner product. We can conceptually think of signals as vectors, and so the inner product of two signals has the same interpretation as the projection of one signal on another, i.e. the inner product $\langle x, y\rangle$ is the projection of y on x. This value determines how "related" the two signals are. Figure 1.12 shows examples of the projection of y on x. If the value of the inner product is 0 ($\langle x, y\rangle = 0$), the signals are orthogonal. Mathematically, given two signals x and y, we define the inner product of x and y as

$$\langle x, y\rangle := \sum_{n=0}^{N-1} x(n)y^*(n), \tag{1.12}$$

Which can be explicitly written as

$$\langle x, y\rangle = \sum_{n=0}^{N-1} x_R(n)y_R(n) + \sum_{n=0}^{N-1} x_I(n)y_I(n) + j\sum_{n=0}^{N-1} x_I(n)y_R(n) - j\sum_{n=0}^{N-1} x_R(n)y_I(n) \tag{1.13}$$

The inner product is a linear operation, such that $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$. Additionally, reversing the order of the inner product means conjugation, $\langle y, x\rangle = \langle x, y\rangle^*$. Remember that for complex signals, conjugation means the negation of the imaginary part of a signal.

Another common operation in signal processing is the norm of a signal, defined as

$$\|x\| := \left[\sum_{n=0}^{N-1} |x(n)|^2\right]^{1/2} = \left[\sum_{n=0}^{N-1} |x_R(n)|^2 + \sum_{n=0}^{N-1} |x_I(n)|^2\right]^{1/2} \tag{1.14}$$

For easier mathematical operations, we put greater emphasis on the energy of a signal, which is simply the norm of the signal, squared,

$$\|x\|^2 := \sum_{n=0}^{N-1} |x(n)|^2 = \sum_{n=0}^{N-1} |x_R(n)|^2 + \sum_{n=0}^{N-1} |x_I(n)|^2 \tag{1.15}$$

Therefore, from the definition of inner product in (1.12), we can see that the inner product of a signal with its own conjugate is equivalent to calculating the energy of the signal,

$$x(n)x^*(n) = |x_R(n)|^2 + |x_I(n)|^2 = |x(n)|^2 \tag{1.16}$$

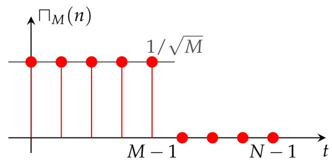

Figure 1.13. Sqaure pulse of duration M goes from $0 to M-1$ with value $1/sqrt(M)$ and other values go from $M to N-1$ with value 0.

An equivalent expression for this energy is $\|x\|^2 = \langle x, x\rangle$. When considering inner products and energy, it is intuitive that for two signals, the largest a projection of a signal can be is when the two vectors are collinear, seen in Figure 1.12. The two signals can point in the same, or opposite directions, and so the inner product can be between the positive and negative of these values,

$$\|x\|\,\|y\| \leq \langle x, y\rangle \leq \|x\|\,\|y\| \tag{1.17}$$

This is known as the Cauchy Schwarz inequality. Here we will simply introduce it, but we will reference this in more detail in later lectures. In terms of energy, we can see the same idea,

$$\langle x, y\rangle^2 \leq \|x\|^2\,\|y\|^2 \tag{1.18}$$

Explicitly written as,

$$\sum_{n=0}^{N-1} x(n)y^*(n) \leq \left[\sum_{n=0}^{N-1} |x(n)|^2\right]\left[\sum_{n=0}^{N-1} |y(n)|^2\right] \tag{1.19}$$

Example 2 *Compute the energy of the square pulse in Figure 1.13, of the signal $\sqcap_M(n)$ that takes values*

$$\sqcap_M(n) = \frac{1}{\sqrt{M}} \qquad \text{if } 0 \leq n < M$$
$$\sqcap_M(n) = 0 \qquad \text{if } M \leq n$$

Solution: We will use the definition from (1.15), such that

$$\| \sqcap_M \|^2 := \sum_{n=0}^{N-1} | \sqcap_M (n)|^2 = \sum_{n=0}^{M-1} \left|(1/\sqrt{M})\right|^2 = \frac{M}{M} = 1$$

This is a unit energy square pulse, because it has the value 1. If the height of the pulse is 1 instead of $1/\sqrt{M}$, then the energy is M. These signals solely live in the digital world, and so there is no defined units to these calculations. ■

To better understand orthogonality, we can consider a shifted version of this square pulse. The shift is a modification such that $\sqcap_M(n-K)$, where $K \geq M$, which indicates that the pulse is centered at K, since originally the pulse was centered at $n = 0$. You can see in Figure 1.14 that this shifted pulse does not directly overlap with the original pulse, but

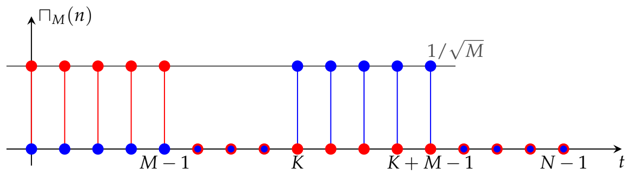

Figure 1.14. Sqaure pulse of duration M goes from $0 to M-1$ with value $1/sqrt(M)$ and other values go from $M to N-1$ with value 0.

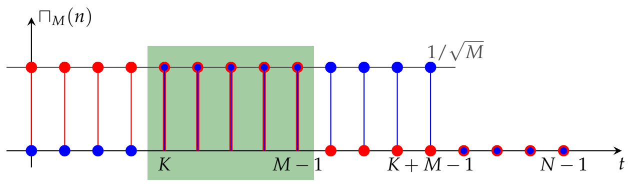

Figure 1.15. Shifted sqaure pulse of duration M goes from $K+M-1$ to K with value $1/sqrt(M)$ and all other values 0.

spikes at values greater than M. Therefore, the inner product of these two pulses shows that these signals are unrelated, and such are orthogonal to each other

$$\langle \sqcap_M(n), \sqcap_M(n-K)\rangle := \sum_{n=0}^{N-1} \sqcap_M(n)\, \sqcap_M(n-K) = 0 \tag{1.20}$$

When the original signal spikes, the shifted signal has a value of 0, and when the shifted signal spikes, the original signal has a value of 0. If we consider a different shifted version, though, we can once again see how the inner product determines the relatedness between two signals. Consider a shifted square pulse, $\sqcap_M(n-K)$, where $K < M$. As seen in Figure 1.15, these signals overlap between K and $M-1$ such that

$$\langle \sqcap_M(n), \sqcap_M(n-K)\rangle := \sum_{n=0}^{N-1} \sqcap_M(n)\, \sqcap_M(n-K) \tag{1.21}$$

When we plug in the overlapping values for $\sqcap_M(n)$ and $\sqcap_M(n-K)$, we get that

$$\langle \sqcap_M(n), \sqcap_M(n-K)\rangle := \sum_{n=K}^{M-1} \left(1/\sqrt{M}\right)\left(1/\sqrt{M}\right) \;=\; \frac{M-K}{M} \;=\; 1-\frac{K}{M} \tag{1.22}$$

This shifted pulse overlaps with the original signal, and so the inner product between these two signals holds information about the relationship between these signals. We can see from (1.21) that the inner product is proportional to the overlap of the two signals. The greater the overlap, the greater the value of the inner product.

1.4 Complex Exponentials

Sines, cosines, and complex exponentials play a very important role in signal and information processing. It is important to gain some experience and intuition on how these

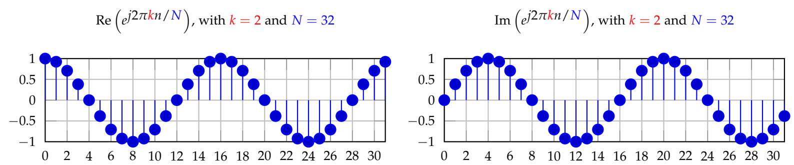

Figure 1.16. Shifted sqaure pulse of duration M goes from $K+M-1$ to K with value $1/sqrt(M)$ and all other values 0.

signals look like and behave. The signals we consider here are discrete because they are indexed by a finite and integer time index $n = 0, 1, \ldots, N-1$. The constant N is referred to as the length of the signal. A complex exponential is simpler way of writing discrete cosines and discrete sines. It represents an oscillation of a specific frequency. These are used to represent how fast signals are changing, i.e. slow and fast oscillations mimic how fast a signal changes. The discrete complex exponential $e_{kN}(n)$ of discrete frequency k and duration N is defined as

$$e_{kN}(n) = \frac{1}{\sqrt{N}}\, e^{j2\pi kn/N} = \frac{1}{\sqrt{N}}\, \exp(j2\pi kn/N). \tag{1.23}$$

The (regular) complex exponential is explicitly given by

$$e^{j2\pi kn/N} = \cos(2\pi kn/N) + j\sin(2\pi kn/N), \tag{1.24}$$

so that if we compute the real and imaginary parts of $e_{kN}(n)$ we can see that the real part is a discrete cosine and the imaginary part is a discrete sine, as seen in Figure 1.16.
As with discrete sines and cosines, complex exponentials have important properties associated with them, which will be referenced throughout the rest of this class. Firstly, for a frequency $k = 0$, the exponential $e_{kN}(n) = e_{0N}(n)$ is a constant so that

$$e_{kN}(n) = \frac{1}{\sqrt{N}} \;= \frac{1}{\sqrt{N}}\, 1 \tag{1.25}$$

Second, we have that for frequency $k = N$, or $k \in \dot{N}$ (multiple of N), the exponential $e_{kN}(n) = e_{NN}(n)$ is a constant so that

$$e^{j2\pi Nn/N}\sqrt{N} \;= \frac{(e^{j2\pi})^n}{\sqrt{N}} \;= \frac{(1)^n}{\sqrt{N}} \;= \frac{1}{\sqrt{N}} \tag{1.26}$$

Additionally, we have that for discrete frequency $k = N/2$, the exponential

$$e_{N/2N}(n) = \frac{e^{j2\pi(N/2)n/N}}{\sqrt{N}} \;= \frac{(e^{j\pi})^n}{\sqrt{N}} \;= \frac{(-1)^n}{\sqrt{N}} \tag{1.27}$$

This represents the fastest possible oscillation with N samples. The simplification of $e^{j2\pi} = 1$ is possible because it follows from $e^{j\pi} = -1$, which is a consequent of the mathematical constant

$$e^{j\pi} + 1 = 0, \tag{1.28}$$

To end this chapter, we will introduce some important theorems that will be referenced throughout this class.

Theorem 1 *If $k-l=N$, the signals $e_{kN}(n)$ and $e_{lN}(n)$ coincide for all n, i.e.,*

$$e_{kN}(n) = \frac{e^{j2\pi kn/N}}{\sqrt{N}} = \frac{e^{j2\pi ln/N}}{\sqrt{N}} = e_{lN}(n) \tag{1.29}$$

To put simply, exponentials with frequencies k and l are equivalent if $k-l=N$. Exponential frequencies are periodic, and so they form a canonical set, which is any set between N consecutive frequencies, e.g. $k=0,1,...N-1$. Typically, we will use canonical set of frequencies from $-N/2+1$ to $N/2$. When going from one canonical set to another, you are chopping the set at some point, and then shifting the set. This theorem of equivalent frequencies is profound, because even though a signal can go from $-\infty$ to ∞, this is not necessary due to the equivalence of canonical sets. Therefore, it is the same to solely look at one canonical set of size N, without losing any information held in the signal. To understand the mathematics behind this theorem, the proof is as follows
Proof: We prove by showing that $e_{kN}(n)/e_{lN}(n)=1$.

$$\frac{e_{kN}(n)}{e_{lN}(n)} = \frac{e^{j2\pi kn/N}}{e^{j2\pi ln/N}} = e^{j2\pi(k-l)n/N} \tag{1.30}$$

Since $k-l=N$, the above simplifies to

$$\frac{e_{kN}(n)}{e_{lN}(n)} = e^{j2\pi Nn/N} = \left[e^{j2\pi}\right]^n = 1^n = 1 \tag{1.31}$$

■

Theorem 2 *Complex exponentials with nonequivalent frequencies are orthogonal.*

$$\langle e_{kN}, e_{lN}\rangle = 0 \tag{1.32}$$

when $k-l<N$, E.g., when $k=0,\ldots N-1$, or $k=-N/2+1,\ldots,N/2$.

This determines that signals of canonical sets are "unrelated," since they contain different rates of change. This is in contrast to signals of equivalent frequencies, which have unit energy, e.g. $|e_{kN}\|^2 = \langle e_{kN}, e_{kN}\rangle = 1$. Exponentials within a canonical set are orthonormal, and such form an orthonormal basis of signal space with N samples, given by

$$\langle e_{kN}, e_{lN}\rangle = \delta(l-k) \tag{1.33}$$

Orthonormality will be explored further later in this course. The proof for orthogonality is as follows
Proof: Use definitions of inner product and discrete complex exponential to write

$$\langle e_{kN}, e_{lN}\rangle = \sum_{n=0}^{N-1} e_{kN}(n)e^*_{lN}(n) = \sum_{n=0}^{N-1} \frac{e^{j2\pi kn/N}}{\sqrt{N}}\frac{e^{-j2\pi ln/N}}{\sqrt{N}} \tag{1.34}$$

Regroup terms to write as geometric series

$$\langle e_{kN}, e_{lN}\rangle = \frac{1}{N}\sum_{n=0}^{N-1} e^{j2\pi(k-l)n/N} = \frac{1}{N}\sum_{n=0}^{N-1}\left[e^{j2\pi(k-l)/N}\right]^n \tag{1.35}$$

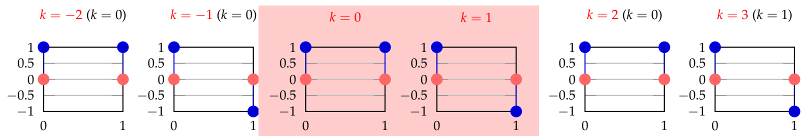

Figure 1.17. Real and imaginary components of complex exponentials for size $N = 2$. There are only 2 distinct frequencies, after which the signal repeats itself.

Recall that a geometric series with basis a sums to $\sum_{n=0}^{N-1} a^n = (1 - a^N)/(1 - a)$. Thus,

$$\langle e_{kN}, e_{lN} \rangle = \frac{1}{N} \frac{1 - \left[e^{j2\pi(k-l)/N}\right]^N}{1 - e^{j2\pi(k-l)/N}} = \frac{1}{N} \frac{1 - 1}{1 - e^{j2\pi(k-l)/N}} = 0 \tag{1.36}$$

The completion of this proof follows from

$$\left[e^{j2\pi(k-l)/N}\right]^N = e^{j2\pi(k-l)} = \left[e^{j2\pi}\right]^{(k-l)} = 1$$

■

Theorem 3 *Opposite frequencies k and $-k$ yield conjugate signals: $e_{-kN} = e^*_{kN}(n)$*

Opposite frequencies have the same real part, but opposite imaginary parts, meaning the cosine stays the same and the sine changes sign. We can prove this with the following proof.
Proof: Following from the definition of complex exponentials given in (1.23)

$$e_{-kN}(n) = \frac{e^{j2\pi(-k)n/N}}{\sqrt{N}} = \frac{e^{-j2\pi kn/N}}{\sqrt{N}} = \left[\frac{e^{j2\pi kn/N}}{\sqrt{N}}\right]^* = e^*_{kN}(n) \tag{1.37}$$

■

Following this theorem of conjugate frequencies, we once again look at the canonical set of N frequencies for complex exponentials. Since we know that opposite frequencies have conjugates, in actuality, then, there are only $N/2 + 1$ distinct frequencies. Consider the canonical set $-N/2 + 1$ to $N/2$. Here we have conjugate pairs for all frequencies but 0 and $N/2$ (e.g. $-N/2 + 1$ and $N/2 - 1$ down to -1 and 1). Recall that only frequencies from 0 to $N/2$ (or $f_s/2$) have physical meaning. The concept of conjugate frequencies then becomes more reasonable to understand, since you can only have up to $N/2$ oscillations in N samples. Consider complex exponentials for different sample sizes N. In Figure 1.17, you can see that for sample size $N = 2$, there are only 2 distinct frequencies for $k = 0$ and $k = 1$. Here, there are no imaginary parts. Now consider complex exponentials of size $N = 4$ in Figure 1.18. Here we have 3 distinct signals of frequencies $k = 0, 1, 2$. You can see that frequency $k = 1$ has conjugate $k = -1$. Figure 1.19 has complex exponentials of size $N = 8$. Once again, there are $N/2 + 1 = 5$ distinct signals. Those that are conjugates are seen in Figure 1.20. One last example in Figure 1.21 is for size $N = 16$, showing only the 9 distinct frequencies.

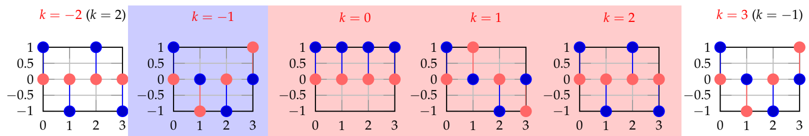

Figure 1.18. Real and imaginary components of complex exponentials for size $N = 4$. There are only 3 distinct frequencies, after which the signal repeats itself.

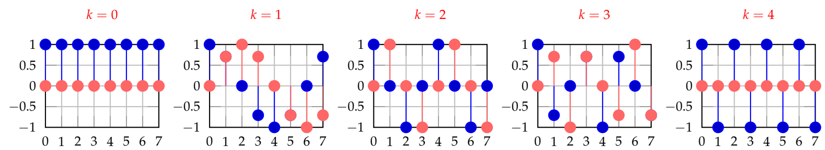

Figure 1.19. Distict real and imaginary components of complex exponentials for size $N = 8$. There are only 5 distinct frequencies, after which the signal repeats itself.

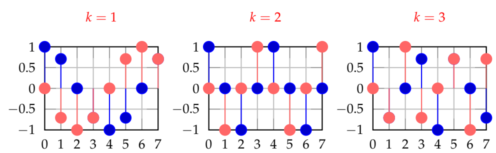

Figure 1.20. Non-distinct, conjugate signals of $k = 1$,$k = 2$ and $k = 3$ for complex exponentials of sample size $N = 8$.

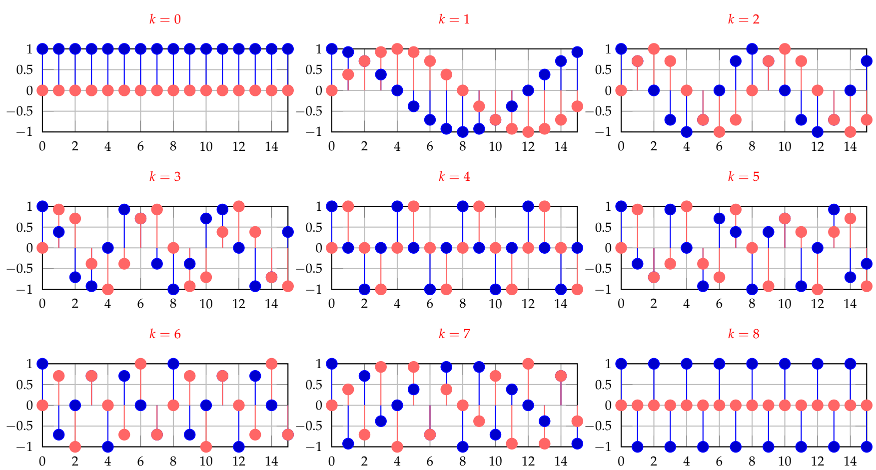

Figure 1.21. Distict real and imaginary components of complex exponentials for size $N = 16$. There are only 9 distinct frequencies, after which the signal repeats itself (non-distinct frequencies are not shown).

Mathematically speaking, the complex exponential, the sine, and the cosine are all different signals. Intuitively speaking, all of them are oscillations of the same frequency. Since complex exponentials have imaginary parts, they don't exist in the real world. Nevertheless, we work with them instead of sines and cosines because they are easier to handle, as we will see later on in this course.