Signal and Information Processing

Alejandro Ribeiro

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Contents

Chapter 1

Discrete Fourier Transform

1.1 Discrete Fourier Transform(DFT), definitions and examples

1.1.1 Definition of the Discrete Fourier Transform

The Discrete Fourier Transform (DFT) is a very useful tool for the analysis of information. It defined as the following for a signal $x(n)$ of duration N with time indices from $n = [0, N-1]$.

$$X(k) := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) \exp(-j2\pi kn/N) \tag{1.1}$$

For all integers $k \in \mathbb{Z}$

We see that the DFT $X(k)$ takes the argument k, referred to as the frequency, instead of the time index n which acts as the argument for the original signal $x(n)$. The value corresponding to each frequency k is a summation involving the original signal and a complex exponential of frequency $-k$. Since each value in X involves a summation with all the values in x, it captures all the information in the original signal. The DFT is therefore a transformation of the original signal, written as $X = \mathcal{F}(x)$.

Because we multiply $x(n)$ with a complex exponential in the calculation of the DFT, the DFT is complex even if the signal $x(n)$ is real. While analyzing a DFT it is customary to focus on its magnitude instead of its real and imaginary parts separately.

$$X(k) = X_R(k) + jX_I(k) \tag{1.2}$$

$$|X(k)| = \left[X_R^2(k) + X_I^2(k)\right]^{1/2} = [X(k)X^*(k)]^{1/2} \tag{1.3}$$

Although the definition of a DFT as an explicit summation is easier to compute, sometimes an alternate definition can be more useful for analysis

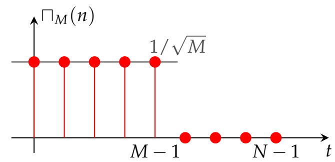

Figure 1.1: Unit Energy Square Pulse ending at M with signal duration N

1.1.2 DFT elements as inner products

We see that exponential factor in the DFT calculation, $e_{-kN}(n) = \frac{1}{\sqrt{N}}e^{-j2\pi kn/N}$, is a complex exponential with a frequency of $-k$. Since complex exponentials with negative frequencies are conjugates of each other, we can rewrite the DFT as the following

$$X(k) := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e_{-kN}(n) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^*_{kN}(n) \tag{1.4}$$

Which is equal to $X(k) = \langle x, e_{kN} \rangle$ from the definition of the inner product.

This gives us another interpretation of the DFT. The value of the DFT at k is the inner product between the signal $x(n)$ and a complex exponential of frequency k. This is a projection of the signal $x(n)$ onto the complex exponential $e_{kN}(n)$, and tells us how much of the signal x is made up of an oscillation of frequency k.

Using these definitions for the calculation of the DFT, we can determine the DFT for several common signals.

Example 1 *The DFT of a square pulse*

Solution: Let us consider the DFT of a square pulse signal. As a reminder, the unit energy square pulse is defined as

$$\begin{aligned} \sqcap_M(n) &= \frac{1}{\sqrt{M}} && \text{if } 0 \le n < M \\ \sqcap_M(n) &= 0 && \text{if } M \le n \end{aligned} \tag{1.5}$$

which can be seen in Figure 1.1. Substituting the square pulse into the definition of the DFT, we have

$$\begin{aligned} X(k) &:= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \sqcap_M(n) e^{-j2\pi kn/N} \\ &= \frac{1}{\sqrt{N}} \sum_{n=0}^{M-1} \frac{1}{\sqrt{M}} e^{-j2\pi kn/N} \end{aligned} \tag{1.6}$$

We can shorten the summation in the DFT since only the first $M-1$ elements of $\sqcap_M(n)$ are not null. Therefore, $X(k)$ is just the sum of the first M components of an exponential of frequency $-k$.

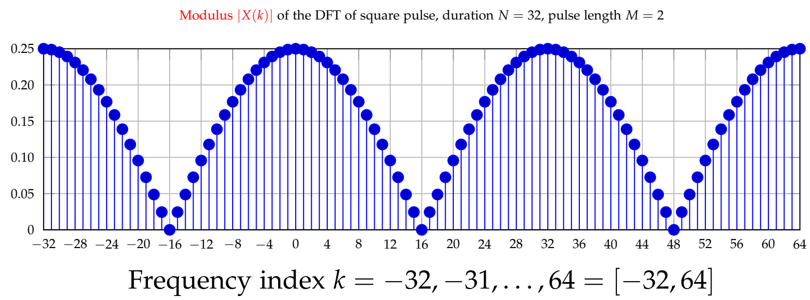

Figure 1.2: DFT of a square pulse of $M = 2$ and $N = 32$

For a square pulse of length $M = 2$ and signal duration $N = 32$, the DFT is

$$\begin{aligned} X(k) &:= \frac{1}{\sqrt{N}} \sum_{n=0}^{M-1} \frac{1}{\sqrt{M}} e^{-j2\pi kn/N} \\ &= \frac{1}{\sqrt{N}} \sum_{n=0}^{2-1} \frac{1}{\sqrt{2}} e^{-j2\pi kn/N} \\ &= \frac{1}{\sqrt{2N}} (1 + e^{-j2\pi k/N}) \\ &= \frac{1}{\sqrt{64}} (1 + e^{-j2\pi k/64}) \end{aligned} \tag{1.7}$$

We can see this DFT plotted in Figure 1.2. From the interpretation in section 1.1.2, the DFT tells us how much of a signal is made up of an oscillation of frequency k. The DFT of the square pulse then shows us the proportion of the signal that is made up of each frequency k

The DFT in 1.2 also has the interesting property of being periodic with period $N = 64$. If we substitute $k = 0, \pm N, \pm 2N \dots$ into equation 1.7, we have

$$\begin{aligned} X(k \pm N) &= \frac{1}{\sqrt{64}} (1 + e^{-j2\pi(k \pm N)/64}) \\ X(0 \pm 64) &= \frac{1}{\sqrt{64}} (1 + e^{-j2\pi(0 \pm 64)/64}) \\ &= \frac{1}{\sqrt{64}} (1 + e^{-j2\pi 0/64} e^{-j2\pi(\pm 64)/64}) \\ &= \frac{1}{\sqrt{64}} (1 + e^{-j2\pi 0/64} e^{\pm j2\pi}) \\ &= \frac{1}{\sqrt{64}} (1 + e^{-j2\pi 0/64}) \\ X(0 \pm 64) &= \frac{1}{\sqrt{64}} \end{aligned} \tag{1.8}$$

Likewise, for $k = N/2 \pm N$, we have $X(k) = 0$. In the next section, we will see that all DFT have the important property of being periodic with period N ■

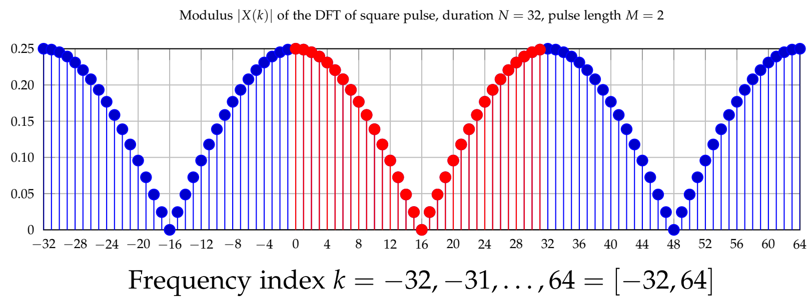

Figure 1.3: DFT of a square pulse with the canonical set $k \in [0, N-1]$

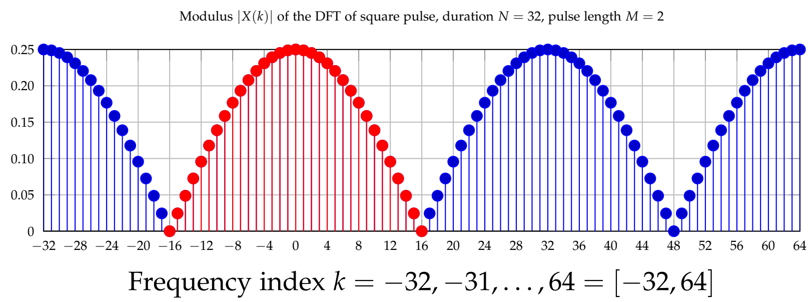

Figure 1.4: DFT of a square pulse with the canonical set $k \in [-N/2, N/2]$

1.1.3 Periodicity of the DFT

Since complex exponentials with frequencies N apart are equivalent, we can use this to prove that for a DFT of $x(n)$ with length N, frequencies k that are N apart have the same DFT values.

$$\begin{aligned} X(k+N) &:= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi(k+N)n/N} \\ &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \\ &= X(k) \end{aligned} \tag{1.9}$$

Because the DFT shows how much of signal $x(n)$ of length N can be represented by a complex exponential of frequency k, and complex exponentials range in frequency from $k = 0$ to $k = N$, it makes sense that the DFT has only N frequencies and is periodic outside of this range.

Thanks to the periodicity of the DFT, we only have to look at N consecutive frequencies, known as a *canonical set*. The two main canonical sets we use are $k \in [0, N-1]$ and $k \in [-N/2, N/2]$ These 2 canonical sets can be seen in Figures 1.3 and 1.4.

The canonical set $k \in [0, N-1]$ is easier to use for computations since we can easily iterate over each frequency k, calculating the DFT values as we do. However, for interpreting the DFT we often use the canonical set $k \in [-N/2, N/2]$ instead. This puts frequency 0 at the center of the DFT, with frequencies k and $-k$ symmetrically on either side.

The 2 canonical sets $k \in [0, N-1]$ and $k \in [-N/2, N/2]$ are related to each other by a "chop and shift". Since frequencies greater than $N/2$ have no physical meaning, we take the frequencies $[N/2, N-1]$ and move them N steps to left to $[-N/2, -1]$, giving us the canonical set $[-N/2, N/2]$. During our chop and shift, we end up copying the value at $N/2$ and moving it to $-N/2$ resulting in a repeated element. Since frequencies $-N/2$ and $N/2$ are repeated, we actually have $N+1$ elements, but we keep both these end elements for the sake of symmetry.

1.1.4 Pulses of Different Lengths

From example 1, we see that the DFT $X(k)$ tells us how much of $x(n)$ is made up of a complex exponential of frequency k. Another interpretation is that the DFT gives us information on how fast the signal x changes.

In Figures 1.5 to 1.9, we have the DFTs of several square pulses with varying length. As the width of the square pulse increases, we can see that the center lobe in the DFT becomes narrower. This shows that more of the DFT's weight becomes concentrated at lower frequencies and reflects the fact that a pulses of longer lengths change more slowly than pulses of shorter lengths. For example, the DFT of a square pulse of length $M = 4$ in Figure 1.5 has its central lobe ranging from $k = -64$ to $k = 64$. However, the DFT of a square pulse of length $M = 8$ in Figure 1.7 has its central lobe over a shorter range of frequencies from $k = -32$ to $k = 32$.

1.1.5 Delta Functions and Complex Exponentials

Delta Functions and Complex Exponentials are also signals we will encounter often in the future, and here we will calculate their DFTs.

Example 2 *The DFT of a delta function*

Solution: The delta function is defined as the following

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

The fact that the delta function is 0 everywhere except for $n = 0$ is a useful property for simplifying the summations in the calculation of the DFT. After substituting the delta function into the definition of the DFT, we have

$$\begin{aligned} X(k) &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \delta(n) e^{-j2\pi kn/N} \\ &= \frac{1}{\sqrt{N}} \delta(0) e^{-j2\pi k(0)/N} \\ &= \frac{1}{\sqrt{N}} \end{aligned} \tag{1.10}$$

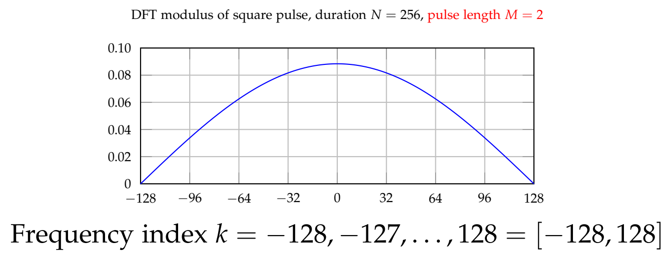

Figure 1.5: DFT of a square pulse with lengths $M = 2$ and signal duration $N = 256$

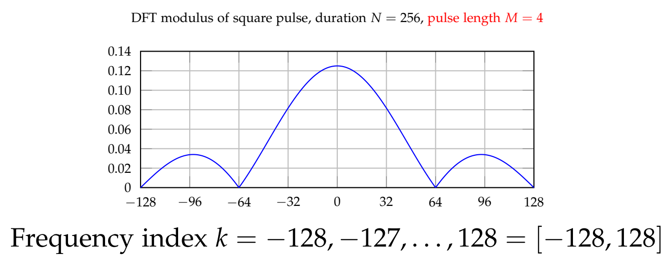

Figure 1.6: DFT of a square pulse with lengths $M = 4$ and signal duration $N = 256$

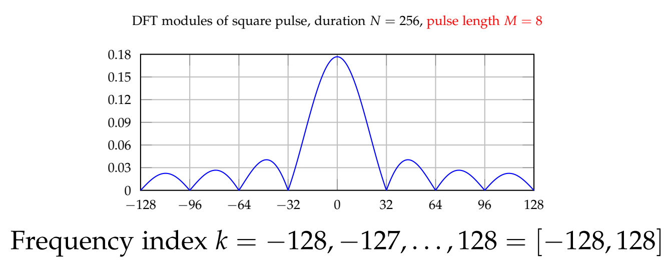

Figure 1.7: DFT of a square pulse with lengths $M = 8$ and signal duration $N = 256$

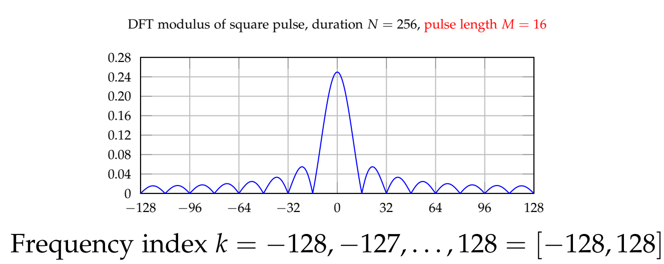

Figure 1.8: DFT of a square pulse with lengths $M = 16$ and signal duration $N = 256$

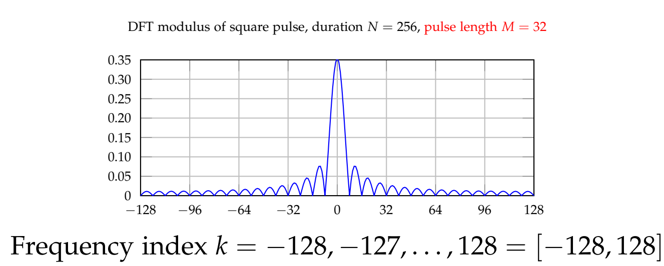

Figure 1.9: DFT of a square pulse with lengths $M = 32$ and signal duration $N = 256$

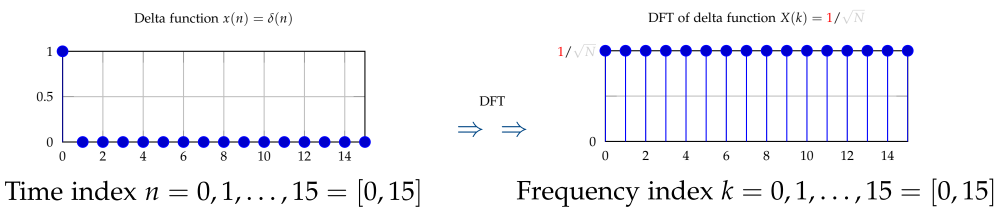

Figure 1.10: DFT of a delta function is constant

We can ignore all terms except for $n = 0$ in the delta function since they will all be 0

We obtain the DFT for the delta function seen in 1.10. The DFT of a delta function is completely constant, with each of its values equal to $1/\sqrt{N}$. If we compute the energy of the DFT by summing the square of all N values, we will see that the DFT has energy equal to 1, the same as the delta function. This energy conservation is an important property of the DFT that we will prove later in section 1.4.2 ■

Example 3 *DFT of a shifted delta function*

Solution:
We have the definition of a shifted delta function, which is similar to a normal delta function except it has a non-zero value at $n = n_0$ instead of $n = 0$.

$$\delta(n - n_0) = \begin{cases} 1 & n = n_0 \\ 0 & n \neq n_0 \end{cases} \tag{1.11}$$

After substituting the shifted delta function into the definition of the DFT

$$\begin{aligned} X(k) &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \delta(n - n_0) e^{-j2\pi kn/N} \\ &= \frac{1}{\sqrt{N}} \delta(n_0 - n_0) e^{-j2\pi kn_0/N} \\ &= \frac{1}{\sqrt{N}} e^{-j2\pi kn_0/N} \\ &= e_{-n_0 N}(k) \end{aligned} \tag{1.12}$$

we can ignore any values in the DFT summation for which $n \neq n_0$ since the delta function will be 0. Observe that we obtain a complex exponential with frequency $-n_0$ for a delta function shifted by $-n_0$, seen in 1.11.

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We can also use the inner product definition of the DFT defined in section 1.1.2 in order to find the DFT of complex exponentials. Their property of orthonormality for means that the inner product of any 2 complex exponentials with different frequencies will be equal to 0, and the inner product of any 2 complex exponentials with the same frequency will be equal to 1. This allows us to simplify much of our calculations.

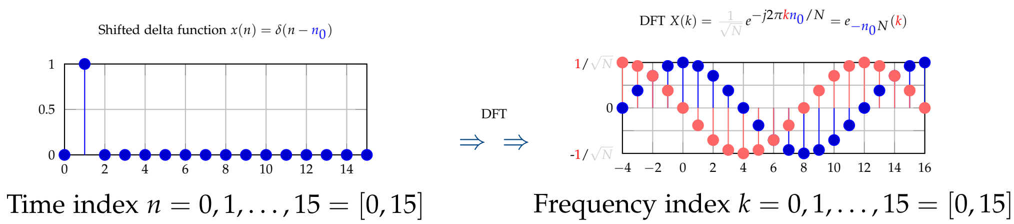

Figure 1.11: DFT of a delta function shifted to the right by n_0 is equal to a complex exponential with a frequency of $-n_0$

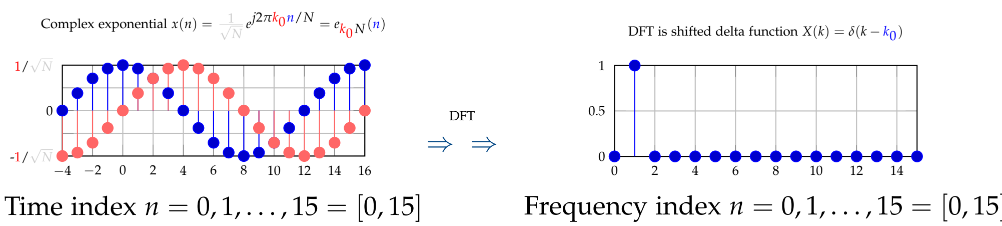

Figure 1.12: DFT of a complex exponential with frequency k_0 is a delta function shifted k_0 to the right

Example 4 *DFT of a complex exponential*

Solution: For a complex exponential with frequency k_0, we can use the inner product notation for the DFT

$$x(n) = \frac{1}{\sqrt{N}} e^{j2\pi k_0 n/N} = e_{k_0 N}(n) \tag{1.13}$$

$$X(k) = \langle e_{k_0 N}, e_{kN} \rangle \quad = \delta(k - k_0) \tag{1.14}$$

The value of the DFT at frequency k is the inner product of a complex exponential of frequency k with the signal. Due to the orthonormality of complex exponentials, we see that the DFT of of our signal is 0 for all frequencies except for k_0. The DFT is therefore a shifted delta function, seen in 1.12

■

Example 5 *DFT of a constant*

Solution: We can express a complex function $x(n)$ as a complex exponential with frequency $k_0 = 0$

$$\begin{aligned} x(n) &= \frac{1}{\sqrt{N}} \\ &= \frac{1}{\sqrt{N}} e^{j2\pi(0)n/N} \\ &= e_{0N}(n) \end{aligned} \tag{1.15}$$

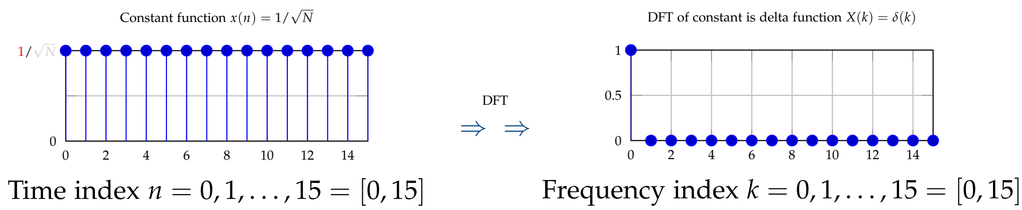

Figure 1.13: DFT of a constant is a delta function

$$\begin{aligned} X(k) = \langle e_{0N}, e_{kN} \rangle &= \delta(k-0) \\ &= \delta(k) \end{aligned} \tag{1.16}$$

Due to the orthonormality of complex exponentials, we see that the DFT of the complex exponential is 0 for all frequencies except for 0. The DFT is therefore a delta function, seen in 1.13

■

1.1.6 Observations

From the above examples, we observe several properties of the DFT

Firstly, from example 1, we see that the DFT of a signal contains information about what frequencies it is made of and its rate of change. A signal that changes faster, such as a short square pulse, will have higher DFT values for higher frequencies. This results in more of the DFT's weight being at high values of k. The opposite happens for signals that change slowly, such as a longer square pulse, which has more of the DFT's weight being concentrated around frequency 0.

Secondly, from example 2, we see that the DFT conserves energy. The energy of a DFT should be equal to the energy of the original signal. For our DFT of a delta function, we obtain a constant with values equal to $1/\sqrt{N}$. If we calculate the energy of this constant, we get an energy equal to 1, the same as our unit energy delta function. We will prove this property in later sections.

Thirdly, from examples 2 to 5, we can see a duality between signals and DFTs. For delta functions and complex exponentials, we see that signals and DFTs seem come in pairs. In Figure 1.10, the DFT of a delta is a constant, and in Figure 1.13, the DFT of a constant is a delta. Likewise, in Figure 1.11, the DFT of a shifted delta is a complex exponential, and in Figure 1.12, the DFT of a complex exponential is a shifted delta. Later, when we introduce the inverse DFT, we will see that signals and DFTs do in fact come in signal - transform pairs.

1.2 Units of the DFT

Up to now, we have considered our signals and DFTs mathematically without any regard to their duration and frequency in the physical world. Now we relate the discrete frequency k to the actual physical frequency f_k.

The discrete and physical frequencies are related to each other by the sampling time T_s and sampling frequency $f_s = 1/T_s$. Every T_s seconds, we sample the value of our physical signal, forming our discrete signal $x[n]$.

If our discrete signal has length N, then the physical signal has duration $T = NT_s$. $x[0]$ is equal to the physical signal at $x(0 \times T_s)$, $x[1]$ is equal to $x(1 \times T_s)$, $x[2]$ is equal to $x(2 \times T_s)$, and so on. The last value in our discrete signal, $x[n-1]$, is equal to the physical signal at $x((n-1)T_s)$. Since each of our samples lasts for T_s seconds, the last sample at $(n-1)T_s$ fills up the remaining T_s seconds for a total duration of $T = NT_s$.

If we have a discrete frequency k, we have k oscillations in a time of NT_s, and the period of each oscillation would be equal to NT_s/k. The real frequency would also be equal to the inverse of the period.

$$f_k = \frac{k}{NT_s} = k\frac{f_s}{N} \tag{1.17}$$

Since discrete frequencies greater than $k = N/2$ have no real meaning, we see that if we convert this to physical frequencies.

$$f_{N/2} = \frac{N/2 f_s}{N} = \frac{f_s}{2} \tag{1.18}$$

For our DFT, we cannot obtain information about frequencies greater than half our sampling frequency.

If we convert our canonical set $k \in [-N/2, N/2]$ into physical frequencies, we have a range of frequencies between $-f_s/2$ and $f_s/2$, with each frequency separated by f_s/N. From this set of frequencies, we see how changing N, the number of samples, and f_s, the sampling frequency, changes the information we can obtain. By increasing the number of samples, we decrease the interval f_s/N, increasing our resolution. By increasing our sampling frequency f_s (and decreasing sampling time T_s), we increase the range of frequencies from $-f_s/2$ to $f_s/2$.

Example 6 *Physical Units for the DFT of a discrete complex exponential*

Solution: From example 4, we see that the DFT of a discrete complex exponential of frequency k_0 is a delta function shifted to k_0. We will now show how the discrete frequencies in this DFT correspond to physical frequencies.

In Figure 1.14, we see that for a delta function shifted to k_0, k_0 corresponds to a frequency of f_0 where $f_0 = k\frac{f_s}{N}$. Therefore, Figure 1.14 is the DFT of a physical complex exponential with a frequency of $f_0 = k\frac{f_s}{N}$ hertz. Notice that only complex exponentials with physical frequencies that are multiples of f_s/N, such as $f_0 = k_0\frac{f_s}{N}$ have DFTs that

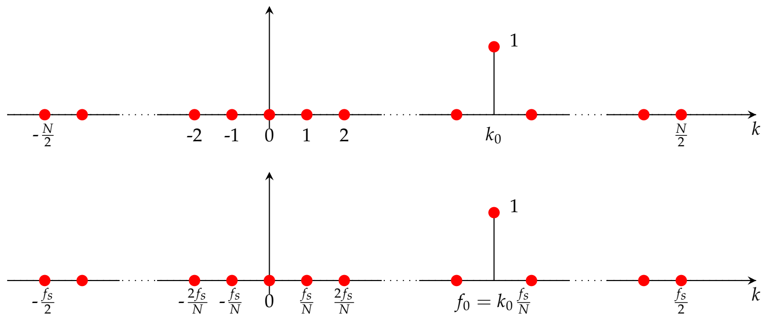

Figure 1.14: DFT of a complex exponential with physical units

are equal to delta functions. For physical frequencies that are not multiples of f_s/N, they will have several non-zero values in their DFT, showing that they are made up of several different frequencies that are multiples of f_s/N. ∎

Example 7 *Physical units for the DFT of a square pulse*

Solution: In example 1, we calculated the DFT of a square pulse $M = 2$ long and with duration $N = 32$. Here, we will calculate the DFT of a physical square pulse of length $T_0 = 4$ seconds with signal duration $T = 32$s, with a sampling frequency of $f_s = 8$ Hz.

A sampling frequency of $f_s = 8$ Hz corresponds to a sampling time of $T_s = 1/8 = 0.125$ seconds. If we sample continuously over the signal duration of $T = 32$s, we will get $N = 32/0.128 = 256$ samples, the duration of our discrete signal. The length of our square pulse is $M = 4/0.128 = 32$ samples long.

Since k ranges from $-N/2$ to $N/2$, our maximum discrete frequency will be $k = N/2 = 256/2 = 128$. This corresponds to a physical frequency of $f_k = kf_s/N = f_s/2 = 4$ Hz. The resolution of our frequencies in the DFT will be $f_s/N = 8/256 = 0.03125$Hz

In Figure 1.16, we can see the DFT of our square pulse with physical frequencies, and the corresponding DFT with discrete frequencies in 1.15.

In Fig 1.17, we take a closer look at our DFT in 1.16. We can see that the distance between every value is equal to the value of $f_s/N = 8/256 = 1/32 = 0.03125$ Hz we calculated earlier, which is the best resolution we can get with a sampling frequency of f_s and N samples.

From the DFT, we see that frequencies 0.25 Hz, 0.5 Hz, 0.75 Hz, and so on are equal to 0. These frequencies also correspond to $1/T_0, 2/T_0, 3/T_0, \ldots,$ where $T_0 = 4s$ is the length of our square pulse. In fact, for any square pulse, there will be zeros at multiples of $1/T_0$. Most of the energy of our DFT is between $-1/T_0$ and $1/T_0$. As the duration of our square pulse increases, T_0 increases resulting in more of our DFT energy being concentrated around frequency 0.

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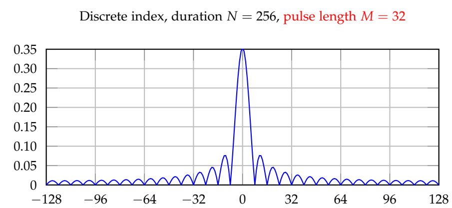
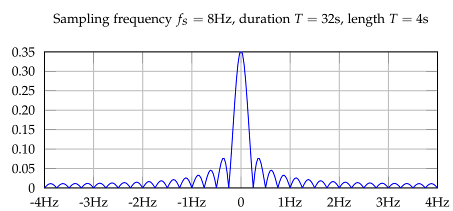


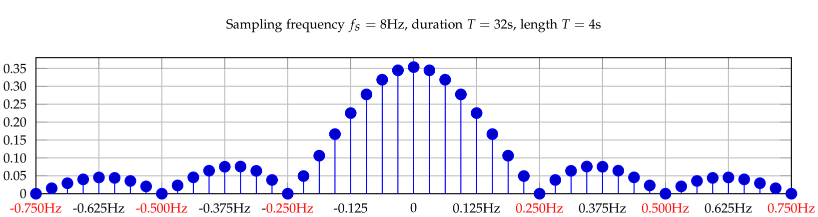
Discrete frequency $k \in [-128, 128]$

Figure 1.15: DFT of a square pulse with discrete frequencies



Frequencies are $0, \pm f_s/N, \pm 2f_s/N, \ldots, \pm f_s/2$

Figure 1.16: DFT of a square pulse with physical units



Frequencies $0, \pm 0.03125Hz, \pm 0.06250Hz, \ldots, \pm 0.750Hz$

Figure 1.17: Closeup of the DFT of a square pulse

1.3 DFT inverse

Now that we have a grasp of the DFT transform, we will define the inverse transform. If the DFT transforms $x \to \mathcal{F}(x) = X$, then the iDFT transforms $X \to \mathcal{F}^{-1}(X) = \mathcal{F}^{-1}\mathcal{F}(x) = x$ back into the original signal.

The idea behind the construction of the iDFT is that any signal can be represented as a sum of complex exponentials, each with a different weight. The weight for a complex exponential of frequency k is equal to the $X(k)$, the DFT of the original signal. With this, we can define the iDFT as

$$x(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \tag{1.19}$$

Notice that this is very similar to the definition of the DFT except for the sign in the exponent. A DFT has a negative sign in the exponent, while the iDFT has a positve sign. Because a DFT is periodic, with $X(k+N) = X(k)$, we only need to consider N different frequencies in a canonical set. Since we can sum over any canonical set of frequencies k, we can redefine the interval of summation as

$$\begin{aligned} x(n) &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \\ &= \frac{1}{\sqrt{N}} \sum_{k=-N/2+1}^{N/2} X(k) e^{j2\pi kn/N} \end{aligned} \tag{1.20}$$

Where the coefficient $X(k)$ multiplying each complex exponential $e^{j2\pi kn/N}$ is the k'th element of the DFT of x

$$X(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \tag{1.21}$$

Example 8 *Proof of the DFT inverse formula*

Solution: Although we have defined the iDFT, we have no proof that it actually is an inverse of the DFT.

Suppose we have a signal x and its DFT $X = \mathcal{F}(x)$. Let the signal $\tilde{x} = \mathcal{F}^{-1}(X)$ be the signal obtained from the iDFT of X. To prove that the iDFT is in fact the inverse of the DFT, we need to prove that the resulting signal $\tilde{x}$ is equal to the original signal x.

From the definition of the iDFT, we have

$$\tilde{x}(\tilde{n}) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X(k) e^{j2\pi k\tilde{n}/N} \tag{1.22}$$

From the definition of the DFT, we have

$$X(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \tag{1.23}$$

If we substitute the definition of the DFT into the definition of the iDFT, we have

$$\tilde{x}(\tilde{n}) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left[\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \right] e^{j2\pi k\tilde{n}/N} \tag{1.24}$$

If we exchange the order of summation to sum first over k then n

$$\begin{aligned} \tilde{x}(\tilde{n}) &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \left[\sum_{k=0}^{N-1} x(n) \frac{1}{\sqrt{N}} e^{-j2\pi kn/N} e^{j2\pi k\tilde{n}/N} \right] \\ &= \sum_{n=0}^{N-1} x(n) \left[\sum_{k=0}^{N-1} \frac{1}{\sqrt{N}} e^{-j2\pi kn/N} \frac{1}{\sqrt{N}} e^{j2\pi k\tilde{n}/N} \right] \end{aligned} \tag{1.25}$$

Since $x(n)$ does not depend on k, we can move it out of the innermost summation. We can also move the factors $\frac{1}{\sqrt{N}}$ anywhere we wish, and we put them next to the exponential factors to turn them into unit energy complex exponentials.

Notice that the innermost sum is an inner product between 2 complex exponentials $e_{\tilde{n}N}$ and e_{nN}. Due to orthonormality, this inner product is equal to a delta function.

$$\sum_{k=0}^{N-1} x(n) \frac{1}{\sqrt{N}} e^{-j2\pi kn/N} e^{j2\pi k\tilde{n}/N} = \delta(\tilde{n} - n) \tag{1.26}$$

If we substitute this back into our equation

$$\begin{aligned} \tilde{x}(\tilde{n}) &= \sum_{n=0}^{N-1} x(n)\delta(\tilde{n} - n) \\ &= x(\tilde{n})\delta(\tilde{n} - \tilde{n}) \\ &= x(\tilde{n}) \end{aligned} \tag{1.27}$$

Because of the delta function in the sum, all terms in the sum except for $n = \tilde{n}$ are equal to 0. The equation then simplifies to $x(\tilde{n})$. We have shown that $\tilde{x} = \mathcal{F}^{-1}(\mathcal{F}(x))$ and therefore proved that our definition of the iDFT is in fact the inverse of the DFT.

■

1.3.1 The Inverse DFT as an Inner Product

Similar to the DFT, we can interpret the iDFT as an inner product with a set of complex exponentials. From the definition of the iDFT

$$\tilde{x}(\tilde{n}) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X(k) e^{j2\pi k\tilde{n}/N} \tag{1.28}$$

We see that $\frac{1}{\sqrt{N}}e^{j2\pi kn/N}$ is a discrete complex exponential $e_{nN}(k)$. We can then rewrite the iDFT as

$$\begin{aligned}\tilde{x}(\tilde{n}) &= \sum_{k=0}^{N-1} X(k)e_{nN}(k) \\ &= \sum_{k=0}^{N-1} X(k)e^{*}_{-nN}(k)\end{aligned} \tag{1.29}$$

Which is the inner product $x(n) = \langle X, e_{-nN}\rangle$ between the DFT and a complex exponential of frequency $-n$.

1.3.2 Inverse DFT as Successive Approximations

We originally developed the iDFT by considering a signal $x(n)$ as a sum of complex exponentials with different frequencies k. Consider the definition of the iDFT over the canonical set $k \in [-N/2+1, N/2]$

$$x(n) = \frac{1}{\sqrt{N}} \sum_{k=-N/2+1}^{N/2} X(k)e^{j2\pi kn/N} \tag{1.30}$$

If we start with frequency $k = 0$, and add frequencies $k = \pm 1$, $k = \pm 2$, and so on up to faster frequencies, we see that the sum in the iDFT is equal to.

$$\begin{array}{llll} x(n) = & X(0)e^{j2\pi 0n/N} & & \text{constant} \\ + & X(1)e^{j2\pi 1n/N} & +X(-1)e^{-j2\pi 1n/N} & \text{single oscillation} \\ + & X(2)e^{j2\pi 2n/N} & +X(-2)e^{-j2\pi 2n/N} & \text{two oscillations} \\ + & X\left(\frac{N}{2}-1\right)e^{j2\pi(\frac{N}{2}-1)n/N} & +X\left(-\frac{N}{2}+1\right)e^{-j2\pi(\frac{N}{2}-1)n/N} & \left(\frac{N}{2}-1\right) \text{oscillations} \\ + & X\left(\frac{N}{2}\right)e^{j2\pi(\frac{N}{2})n/N} & & \frac{N}{2} \text{ oscillations} \end{array} \tag{1.31}$$

This is the basis behind using the iDFT to approximate and reconstruct different signals. We start with a constant equal to the average value of the signal, and add on faster and faster oscillations until finally reaching oscillations with frequency $k = N/2$. To illustrate this, we will try reconstructing a square pulse.

Example 9 *Reconstruction of a square pulse*

Consider a square pulse with $N = 256$ and $M = 128$. Starting with $k = 0$, we have only the DC component of the pulse, seen in Figure 1.18. This is just the average value, and not a good approximation.

But as we add more and more frequencies, from 1.18 to 1.23, we get a better and better approximation. A common way of compressing signals is to store the DFT values instead of the actual signal values, and reconstructing the original signal when needed.

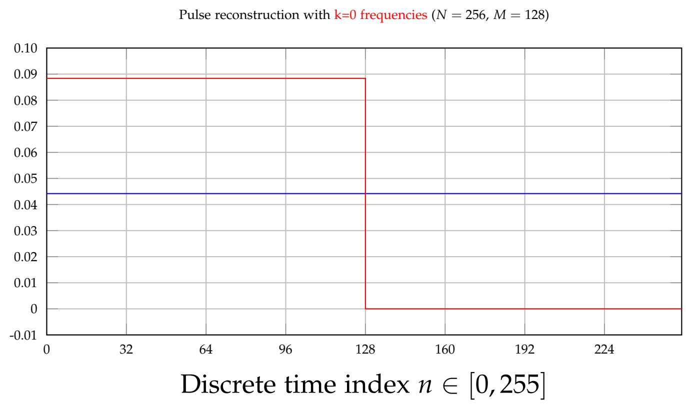


Figure 1.18: Closeup of the DFT of a square pulse

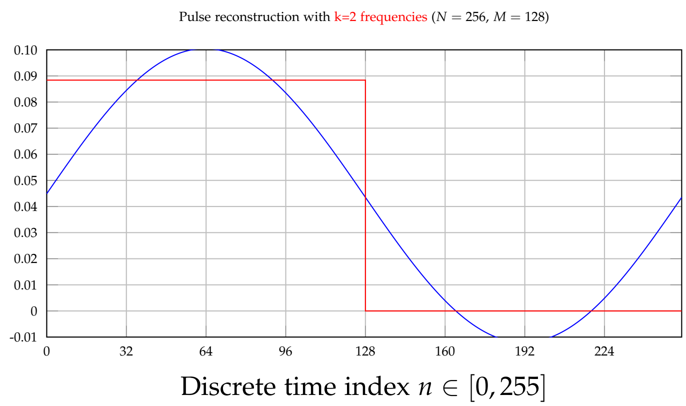


Figure 1.19: Closeup of the DFT of a square pulse

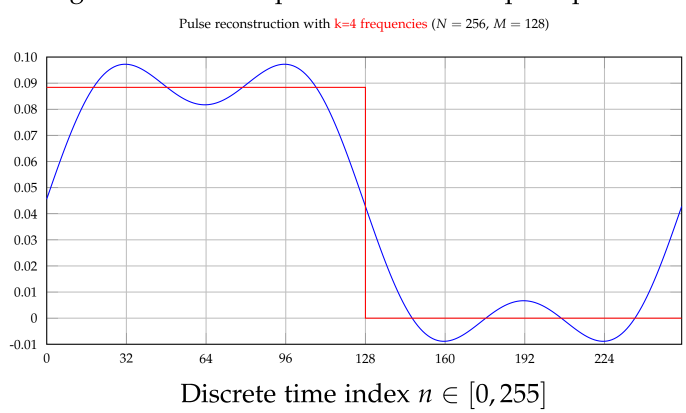
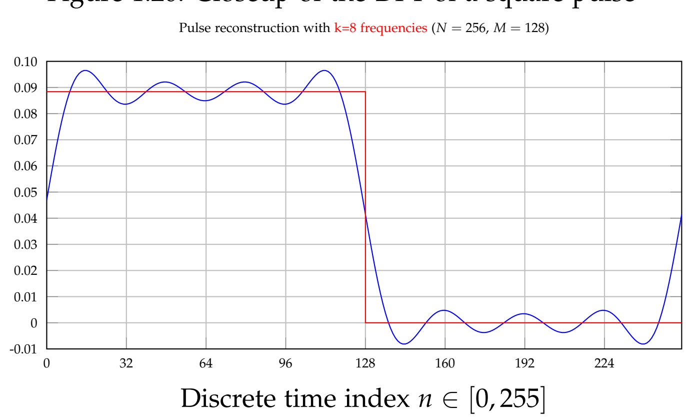


Figure 1.20: Closeup of the DFT of a square pulse



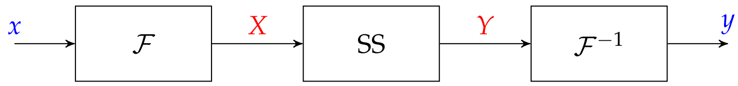


Figure 1.24: Process of Spectrum Reshaping

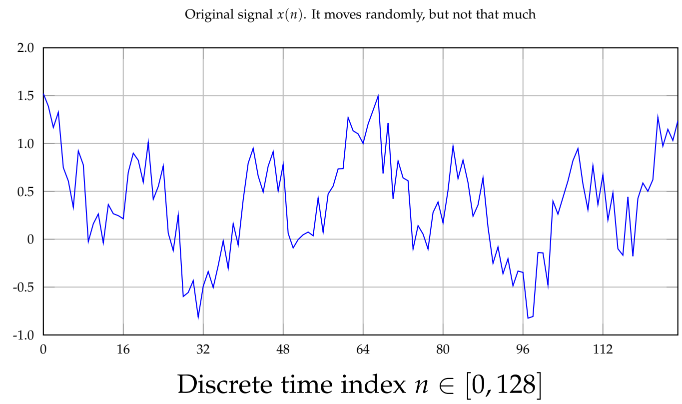


Figure 1.25: Process of Spectrum Reshaping

For example, sending the DFT of a song over the Internet instead of the actual sound samples. In Figure 1.23, we only have to send $k = 32$ DFT values instead of $N = 256$ samples. If we want a better reconstruction of the original signal, we can send more DFT values, but this results in a lower compression ratio and a tradeoff between compression and accurate reconstruction.

1.3.3 Spectrum Shaping

Now that we can transform a signal $x(n)$ into the frequency domain $X(k)$ with the DFT, and transform it back into the time domain iDFT, we can do a lot of interesting analysis and processing in the frequency domain that we would be able to do in the time domain. A common procedure is Spectrum (Re)shaping, where we modify X and transform it into Y before converting it back into the time domain, as seen in Figure 1.24.

Example 10 *Spectrum reshaping to remove noise*

Solution:
A common application of spectrum reshaping is cleaning a noisy signal. Consider the signal in Figure 1.25, which has an underlying trend we want to analyse which is distorted by noise. In the time domain, it is difficult to see what the original signal should be. However, we can look at the spectrum(DFT) of this signal.

We can see the DFT in Figure 1.26. Here, the trend we are looking for appears as spikes in the DFT, which is clearly separated from the high frequency noise that ranges over the entire spectrum. To remove the noise, we reshape the spectrum into Figure 1.27 by removing all frequencies $k > 8$.

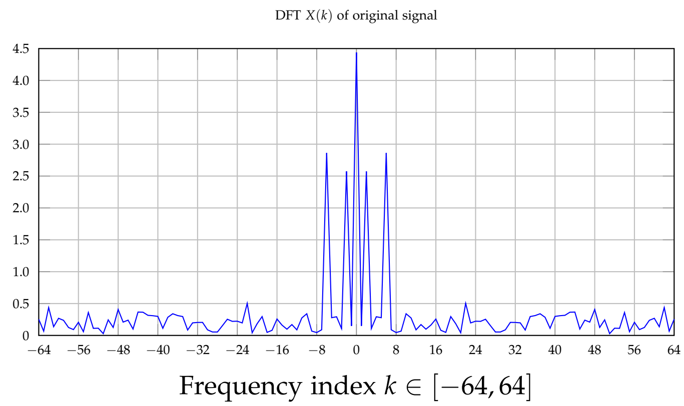


Figure 1.26: Process of Spectrum Reshaping

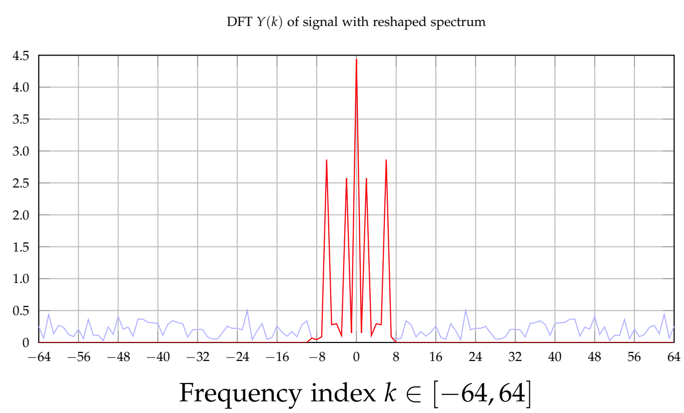


Figure 1.27: Process of Spectrum Reshaping

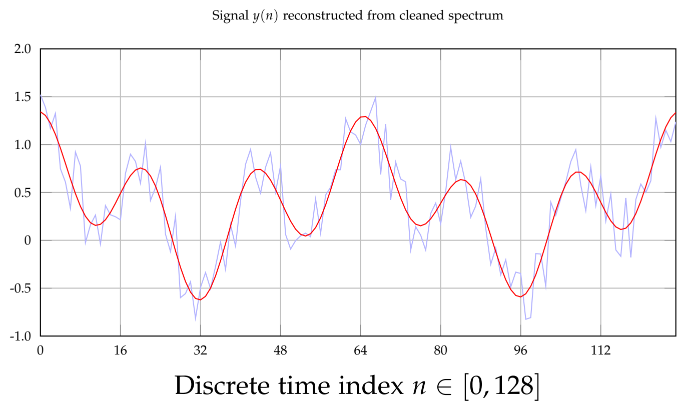


Figure 1.28: Process of Spectrum Reshaping

Now that we have our clean spectrum, we can recover our signal by transforming it back into the time domain using the iDFT, as seen in Figure 1.28. Here, the noise has been removed and the signal trend is clearly visible.

The spectrum reshaping to clean high frequency noise from a signal here is a very common procedure used in a variety of communications and electronics applications. ■

1.4 Properties of the DFT

DFTs have 3 important mathematical properties that simplify how we analyse and apply them in many cases.

1.4.1 Symmetry

The DFT of a real signal is conjugate symmetric. That is

$$X(-k) = X^*(k) \tag{1.32}$$

Therefore, instead of storing the DFT components for all frequencies $k \in [-N/2, N/2]$, if we know our signal is a real signal, we only need to store the DFT components for $k \in [0, N/2]$. We can recover the frequencies in $k \in [-N/2, -1]$ just by taking the conjugate of frequencies in $k \in [0, N/2]$. Since most of the signals we work with are physical signals containing only real values, the symmetry property halves the number of DFT components we need to store and transmit.

Example 11 *Proof of the symmetry property*

Solution: We can write the DFT $X(-k)$ as

$$\begin{aligned}
X(-k) &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi(-k)n/N} \\
&= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{j2\pi kn/N} \\
&= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x^*(n) \left(e^{j2\pi(-k)n/N}\right)^*
\end{aligned} \tag{1.33}$$

Since the conjugate of a real signal remains the same, we can replace $x(n)$ with $x^*(n)$. Conjugating a complex exponential just changes the sign of its imaginary exponent, so $e^{j2\pi kn/N} = \left(e^{j2\pi(-k)n/N}\right)^*$. Next we change the order of conjugation with multiplication. We can do this since

$$\begin{aligned}
(a+bj)^*(c+dj)^* &= (a-bj)(c-dj) \\
&= ((ac-bd) - j(ad+bc)) \\
&= ((ac-bd) + j(ad+bc))^* \\
&= ((a+bj)(c+dj))^*
\end{aligned} \tag{1.34}$$

Multiplying 2 conjugates is just the conjugate of the product of the 2 original numbers.

$$\begin{aligned}
X(-k) &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \left[x(n) e^{-j2\pi kn/N}\right]^* \\
&= \left[\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}\right]^* \\
&= X^*(k)
\end{aligned} \tag{1.35}$$

■

1.4.2 Energy conservation

The energy of the original signal is the same is the energy of the DFT. This property is also known as Parseval's Theorem. If $X = \mathcal{F}(x)$, then if we calculate their energies

$$\sum_{n=0}^{N-1} |x(n)|^2 = \|x\|^2 = \|X\|^2 = \sum_{k=0}^{N-1} \|X(k)\|^2 \tag{1.36}$$

Since the DFT is periodic, we can calculate its energy using any canonical set. Therefore, the theorem can also be written as

$$\sum_{n=0}^{N-1} |x(n)|^2 = \|x\|^2 = \|X\|^2 = \sum_{k=-N/2+1}^{N/2} \|X(k)\|^2 \tag{1.37}$$

Example 12 *Proof of Parseval's Theorem*

Solution: The definition of energy, we can calculate the DFT's energy as

$$\|X\|^2 = \sum_{k=0}^{N-1} X(k)X^*(k) \tag{1.38}$$

If we substitute the definition of the DFT $X(k)$ of $x(n)$ into our energy calculation, we have

$$\|X\|^2 = \sum_{k=0}^{N-1} \left[\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}\right] \left[\frac{1}{\sqrt{N}} \sum_{\tilde{n}=0}^{N-1} x(\tilde{n})e^{-j2\pi k\tilde{n}/N}\right]^* \tag{1.39}$$

If we interchange conjugation with multiplication in the term to the right, we have

$$\|X\|^2 = \sum_{k=0}^{N-1} \left[\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}\right] \left[\frac{1}{\sqrt{N}} \sum_{\tilde{n}=0}^{N-1} x(\tilde{n})^*e^{+j2\pi k\tilde{n}/N}\right] \tag{1.40}$$

We can simplify this complicated summation by changing the order to sum over k first instead of n and $\tilde{n}$. In doing so, we can also pull $x(n)$ and $x(\tilde{})$ out of the inner summation since they don't depend on k.

$$\|X\|^2 = \sum_{n=0}^{N-1}\sum_{\tilde{n}=0}^{N-1} x(n)x^*(\tilde{n}) \left[\sum_{k=0}^{N-1} \frac{1}{\sqrt{N}}e^{-j2\pi kn/N}\frac{1}{\sqrt{N}}e^{+j2\pi k\tilde{n}/N}\right] \tag{1.41}$$

Notice that the innermost summation is the inner product between 2 complex exponentials, which is a delta function due to orthonormality.

$$\begin{aligned}\|X\|^2 &= \sum_{n=0}^{N-1}\sum_{\tilde{n}=0}^{N-1} x(n)x^*(\tilde{n}) \left[\sum_{k=0}^{N-1} \frac{1}{\sqrt{N}}e^{-j2\pi kn/N}\frac{1}{\sqrt{N}}e^{+j2\pi k\tilde{n}/N}\right] \\ &= \sum_{n=0}^{N-1}\sum_{\tilde{n}=0}^{N-1} x(n)x^*(\tilde{n})\langle e_{\tilde{n}N}e_{nN}\rangle \\ &= \sum_{n=0}^{N-1}\sum_{\tilde{n}=0}^{N-1} x(n)x^*(\tilde{n})\delta(\tilde{n}-n)\end{aligned} \tag{1.42}$$

Since all terms where $\tilde{n} - n \neq 0$ are equal to zero due to the delta function, we can remove the summation in $\tilde{n}$ to get

$$\begin{aligned}\|X\|^2 &= \sum_{n=0}^{N-1}\sum_{\tilde{n}=0}^{N-1} x(n)x^*(\tilde{n})\delta(\tilde{n}-n) \\ &= \sum_{n=0}^{N-1} x(n)x^*(n)\delta(\tilde{n}-n) \\ &= \|x\|^2\end{aligned} \tag{1.43}$$

■

Parseval's theorem explains why in examples 1 to 5 earlier on, our DFTs of unit energy square pulses, delta functions, and exponential functions also resulted in DFTs that were functions with unit energy. Another interpretation from Parseval's theorem is that the value of the DFT $X(k)$ at k represents how much of $x(n)$'s energy is contained in oscillations of frequency k. In example 10 where we use spectrum reshaping to remove noise, Figure 1.26 shows that most of the signal's energy is contained in several distinct frequency peaks, while the high frequency noise contains only a small proportion of the total signal's energy. By removing frequencies that make up only a little of the signal's energy, we isolate the frequencies that account for most of the signal's energy and recover the original signal.

1.4.3 Linearity

Linearity is a property that is useful for modifying and combining different signals. It can be defined as the following

$$\mathcal{F}(ax+by) = a\mathcal{F}(x) + b\mathcal{F}(y) \tag{1.44}$$

Two useful properties we can draw from linearity is that the spectrum of the sum of 2 signals, $z = x + y$, is the sum of their spectrums $Z = X + Y$, and scaling a signal $y = ax$ scales the spectrum proportionally, $Y = aX$. These properties simplify many calculations and make a variety of signal processing techniques possible.

Example 13 *Proof of Linearity*

Solution: Let $Z = \mathcal{F}(ax+by)$. We can the substitute $ax + by$ into the definition of the DFT.

$$\begin{aligned} X(k) &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \left[ax(n) + by(n)\right] e^{-j2\pi kn/N} \\ &= \frac{a}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} + \frac{b}{\sqrt{N}} \sum_{n=0}^{N-1} y(n) e^{-j2\pi kn/N} \\ &= a\mathcal{F}(x) + b\mathcal{F}(y) \\ &= aX + bY \end{aligned} \tag{1.45}$$

By expanding the product and reordering terms using algebraic manipulations, we see that the DFT is in fact a linear transformation. ■

By taking advantage of the linearity of the DFT, we can quickly determine the DFT of signals that are combinations of what we already know the DFTs of. For example, consider the discrete cosine and sine signals

Example 14 *DFT of a discrete cosine*

Solution: A discrete cosine of frequency k_0 is defined as

$$x(n) = \frac{1}{\sqrt{N}} \cos(2\pi k_0 n/N) \tag{1.46}$$

We can also write the discrete cosine as a sum of discrete complex exponentials using the identity $\cos(x) = \frac{e^{jx}+e^{-jx}}{2}$

$$\begin{aligned} x(n) &= \frac{1}{2\sqrt{N}}\left[e^{j2\pi k_0 n/N} + e^{-j2\pi k_0 n/N}\right] \\ &= \frac{1}{2}\left[e_{k_0 N}(n) + e_{-k_0 N}\right] \end{aligned} \tag{1.47}$$

Due to the linearity of the DFT

$$\begin{aligned} \mathcal{F}(x(n)) &= \mathcal{F}\left(\frac{1}{2}\left[e_{k_0 N}(n) + e_{-k_0 N}\right]\right) \\ &= \frac{1}{2}\left[\mathcal{F}\left(e_{k_0 N}(n)\right) + \mathcal{F}\left(e_{-k_0 N}\right)\right] \end{aligned} \tag{1.48}$$

Since the DFT of a complex exponential with frequency k_0 is a delta function $\delta(k-k_0)$ shifted to k_0, the DFT if the discrete cosine is then

$$X(k) = \frac{1}{2}\left[\delta(k-k_0) + \delta(k+k_0)\right] \tag{1.49}$$

We see that the DFT of the discrete cosine with frequency k_0 is a pair of deltas located at positive and negative frequencies k_0, seen in the upper half of Figure 1.29

■

Example 15 *DFT of a discrete sine*

Solution:
Using a process similar to the one used to find the DFT of a discrete cosine, we start with

$$x(n) = \frac{1}{\sqrt{N}}\sin(2\pi k_0 n/N) \tag{1.50}$$

We can discrete sine as a sum of discrete complex exponentials using the identity $\sin(x) = \frac{e^{jx}-e^{-jx}}{2j}$

$$\begin{aligned} x(n) &= \frac{1}{2j\sqrt{N}}\left[e^{j2\pi k_0 n/N} - e^{-j2\pi k_0 n/N}\right] \\ &= \frac{j}{-2}\left[e_{k_0 N}(n) - e_{-k_0 N}\right] \end{aligned} \tag{1.51}$$

Due to the linearity of the DFT

$$\begin{aligned} \mathcal{F}(x(n)) &= \mathcal{F}\left(\frac{-j}{2}\left[e_{k_0 N}(n) - e_{-k_0 N}\right]\right) \\ &= \frac{-j}{2}\left[\mathcal{F}\left(e_{k_0 N}(n)\right) - \mathcal{F}\left(e_{-k_0 N}\right)\right] \\ &= \frac{j}{2}\left[\mathcal{F}\left(e_{-k_0 N}(n)\right) - \mathcal{F}\left(e_{k_0 N}\right)\right] \end{aligned} \tag{1.52}$$

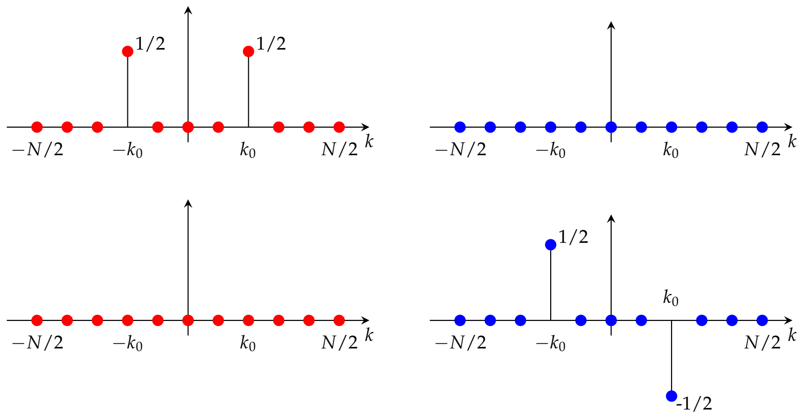


Figure 1.29: DFT of discrete cosine and sine

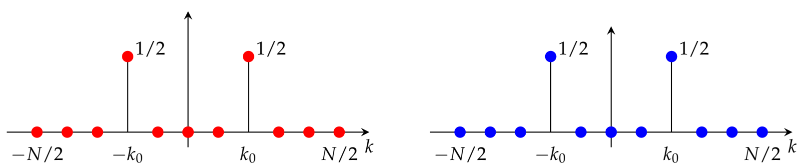


Figure 1.30: Modulus of the DFT of discrete cosine and sine

Since the DFT of a complex exponential with frequency k_0 is a delta function $\delta(k-k_0)$ shifted to k_0, the DFT if the discrete sine is then

$$X(k) = \frac{j}{2}\left[\delta(k+k_0) - \delta(k-k_0)\right] \tag{1.53}$$

We see that the DFT of the discrete sine with frequency k_0 is a pair of opposite complex deltas located at positive and negative frequencies k_0, seen in the lower half of Figure 1.29 ■

DFTs of discrete sines and cosines with frequency k_0 are made up of only complex exponentials with frequency k_0. The discrete cosine has only real values for its DFT while the discrete sine has only imaginary values, despite the fact that both are real signals. Also, the discrete cosine produces an even DFT symmetric around $k = 0$, while the discrete sine produces an odd DFT antisymmetric arouond $k = 0$. If we consider the DFT's property of symmetry in section 1.4.1, we see that the DFT at $-k_0$ is in fact the conjugate of the DFT at k_0 for both the discrete sine and cosine.

Despite the differences in the real and imaginary parts of the discrete sine and cosine, we see that they both have the same DFT moduli, as seen in Figure 1.30. Both these signals are essentially the same in terms of frequency, just shifted in phase. The information about their phase difference is captured by the phase of the complex numbers at $X(\pm k_0)$.