SYSTEMS 302
LECTURE 2

- **DISCRETE RANDOM VARIABLES**
  - **Bernoulli Random Variables**
  - **Binomial Random Variables**

- **SUMS OF RANDOM VARIABLES**
  - **Means of Sums**
  - **Variance of Sums**

- **For next time:**
  - **Devore, Sections 5.1,5.2,5.5**

ASSERTION:

"The probability of heads on any toss is ½."

QUESTION:

What does it mean?

LAPLACE HYPOTHESIS

If there is no reason to believe that one outcome is more likely than the other, then they must be treated as equally likely outcomes.

$$p_1 = p_2$$
$$p_1 + p_2 = 1$$

$$\Rightarrow 2p_1 = 1$$

$$\Rightarrow p_1 = \tfrac{1}{2} = p_2$$

VON MISES HYPOTHESIS

There exists a population (universe) of ‘potential states of the world’, each including a ‘Head’ or ‘Tail’ outcome. The realized coin flip is treated as a sample from this population.

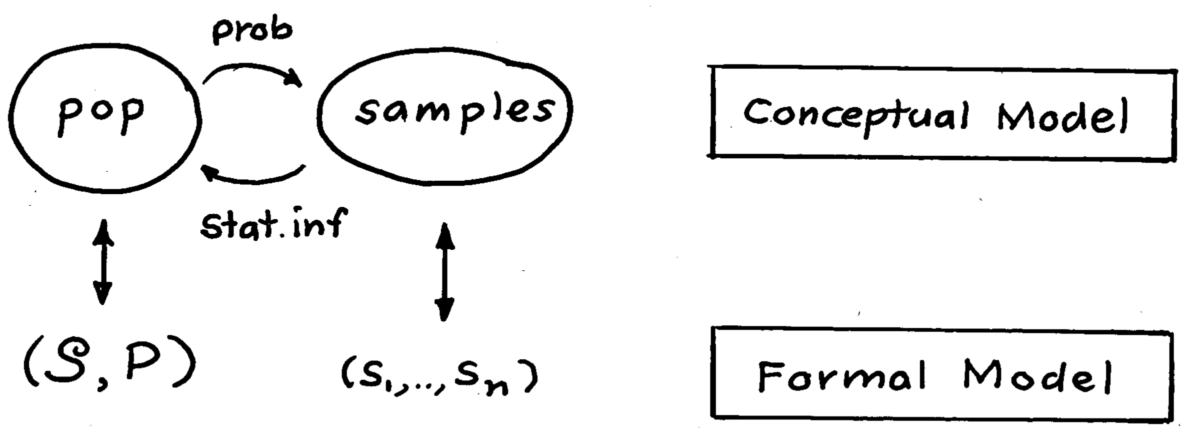
• FORMAL STATISTICAL POPULATIONS

s = possible <u>sample</u> (or 'state of the world')

$\mathcal{S}$ = <u>sample space</u> (set of all possible 'states')

$P(s)$ = <u>probability</u> of sample s

$(\mathcal{S}, P)$ = <u>statistical population</u> (or 'probability space')



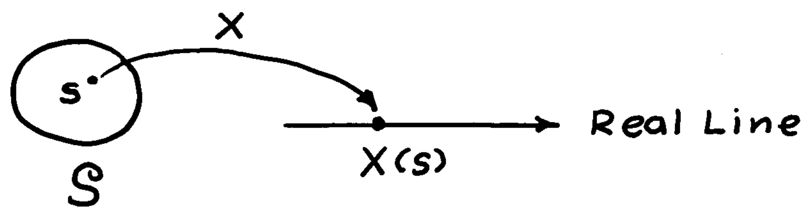
• <u>EXAMPLE</u> :

$\mathcal{S}$ = sys 302 <u>class</u>

s = individual <u>student</u>

$P(s)$ = probability that student s is <u>sampled</u>

• DISCRETE RANDOM VARIABLES



X = <u>random variable</u> on $(\mathcal{S}, P)$

= numerical property of samples, s.

• <u>Example</u>:

$X(s)$ = age (weight, etc.) of student, s.

x = <u>realized value</u> of X (data!)

$\mathcal{X} = \{x_1, x_2, \ldots\}$

= set of possible <u>X-realizations</u>

$P_X(x) = P(X = x) = P\{s \in \mathcal{S} : X(s) = x\}$

= probability that value x <u>occurs</u>

P_X = <u>probability distribution</u> on $\mathcal{X}$

• THE X-POPULATION MODEL

$$
\begin{tikzcd}
(\mathcal{S}, P) \arrow[rr, bend left, "\text{prob}"] \arrow[d, leftrightarrow] & & (s_1, \ldots, s_n) \arrow[ll, bend left, "\text{stat inf}"] \arrow[d, leftrightarrow] \\
(\mathcal{X}, P_X) \arrow[rr, bend left, "\text{prob}"] & & (x_1, \ldots, x_n) \arrow[ll, bend left, "\text{stat inf}"]
\end{tikzcd}
$$

• <u>EXPECTED VALUES</u>

$p_x = p(x) = P_X(x)$ (equivalent notation)

$$\boxed{\mu = E(X) = \Sigma_x \, x \, p(x)}$$

= <u>mean</u> of X (<u>first moment</u> of X)

$$\boxed{\sigma^2 = E[(X-\mu)^2] = \Sigma_x (x-\mu)^2 p(x)}$$

= <u>variance</u> of X (<u>second moment</u> of X)

➡ X- <u>population properties</u>

• BERNOULLI RANDOM VARIABLES

• BERNOULLI EVENTS: Voting : 1 = 'Obama Wins'
Testing: 1 = 'Pass'

$$\boxed{X(s) = \begin{cases} 1 & , \text{ event occurs in state } s \\ 0 & , \text{ event does not occur in state } s \end{cases}}$$

$$p = P(X = 1) = P\{s \in \mathcal{S} : X(s) = 1\}$$

$$1-p = P(X = 0) = P\{s \in \mathcal{S} : X(s) = 0\}$$

$$(\mathcal{X}, P_X) \rightarrow \boxed{\begin{matrix} 1 & p \\ 0 & 1-p \end{matrix}}$$

$$\boxed{X \sim \text{Bernoulli}(p)}$$

• MOMENTS

$$\mu = 1 \cdot p + 0 \cdot (1-p) = \boxed{p}$$

$$\sigma^2 = (1-p)^2 p + (0-p)^2 (1-p) = \boxed{p(1-p)}$$

→ Population Values

BATCH SAMPLING PROBLEM:

Given a shipment of 1000 light bulbs with 50 defective bulbs, take an inspection sample of 20 bulbs.

Q1. **What is the chance of getting at least one defective bulb in the sample?**

Q2. **What is the expected number of defective bulbs in the sample?**

Q3. **What is the variance of this number?**

• BINOMIAL RANDOM VARIABLES

→ Given n <u>independent</u> <u>Bernoulli random variables</u>

$$X_i \sim \text{Bernoulli}(p) \ , \quad i=1,\dots,n$$

the associated <u>Binomial random variable</u> is:

$$\boxed{X = \Sigma_{i=1}^{n} X_i = \underline{\text{number}} \text{ of event occurrences}}$$

$$P(X_1=1,\dots,X_k=1,X_{k+1}=0,\dots,X_n=0) = p^k(1-p)^{n-k}$$

$$\boxed{P(X=k\,|\,n,p) = \binom{n}{k} p^k (1-p)^{n-k} \ , \quad k=0,1,\dots,n}$$

$$\rightarrow \quad \boxed{X \sim \text{Bin}(n,p)}$$

• <u>MOMENTS</u>

$$\mu = \Sigma_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = \boxed{np}$$

$$\sigma^2 = \Sigma_{k=0}^{n} (k-np)^2 \binom{n}{k} p^k (1-p)^{n-k} = \boxed{np(1-p)}$$