Stochastic Systems Analysis and Simulations

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Who are us, where to find me, lecture times

- Alejandro Ribeiro
- Assistant Professor, Dept. of Electrical and Systems Engineering
- GRW 276, aribeiro@seas.upenn.edu,
- `http://alliance.seas.upenn.edu/~aribeiro/wiki`
- Felicia Lin
- Teaching assistant, life@seas.upenn.edu
- We also have a separate grader
- We meet on Moore 216
- Mondays, Wednesdays, Fridays 10 am to 11 am
- My office hours, Fridays at 2 pm
- Anytime, as long as you have something interesting to tell me
- `http://alliance.seas.upenn.edu/~ese303/wiki`

Prerequisites

- Probability theory
- Stochastic processes are time-varying random entities
- If unknown, need to learn as we go
- Will cover in first seven lectures

- Linear algebra
- Vector matrix notation, systems of linear equations, eigenvalues

- Programming in Matlab
- Needed for homework.
- If you know programming you can learn Matlab in one afternoon
- But it has to be this afternoon

- Differential equations, Fourier transforms
- Appear here and there. Should not be a problem

Homework and grading

- 14 homework sets in 14 weeks
- Collaboration accepted, welcomed, and encouraged
- Sets graded as 0 (bad), 1 (good), 2 (very good) and 3 (outstanding)
- We'll use the 3 sparingly. Goal is to earn 28 homework points

- Midterm examination starts on Friday October 21 worth 36 points
- In class-piece + take home piece due on Monday October 24
- Work independently. No collaboration, no discussion
- If things are going well, no in-class piece

- Final examination on December 14-21 worth 36 points

- At least 60 points are required for passing.
- C requires at least 70 points. B at least 80. A at least 90
- Goal is for everyone to earn an A

Textbooks

- Textbook for the class is (older or newer editions acceptable)
- Sheldon M. Ross "Introduction to Probability Models", Academic Press, 10th ed.

- Same topics at advanced level (more rigor, includes proofs)
- Sheldon Ross "Stochastic Processes", John Wiley & sons, 2nd ed.

- Stohastic processes in systems biology
- Darren J. Wilkinson "Stochastic Modelling for Systems Biology", Chapman & Hall/CRC, 1st ed.
- Part on simulation of chemical reactions taken from here

- Use of stochastic processes in finance
- Masaaki Kijima "Stochastic Processes with Applications to Finance", Chapman & Hall/CRC, 1st ed.

Be nice

- I am a very emotional person. I will love most of you, despise a few
- Seriously though, I work hard for this course, please do the same

- Come to class, be on time, pay attention
- Do all of your homework
- Do not hand in as yours my own solution
- Do not collaborate in the take-home midterm

- A little bit of probability ...
- Probability of getting an F in this class is 0.04
- Probability of getting an F given you skip 4 homework sets is 0.7
- I'll give you three notices, afterwards, I'll give up on you

- Come and learn. Useful. Very good student ratings

Class contents

Stochastic systems

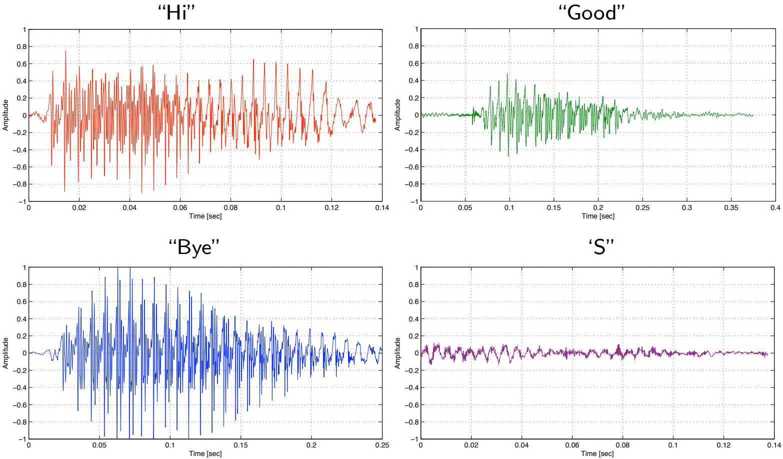
- Anything random that evolves in time
- Time can be discrete $(0, 1, \ldots)$ or continuous

- More formally, assign a function to a random event
- Compare with "random variable assigns a value to a random event"
- Generalizes concept of random vector to functions
- Or generalizes the concept of function to random settings

- Can interpret a stochastic process as a set of random variables
- Not always the most appropriate way of thinking

A voice recognition system

- Random event ∼ word spoken. Stochastic process ∼ the waveform
  - Try the file speech_signals.m



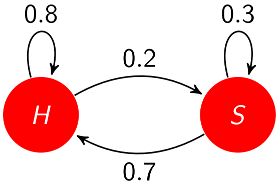
Five blocks

- Probability theory review (6 lectures)
  - Probability spaces
  - Conditional probability: time $n+1$ given time n, future given past ...
  - Limits in probability, almost sure limits: behavior as $t \to \infty$ ...
  - Common probability distributions (binomial, exponential, Poisson, Gaussian)
- Stochastic processes are complicated entities
- Restrict attention to particular classes that are somewhat tractable
- Markov chains (9 lectures)
- Continuous time Markov chains (12 lectures)
- Stationary random processes (9 lectures)
- Midterm covers up to Markov chains

Markov chains

- A set of states $1, 2, \ldots$. At time n, state is X_n
- Memoryless property
  - $\Rightarrow$ Probability of next state X_{n+1} depends on current state X_n
  - $\Rightarrow$ But not on past states X_{n-1}, X_{n-2}, ...

- Can be happy ($X_n = 0$) or sad ($X_n = 1$)
- Happiness tomorrow affected by happiness today only
- Whether happy or sad today, likely to be happy tomorrow
- But when sad, a little less likely so

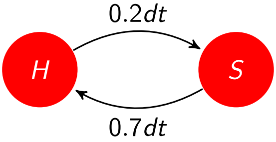


- Classification of states, ergodicity, limiting distributions
- Google's page rank, machine learning, virus propagation, queues ...

Continuous time Markov chains

- A set of states $1, 2, \ldots$. Continuous time index t
- Transition between states can happen at any time
- Future depends on present but is independent of the past

- Probability of changing state in an infinitesimal time dt



- Poisson processes, exponential distributions, transition probabilities, Kolmogorov equations, limit distributions
- Chemical reactions, queues, communication networks, weather forecasting ...

Stationary random processes

- Continuous time t, continuous state $x(t)$, not necessarily memoryless
- System has a steady state in a random sense
- Prob. distribution of $x(t)$ constant or becomes constant as t grows

- Brownian motion, white noise, Gaussian processes, autocorrelation, power spectral density.

- Black Scholes model for option pricing, speech, noise in electric circuits, filtering and equalization ...

An interesting betting game

- There is a certain game in a certain casino in which ...

  $\Rightarrow$ your chances of winning are $p > 1/2$
- You place $\$b$ bets
  (a) With probability p you gain $\$b$ and
  (b) With probability $(1-p)$ you loose your $\$b$ bet
- The catch is that you either
  (a) Play until you go broke (loose all your money)
  (b) Keep playing forever
- You start with an initial wealth of $\$w_0$
- Shall you play this game?

Modeling

- Let t be a time index (number of bets placed)
- Denote as $x(t)$ the outcome of the bet at time t
  - $x(t) = 1$ if bet is won (with probability p)
  - $x(t) = 0$ if bet is lost (probability $(1 - p)$)
- $x(t)$ is called a Bernoulli random varible with parameter p
- Denote as $w(t)$ the player's wealth at time t
- At time $t = 0$, $w(0) = w_0$
- At times $t > 0$ wealth $w(t)$ depends on past wins and losses
- More specifically we have
  - When bet is won $w(t) = w(t - 1) + b$
  - When bet is lost $w(t) = w(t - 1) - b$

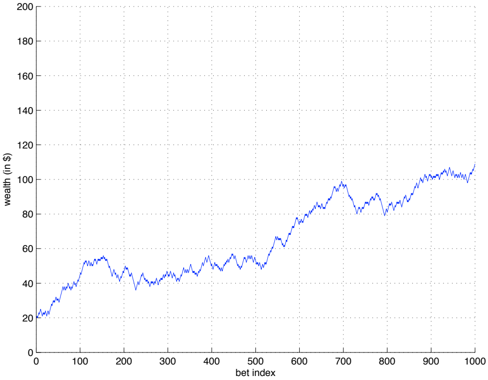
Coding

$t = 0$; $w(t) = w_0$; $max_t = 10^3$; `// Initialize variables`
% repeat while not broke up to time max_t
while $(w(t) > 0)$ & $(t < max_t)$ **do**
    $x(t) = \text{random}(\text{'bino'},1,p)$; % Draw Bernoulli random variable
    if $x(t) == 1$ **then**
        $w(t+1) = w(t) + b$; % If $x = 1$ wealth increases by b
    else
        $x(t+1) = w(t) - b$; % If $x = 0$ wealth decreases by b
    end
    $t = t + 1$;
end

- Initial wealth $w_0 = 20$, bet $b = 1$, win probability $p = 0.55$
- Shall we play?

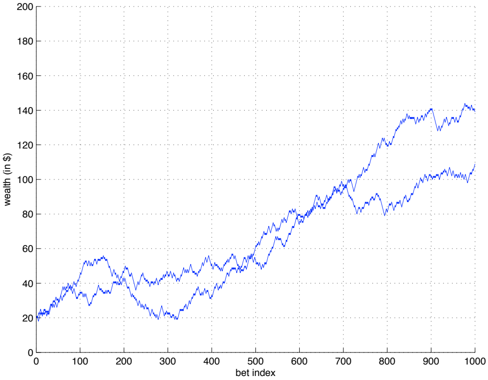
One lucky player

- She didn't go broke. After $t = 1000$ bets, her wealth is $w(t) = 109$
- Less likely to go broke now because wealth increased



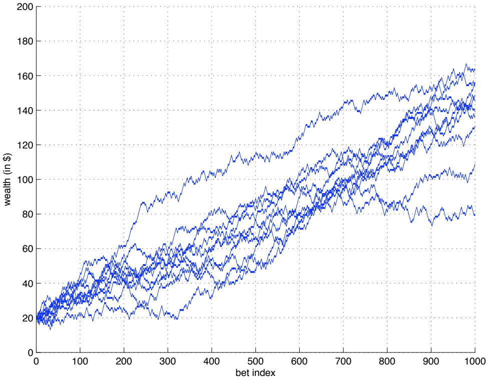
Two lucky players

- Wealths are $w_1(t) = 109$ and $w_2(t) = 139$
- Increasing wealth seems to be a pattern



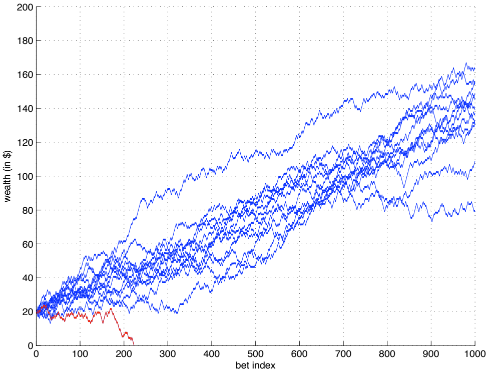
Ten lucky players

- Wealths $w_j(t)$ between 78 and 139
- Increasing wealth is definitely a pattern



One unlucky player

- But this does not mean that all players will turn out as winners
- The twelfth player $j = 12$ goes broke



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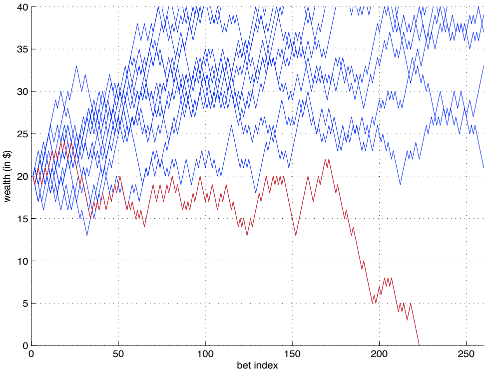
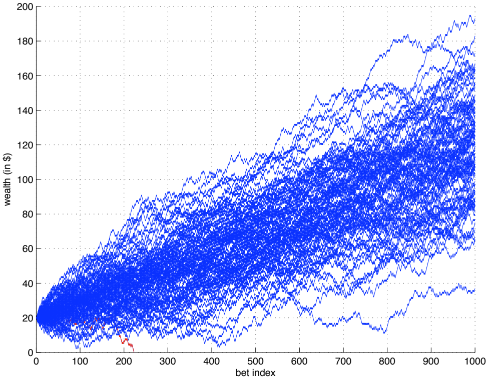


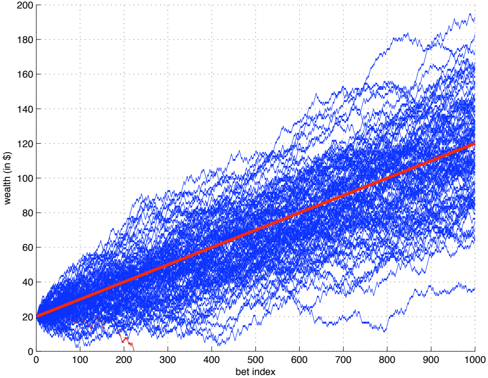
One hundred players

- Only one player ($j = 12$) goes broke
- All other players end up with substantially more money



Average tendency

- It is not difficult to find a line estimating the average of $w(t)$
- $\bar{w}(t) \approx w_0 + (2p - 1)t \approx w_0 + 0.1t$



Where does the average tendency comes from?

- To discover average tendency $\bar{w}(t)$ assume $w(t-1) > 0$ and note

$$\begin{aligned}\mathbb{E}\left[w(t) \,\middle|\, w(t-1)\right] &= w(t-1) + b\mathrm{P}\left[x(t) = 1\right] - b\mathrm{P}\left[x(t) = 0\right] \\ &= w(t-1) + bp \qquad\qquad\quad - b(1-p) \\ &= w(t-1) + (2p-1)b\end{aligned}$$

- Now, condition on $w(t-2)$ and use the above expression once more

$$\begin{aligned}\mathbb{E}\left[w(t) \,\middle|\, w(t-2)\right] &= \mathbb{E}\left[w(t-1) \,\middle|\, w(t-2)\right] + (2p-1)b \\ &= w(t-2) + (2p-1)b \quad + (2p-1)b\end{aligned}$$

- Proceeding recursively t times, yields

$$\mathbb{E}\left[w(t) \,\middle|\, w(0)\right] = w_0 + t(2p-1)b$$

- This analysis is not entirely correct because $w(t)$ might be zero

Analysis of outcomes: mean

- For a more accurate analysis analyze simulation's outcome

- Consider J experiments
- For each experiment, there is a wealth history $w_j(t)$
- We can estimate the average outcome as

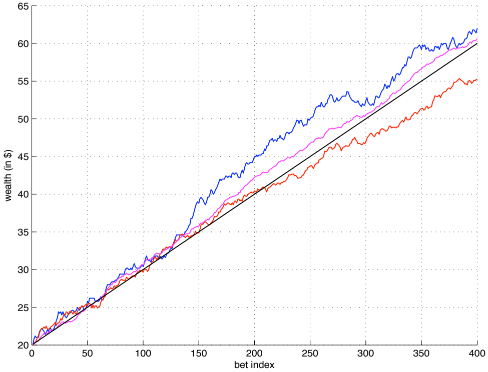
$$\bar{w}_J(t) = \frac{1}{J}\sum_{j=1}^{J} w_j(t)$$

- $\bar{w}_J(t)$ is called the sample average

- Do not confuse $\bar{w}_J(t)$ with $\mathbb{E}[w(t)]$
  - $\bar{w}_J(t)$ is computed from experiments, it is a random quantity in itself
  - $\mathbb{E}[w(t)]$ is a property of the random variable $w(t)$
  - We will see later that for large J, $\bar{w}_J(t) \to \mathbb{E}[w(t)]$

Analysis of outcomes: mean

- Expected value $\mathbb{E}[w(t)]$ in black (approximation)
- Sample average for $J = 10$ (blue), $J = 20$ (red), and $J = 100$ (magenta)



Histogram

- There is more information in the simulation's output
- Estimate the probability distribution function (pdf) $\Rightarrow$ Histogram

- Consider a set of points $w^{(1)}, \ldots, w^{(N)}$
- Indicator function of the event $w^{(n)} \leq w_j < w^{(n+1)}$
  - $\mathbb{I}\left[w^{(n)} \leq w_j < w^{(n+1)}\right] = 1$ when $w^{(n)} \leq w_j < w^{(n+1)}$
  - $\mathbb{I}\left[w^{(n)} \leq w_j < w^{(n+1)}\right] = 0$ else

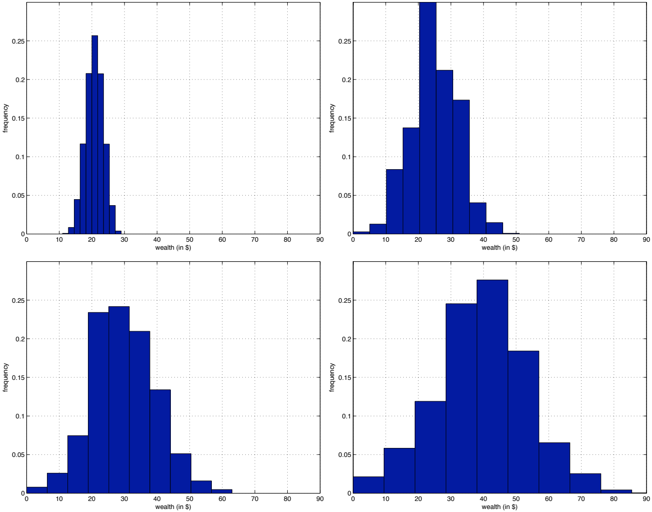
- Histogram is then defined as

$$H\left[t; w^{(n)}, w^{(n+1)}\right] = \frac{1}{J} \sum_{j=1}^{J} \mathbb{I}\left[w^{(n)} \leq w_j(t) < w^{(n+1)}\right]$$

- Fraction of experiments with wealth $w_j(t)$ between $w^{(n)}$ and $w^{(n+1)}$

Histogram

- The pdf broadens and shifts to the right ($t = 10, 50, 100, 200$)



What is this class about

- Analysis and simulation of stochastic systems

  $\Rightarrow$ A system that evolves in time with some randomness

- They are usually quite complex $\Rightarrow$ Simulations

- We will learn how to model stochastic systems, e.g.,
  - $x(t)$ Bernoulli with parameter p
  - $w(t) = w(t-1) + b$ when $x(t) = 1$
  - $w(t) = w(t-1) - b$ when $x(t) = 0$

- ... how to analyze, e.g., $\mathbb{E}\left[w(t) \,\middle|\, w(0)\right] = w_0 + t(2p-1)b$

- ... and how to interpret simulations and experiments, e.g,
  - Average tendency through sample average