Probability review

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Definitions

Sigma algebras and probability spaces

Conditional probability, independence, total probability, Bayes's rule

Random variables

Discrete random variables
- Commonly used discrete random variables

Continuous random variables
- Commonly used continuous random variables

Expected values
- Discrete random variables
- Continuous random variables
- Functions of random variables

Probability

- An event is a thing that happens
- A random event is one that is not certain
- The probability of an event measures how likely it is to occur

Example

- I've written a student's name in a piece of paper. Who is she/he?
- Event(s): Student x's name is written in the paper
- Probability(ies): $P(x)$ how likely is x's name to be the one written

- Probability is a measurement tool

Sigma-Algebra

- Given a space or universe S
  - E.g., all students in the class $S = \{x_1, x_2, \ldots, x_N\}$ (x_n denote names)
- An event E is a subset of S
  - E.g. $\{x_1\}$, student with name x_1,
  - Or in general $\{x_n\}$, student with name x_n
  - But also $\{x_1, x_4\}$, students with names x_1 and x_4
- A sigma-Algebra $\mathcal{F}$ is a collection of events $E \subseteq S$ such that
  - Not empty: $\mathcal{F} \neq \emptyset$
  - Closed under complement: If $E \in \mathcal{F}$, then $E^c \in \mathcal{F}$
  - Closed under countable unions: If $E_i \in \mathcal{F}$ $\cup_{i=1}^{\infty} E_i \in \mathcal{F}$
- Note that $\mathcal{F}$ is a set of sets

Examples of Sigma-Algebras

Example

- No student and all students, i.e., $\mathcal{F}_0 := \{\emptyset, S\}$

Example

- Empty set, women, men, all students, i.e., $\mathcal{F}_1 := \{\emptyset, \text{Women}, \text{Men}, S\}$

Example

- $\mathcal{F}$ including the empty set plus
- All events (sets) with one student $\{x_1\}, \ldots, \{x_N\}$ plus
- All events with two students $\{x_1, x_2\}$, $\{x_1, x_3\}$, ..., $\{x_1, x_N\}$,
  $\{x_2, x_3\}$, ..., $\{x_2, x_N\}$,
  ...
  $\{x_{N-1}, x_N\}$ plus
- All events with three students, four, ..., N students.

Axioms of probability

- Define a function $P(E)$ from a sigma-Algebra $\mathcal{F}$ to the real numbers
- $P(E)$ is a probability if

  $\Rightarrow$ Probability range $\Rightarrow 0 \leq P(E) \leq 1$

  $\Rightarrow$ Probability of universe $\Rightarrow P(S) = 1$

  $\Rightarrow$ Additivity $\Rightarrow$ Given sequence of disjoint events $E_1, E_2, \ldots$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P\left(E_i\right)$$

  $\Rightarrow$ Probability of union is the sum of individual probabilities

- In additivity property number of events is possibly infinite
- Disjoint events means $E_i \cap E_j = \emptyset$

Probability example

- Sigma-algebra with all combinations of students
- Names are equiprobable $\Rightarrow P(x_n) = 1/N$ for all n.

  $\Rightarrow$ Is this function a probability? Is there enough information given?

- Sets with two students (for $n \neq m$):

$$P(\{x_n, x_m\}) = P(\{x_n\}) + P(\{x_m\}) = 2/N$$

  $\Rightarrow$ Is this function a probability? Is there enough information given?

- Have to specify probability for all elements of the sigma-algebra

  $\Rightarrow$ Sets with 3 students $\Rightarrow 3/N$. Sets with 4 students $\Rightarrow 4/N$ ...

  $\Rightarrow$ For universe S $\Rightarrow P(S) = P\left(\bigcup_{n=1}^{N}\{x_n\}\right) = 1$

- Is this function a probability? $\Rightarrow$ Verify properties (range, universe, additivity)

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Conditional probability

- Partial information about the event (E.g. Name is male)
- The event E belongs to a set F

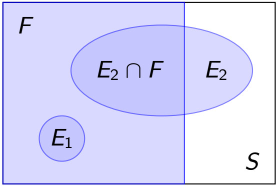
- Define the conditional probability of E given F as

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

- Renormalize probabilities to the set F

- Discard a piece of S
- May discard a piece of E as well

- Need to have $P(F) > 0$



Conditional probability example

- The name I wrote is male. What is the probability of name x_n?
- Assume male names are $F = \{x_1, \ldots, x_M\}$
- Probability of F is $P(F) = M/N$ (true by definition)

- If name is male, $x_n \in F$ and we have for event $E = \{x_n\}$

$$P(E \cap F) = P(\{x_n\}) = 1/N$$

- Conditional probability is as you would expect

$$P(E \,\big|\, F) = \frac{P(E \cap F)}{P(F)} = \frac{1/N}{M/N} = \frac{1}{M}$$

- If name is female $x_n \notin F$, then $P(E \cap F) = P(\emptyset) = 0$
- As you would expect, then $P(E \,\big|\, F) = 0$

Independence

- Events E and F are said independent if $P(E \cap F) = P(E)P(F)$
- According to definition of conditional probability

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)P(F)}{P(F)} = P(E)$$

- Knowing F does not alter our perception of E
- F has no information about E
- The symmetric is also true $P(F \mid E) = P(F)$
- Events that are not independent are dependent

Independence example

- Wrote one name, asked a friend to write another (possibly the same)
- Space S is sets of all pairs of names $[x_n(1), x_n(2)]$
- Sigma-algebra is cartesian product $\mathcal{F} \times \mathcal{F}$
- Pair of names chosen without coordination

$$P\big(\{(x_1, x_2)\}\big) = P\big(\{x_1\}\big)P\big(\{x_2\}\big) = \frac{1}{N^2}$$

- Dependent events: I wrote one name, then another name

Total probability

- Consider event E and events F and F^c
- F and F^c are a partition of the space S ($F \cup F^c = S$, $F \cap F^c = \emptyset$)
- Because $F \cup F^c = S$ cover space S can write the set E as

$$E = E \cap S = E \cap (F \cup F^c) = (E \cap F) \cup (E \cap F^c)$$

- Because $F \cap F^c = \emptyset$ are disjoint, so is $(E \cap F) \cap (E \cap F^c) = \emptyset$. Thus

$$\mathrm{P}[E] = \mathrm{P}[(E \cap F) \cup (E \cap F^c)] = \mathrm{P}[E \cap F] + \mathrm{P}[E \cap F^c]$$

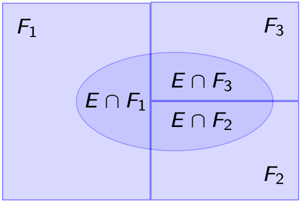
- Use definition of conditional probability

$$\mathrm{P}[E] = \mathrm{P}\left[E \,\middle|\, F\right] \mathrm{P}[F] + \mathrm{P}\left[E \,\middle|\, F^c\right] \mathrm{P}[F^c]$$

- Translate conditional information, $\mathrm{P}\left[E \,\middle|\, F\right]$ and $\mathrm{P}\left[E \,\middle|\, F^c\right]$
  $\Rightarrow$ Into unconditional information $\mathrm{P}[E]$

Total probability - continued

- In general, consider (possibly infinite) partition F_i, $i = 1, 2, \ldots$ of S
- Sets are disjoint $\Rightarrow F_i \cap F_j = \emptyset$, $i \neq j$
- Sets F_i cover the space $\Rightarrow \cup_{i=1}^{\infty} F_i = S$



- As before, because $\cup_{i=1}^{\infty} F_i = S$ cover space S can write the set E as

$$E = E \cap S = E \cap \left(\bigcup_{i=1}^{\infty} F_i \right) = \bigcup_{i=1}^{\infty} E \cap F_i$$

- Because $F_i \cap F_j = \emptyset$ are disjoint, so is $(E \cap F_i) \cap (E \cap F_j) = \emptyset$. Thus

$$\mathrm{P}\left[E\right] = \mathrm{P}\left[\bigcup_{i=1}^{\infty} E \cap F_i\right] = \sum_{i=1}^{\infty} \mathrm{P}\left[E \cap F_i\right] = \sum_{i=1}^{\infty} \mathrm{P}\left[E \,\middle|\, F_i\right] \mathrm{P}\left[F_i\right]$$

Total probability example

- In this class seniors get an A with probability 0.9
- Juniors get an A with probability 0.8
- For a exchange student, we estimate its standing as being senior with prob. 0.7 and junior with prob. 0.3
- What is the probability of the exchange student scoring an A?

- Let $A =$ "exchange student gets an A," S denote senior standing and J junior standing
- Use total probability

$$\mathrm{P}[A] = \mathrm{P}\left[A \,\middle|\, S\right] \mathrm{P}[S] + \mathrm{P}\left[A \,\middle|\, J\right] \mathrm{P}[J]$$

- Or in numbers

$$\mathrm{P}[A] = 0.9 \times 0.7 + 0.8 \times 0.3 = 0.87$$

Bayes's Rule

- From the definition of conditional probability

$$P(E \mid F)P(F) = P(E \cap F)$$

- Likewise, for F conditioned on E, we have

$$P(F \mid E)P(E) = P(F \cap E)$$

- Quantities above are equal, then

$$P(E \mid F) = \frac{P(F \mid E)P(E)}{P(F)}$$

- Bayes's rule allows time reversion. If F (future) comes after E (past),
  - $\Rightarrow P(E \mid F)$, probability of past (E) having seen the future (F)
  - $\Rightarrow P(F \mid E)$, probability of future (F) having seen past (E)
- Models often describe present $\mid$ past. Interest is often in past $\mid$ present

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Random variables (RV) definition

- A RV X is a function that assigns a number to a random event

- Think of RVs as measurements.
- Event is something that happens, RV is an associated measurement

- Probabilities of RVs inferred from probabilities of underlying events

Example

- Throw a ball inside a $1m \times 1m$ square. Interested in ball position
- Random event is the place where the ball falls
- Random variables are x and y position coordinates

Example 1

- Throw coin for head (H) or tail (T). Coin is fair $\mathrm{P}[H] = 1/2$, $\mathrm{P}[T] = 1/2$. Pay \$1 for H, charge \$1 for T. Earnings?
- Events are H and T
- To measure earnings define RV X with values

$$X(H) = 1, \qquad X(T) = -1$$

- Probabilities of the RV are

$$\begin{aligned} \mathrm{P}[X = 1] \quad &= \mathrm{P}[H] = 1/2, \\ \mathrm{P}[X = -1] &= \mathrm{P}[T] = 1/2 \end{aligned}$$

- We also have $\mathrm{P}[X = a] = 0$ for all other $a \neq 1$, $a \neq -1$

Example 2

- Throw 2 coins. Pay \$1 for each H, charge \$1 for each T.
- Events are HH and HT, TH, TT
- To measure earnings define RV Y with values

$$Y(HH) = 2, \quad Y(HT) = 0, \quad Y(TH) = 0, \quad Y(TT) = -2$$

- Probabilities are

$$\begin{aligned} \mathrm{P}[X = 2] &= \mathrm{P}[HH] &&= 1/4, \\ \mathrm{P}[X = 0] &= \mathrm{P}[HT] + \mathrm{P}[TH] &&= 1/2, \\ \mathrm{P}[X = -2] &= \mathrm{P}[HT] &&= 1/4, \end{aligned}$$

About Examples 1 & 2

- RVs are easier to manipulate than events
- Let $E_1 \in \{H, T\}$ be outcome of coin 1 and $E_2 \in \{H, T\}$ of coin 2
- Can relate X and Y as

$$Y(E_1, E_2) = X(E_1) + X(E_2)$$

- Throw N coins. Earnings?
- Enumeration becomes cumbersome
- Let $E_n \in \{H, T\}$ be outcome of n-th coin and define

$$Y(E_1, E_2, \ldots, E_n) = \sum_{n=1}^{N} X(E_n)$$

Example 3

- Throw a coin until landing heads for the first time. $P(H) = p$
- Number of throws until the first head?
- Events are H, TH, TTH, $TTTH$, ...
  - We stop throwing coins at first head (thus THT not a possible event)
- Let N be RV with number of throws.
- $N = n$ if we land T in the first $n-1$ throws and and H in the n-th

$$
\begin{aligned}
\mathrm{P}[N=1] &= \mathrm{P}[H] && = p \\
\mathrm{P}[N=2] &= \mathrm{P}[TH] && = (1-p)p \\
&\vdots \\
\mathrm{P}[X=n] &= \mathrm{P}[TT\ldots TH] && = (1-p)^{n-1}p
\end{aligned}
$$

Example 3 - continued

- It should be $\sum_{n=1}^{\infty} \mathrm{P}\left[N = n\right] = 1$
- This is true because $\sum_{n=1}^{\infty}(1-p)^{n-1}$ is a geometric sum. Then

$$\sum_{n=1}^{\infty}(1-p)^{n-1} = 1 + (1-p) + (1-p)^2 + \ldots = \frac{1}{1-(1-p)} = \frac{1}{p}$$

- Using this for the sum of probabilities

$$\sum_{n=1}^{\infty} \mathrm{P}\left[N = n\right] = p\sum_{n=1}^{\infty}(1-p)^{n} = p\,\frac{1}{p} = 1.$$

Indicator function

- The indicator function is a random variable
- Let E be an event. Let e be the outcome of a random event

$$\mathbb{I}\{E\} = 1 \quad \text{if } e \in E$$
$$\mathbb{I}\{E\} = 0 \quad \text{if } e \notin E$$

- It indicates that outcome e belongs to set E, by taking value 1

Example

- Number of throws N until first H. Interested on N exceeding N_0
- Event is $\{N : N > N_0\}$. Possible outcomes are $N = 1, 2, \ldots$
- Denote indicator function as $\Rightarrow \mathbb{I}_{N_0} = \mathbb{I}\{N : N > N_0\}$
- The probability $\mathrm{P}\left[\mathbb{I}_{N_0} = 1\right] = \mathrm{P}\left[N > N_0\right] = (1-p)^{N_0}$

  $\Rightarrow$ For N to exceed N_0 need N_0 consecutive tails

  $\Rightarrow$ Doesn't matter what happens afterwards

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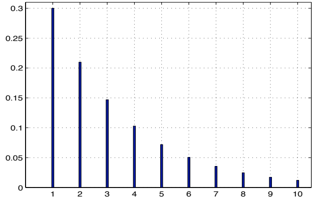
- A discrete RV takes on, at most, a countable number of values
- Probability mass function (pmf) $p_X(x) = \text{P}[X = x]$
  - If the RV is clear from context we just write $p_X(x) = p(x)$

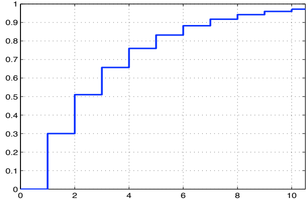
- If X take values in $\{x_1, x_2, \ldots\}$ pmf satisfies
  - $p(x_i) > 0$ for $i = 1, 2, \ldots$
  - $p(x) = 0$ for all other $x \neq x_i$
  - $\sum_{i=1}^{\infty} p(x_i) = 1$
- Pmf for "throw to first head" (p=0.3)



- Cumulative distribution function (cdf) is

$$F_X(x) = \text{P}[X \leq x] = \sum_{i: x_i \leq x} p(x_i)$$

- Staircase function with jumps at each x_i
- Cdf for "throw to first head" (p=0.3)

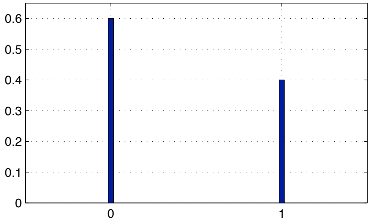


- An experiment/bet can succeed with probability p or fail with probability $(1-p)$ (e.g., coin throws, any indication of an event)
- Bernoulli X can be 0 or 1. Pmf values $\Rightarrow p(1) = p$

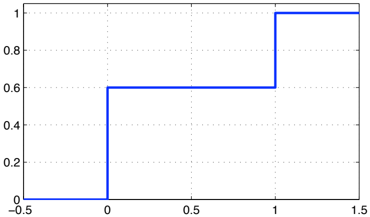
  $\Rightarrow p(0) = q = 1 - p$
- For the cdf we have

$$\begin{aligned} &\Rightarrow F(x) = 0 \quad \text{for} \quad x < 0 \\ &\Rightarrow F(x) = q \quad \text{for} \quad 0 \leq x < 1 \\ &\Rightarrow F(x) = 1 \quad \text{for} \quad 1 < x \end{aligned}$$

pmf

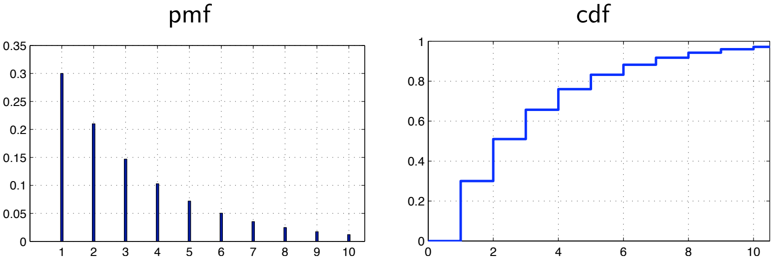


cdf



Geometric

- Count number of Bernoulli trials needed to register first success
- Trials succeed with probability p
- Number of trials X until success is geometric with parameter p
- Pmf is $\Rightarrow p(i) = p(1-p)^{i-1}$
  - $i-1$ failures plus one success. Throws are independent
- Cdf is $\Rightarrow F(i) = 1-(1-p)^{i}$
  - reaches i only if first $i-1$ trials fail; or just sum the geometric series



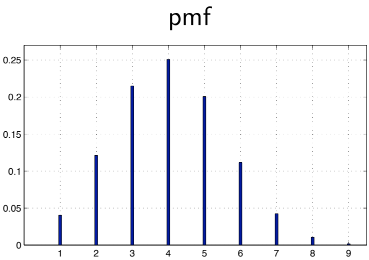
Binomial

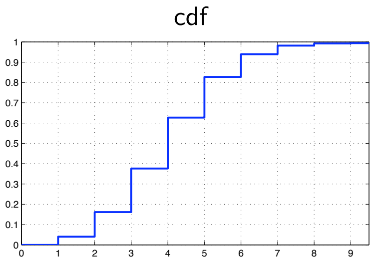
- Count number of successes X in n Bernoulli trials

- n trials. Probability of success p. Probability of failure $q = 1 - p$
- Then, binomial X with parameters (n, p) has pmf

$$p(i) = \binom{n}{i} p^i (1-p)^{n-i} = \frac{n!}{(n-i)!i!} p^i (1-p)^{n-i}$$

- $X = i$ if there are i successes (p^i) and $n - i$ failures ($(1-p)^{n-i}$).
- There are $\binom{n}{i}$ ways of drawing i successes and $n - i$ failures





Binomial continued

- Let Y_i for $i = 1, \ldots n$ be Bernoulli RVs with parameter p

  $\Rightarrow$ Y_i associated with independent events

- Can write binomial X with parameters (n, p) as $\Rightarrow X = \sum_{i=1}^{n} Y_i$

- Consider binomials Y and Z with parameters (n_Y, p) and (n_Z, p)
- Probability distribution of $X = Y + Z$?
- Write $Y = \sum_{i=1}^{n_Y} Y_i$ and $Z = \sum_{i=1}^{n_Z} Z_i$ with Y_i and Z_i Bernoulli with parameter p. Write X as

$$X = \sum_{i=1}^{n_Y} Y_i + \sum_{i=1}^{n_Z} Z_i$$

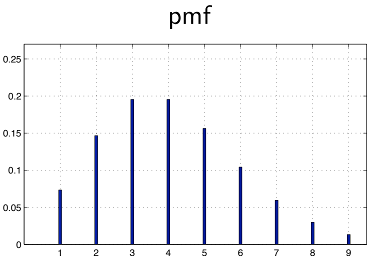
- Then X is binomial with parameter $(n_Y + n_Z, p)$

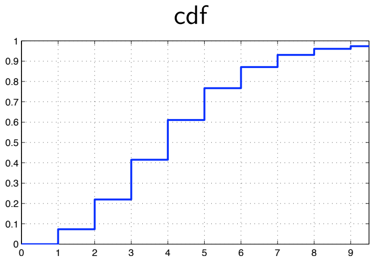
- Approximate a Binomial variable for large n

$$p(i) = e^{-\lambda}\frac{\lambda^i}{i!}$$

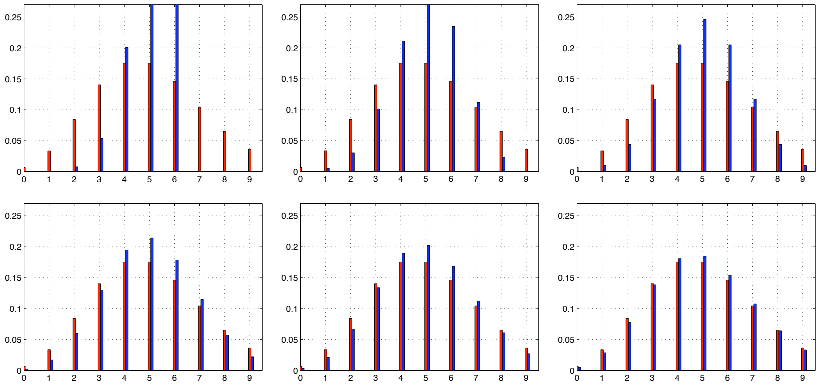
- Is this a properly defined pmf? Yes
- Taylor's expansion of $e^x = 1 + x + x^2/2 + \ldots + x^i/i! + \ldots$. Then

$$\sum_{i=0}^{\infty} p(i) = e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!} = e^{-\lambda}e^{\lambda} = 1$$





- X is binomial with parameters (n, p)
- Let $n \to \infty$ while maintaining a constant product $np = \lambda$
  - If we just let $n \to \infty$ number of successes diverges. Boring.
- Compare with Poisson distribution with parameter λ
  - $\lambda = 5$ $n = 6, 8, 10, 15, 20, 50$



Poisson and binomial - continued

- This is, in fact, the motivation for the definition of a Poisson RV
- Substituting $p = \lambda/n$ in the pmf of a binomial RV

$$
\begin{aligned}
p_n(i) &= \frac{n!}{(n-i)!i!}\left(\frac{\lambda}{n}\right)^i\left(1-\frac{\lambda}{n}\right)^{n-i} \\
&= \frac{n(n-1)\dots(n-i+1)}{n^i}\,\frac{\lambda^i}{i!}\,\frac{(1-\lambda/n)^n}{(1-\lambda/n)^i}
\end{aligned}
$$

- Factorials' defs., $(1-\lambda/n)^{n-i} = (1-\lambda/n)^n/(1-\lambda/n)^i)$, reorder terms
- By definition red term is $\lim_{n\to\infty}(1-\lambda/n)^n = e^{-\lambda}$
- Black and blue terms converge to 1. From both observations

$$
\lim_{n\to\infty} p_n(i) = 1\,\frac{\lambda^i}{i!}\,\frac{e^{-\lambda}}{1} = e^{-\lambda}\frac{\lambda^i}{i!}
$$

- Limit is the pmf of a Poisson RV

Poisson and binomial - continued

- Binomial distribution is justified by counting successes
- The Poisson is an approximation for large number of trials n
- Poisson distribution is more tractable
- Sometimes called "law of rare events"
  - Individual events (successes) happen with small probability $p = \lambda/n$
  - The aggregate event, though, (number of successes) need not be rare
- Notice that all four RVs are related to coin tosses.

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Continuous RVs, probability density function

- Possible values for continuous RV X form a dense subset $\mathcal{X} \in \mathbb{R}$
- Uncountable infinite number of possible values

  $\Rightarrow$ May have $\text{P}\left[X = x\right] = 0$ for all $x \in \mathcal{X}$ (most certainly will)

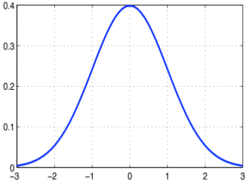
- The probability density function (pdf) is a function such that for any subset $\mathcal{X} \in \mathbb{R}$ (Normal pdf to the right)

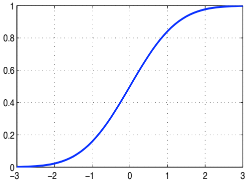
$$\text{P}\left[X \in \mathcal{X}\right] = \int_{\mathcal{X}} f_X(x)$$

- Cdf can be defined as before and related to the pdf (Normal cdf to the right)

$$F_X(x) = \text{Pr}\left[X \leq x\right] = \int_{-\infty}^{x} f_X(u)\, du$$

- $\text{P}\left[X \leq \infty\right] = F_X(\infty) = \lim_{x \to \infty} F_X(x) = 1$





More on cdfs and pdfs

- When the set $\mathcal{X} = [a, b]$ is an interval of the real line

$$\mathrm{P}\left[X \in [a, b]\right] = \mathrm{P}\left[X \leq b\right] - \mathrm{P}\left[X \leq a\right] = F_X(b) - F_X(a)$$

- Or in terms of the pdf can be written as

$$\mathrm{P}\left[X \in [a, b]\right] = \int_a^b f_X(x)\, dx$$

- For small interval $[x_0, x_0 + \delta x]$, in particular

$$\mathrm{P}\left[X \in [x_0, x + \delta x]\right] = \int_{x_0}^{x+\delta x} f_X(x)\, dx \approx f_X(x_0)\delta x$$

- Probability is the "area under the pdf" (thus "density")

- Another relationship between pdf and cdf is $\Rightarrow \dfrac{\partial F_X(x)}{\partial x} = f_X(x)$
- From fundamental theorem of calculus ("derivative inverse of integral")

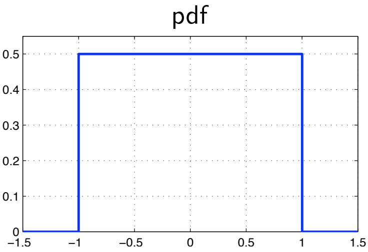
Uniform

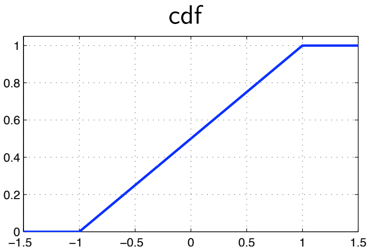
- Model problems with equal probability of landing on an interval $[a, b]$
- Pdf is $f(x) = 0$ outside the interval $[a, b]$ and

$$f(x) = \frac{1}{b-a}, \text{ for } a \leq x \leq b$$

- Cdf is $F(x) = (x-a)/(b-a)$ in the interval $[a, b]$ (0 before, 1 after)
- Prob. of interval $[\alpha, \beta] \subseteq [a, b]$ is $\int_\alpha^\beta f(x) = (\beta - \alpha)/(b-a)$

  $\Rightarrow$ Depends on interval's width $\beta - \alpha$ only, Not on its position

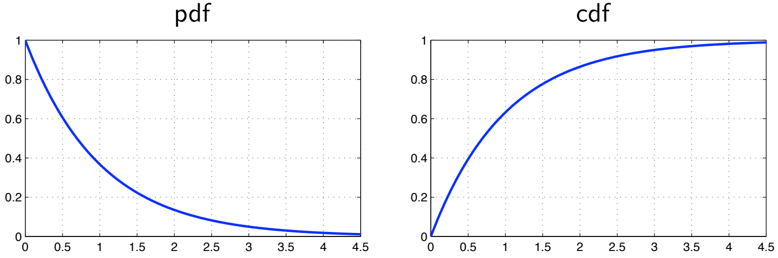




Exponential

- Model memoryless times (more later)
- Pdf is $f(x) = 0$ for $x < 0$ and $f(x) = \lambda e^{-\lambda x}$ for $0 \leq x$
- CDF obtained by integrating pdf

$$F(x) = \int_{-\infty}^{x} f(u)\, du = \int_{0}^{x} \lambda e^{-\lambda u}\, du = -e^{-\lambda u}\Big|_{0}^{x} = 1 - e^{-\lambda x}$$

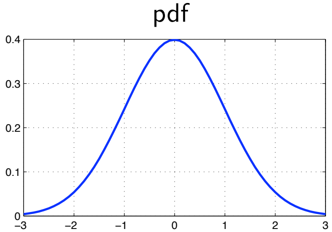


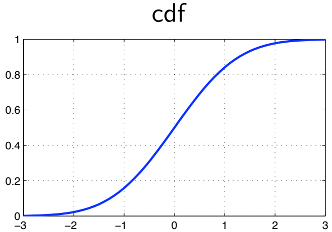
Normal / Gaussian

- Appears in phenomena where randomness arises from a large number of small random effects. Pdf is

$$f(x) = \frac{1}{\sqrt{2/\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

- μ is the mean of the Normal RV. Shifts pdf to right ($\mu > 0$) or left
- σ^2 is the variance, σ the standard deviation. Controls width of pdf
  - 0.68 prob. between $\mu \pm \sigma$, 0.997 prob. in $\mu \pm 3\sigma$
- The cdf $F(x)$ cannot be expressed in terms of elementary functions





Expected values

Sigma algebras and probability spaces

Conditional probability, independence, total probability, Bayes's rule

Random variables

Discrete random variables

- Commonly used discrete random variables

Continuous random variables

- Commonly used continuous random variables

Expected values

- Discrete random variables
- Continuous random variables
- Functions of random variables

Expected values

- We are asked to condense information about a RV in a single value
- What should this value be?
- If we are allowed a description with a few values
- What should they be?

- Expected values are convenient answers to these questions

- Beware: Expectations are condensed descriptions
- They necessarily overlook some aspects of the random phenomenon

Definition for discrete RVs

- RV X taking on values x_i, $i = 1, 2, \ldots$ with pmf $p(x)$
- The expected value of the RV X is

$$\mathbb{E}[X] := \sum_{i=1}^{\infty} x_i p(x_i) = \sum_{x:p(x)>0} xp(x)$$

- Weighted average of possible values x_i
- Common average if RV takes values x_i, $i = 1, \ldots, N$ equiprobably

$$\mathbb{E}[X] = \sum_{i=1}^{N} x_i p(x_i) = \sum_{i=1}^{N} x_i \frac{1}{N} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

Expected value of Bernoulli and geometric RVs

- For a Bernoulli RV $p(1) = p$, $p(0) = 1 - p$ $p(x) = 0$ elsewhere
- Expected value $\Rightarrow \mathbb{E}[X] = (1)p + (0)q = p$

- For a geometric RV $p(x) = p(1-p)^{x-1} = pq^{x-1}$, with $q = 1 - p$
- Note that $\partial q^x / \partial q = xq^{x-1}$ and that derivatives are linear operators

$$\mathbb{E}[X] = \sum_{x=1}^{\infty} xpq^{x-1} = p \sum_{x=1}^{\infty} \frac{\partial q^x}{\partial q} = p \frac{\partial}{\partial q} \left(\sum_{x=1}^{\infty} q^x \right)$$

- Sum inside derivative is geometric. Sums to $q/(1-q)$

$$\mathbb{E}[X] = p \frac{\partial}{\partial q} \left(\frac{q}{1-q} \right) = \frac{p}{(1-q)^2} = \frac{1}{p}$$

- Time to first success is inverse of success probability. Reasonable

Expected value of Poisson RV

- For a Poisson RV $p(x) = e^{-\lambda}(\lambda^x/x!)$. Expected value is (First term of sum is 0, pull λ out, use $x/x! = 1/(x-1)!$)

$$\mathbb{E}[X] = \sum_{x=0}^{\infty} x e^{-\lambda} \frac{\lambda^x}{x!} = \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = \lambda$$

- Sum is Taylor's expansion of $e^{\lambda} = 1 + \lambda + \lambda^2/2! + \ldots \lambda^x/x!$

$$\mathbb{E}[X] = \lambda e^{-\lambda} e^{-\lambda} = \lambda$$

- Poisson is limit of binomial for large number of trials n with $\lambda = np$
- Counts number of successes in n trials that succeed with prob. p
- Expected number of successes is $\lambda = np$,

  $\Rightarrow$ Number of trials $\times$ probability of individual success

Definition for continuous RVs

- Continuous RV X taking values on $\mathbb{R}$ with pdf $f(x)$
- The expected value of the RV X is

$$\mathbb{E}[X] := \int_{-\infty}^{\infty} x f(x)\, dx$$

- Compare with $\mathbb{E}[X] := \sum_{x:p(x)>0} x p(x)$ in the discrete RV case

Expected values of normal RV

- For a normal RV (add and subtract μ, separate integrals)

$$
\begin{aligned}
\mathbb{E}[X] &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x e^{\frac{(x-\mu)^2}{2\sigma^2}} \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x+\mu-\mu) e^{\frac{(x-\mu)^2}{2\sigma^2}} \\
&= \mu\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{\frac{(x-\mu)^2}{2\sigma^2}} + \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-\mu) e^{\frac{(x-\mu)^2}{2\sigma^2}}
\end{aligned}
$$

- First integral is 1 because it integrates a pdf in the whole real line.
- Second integral is 0 by symmetry
- Then $\Rightarrow \mathbb{E}[X] = \mu$

- The mean of a RV with a symmetric pdf is the point of symmetry

Expected value of uniform and exponential RVs

- For a uniform RV $f(x) = 1/(b-a)$ between a and b. Expectation is

$$\mathbb{E}[X] := \int_{-\infty}^{\infty} xf(x)\,dx = \int_a^b \frac{x}{b-a}\,dx = \frac{b^2-a^2}{2(b-a)} = (a+b)/2$$

- Of course, since pdf is symmetric around $(a+b)/2$

- For an exponential RV (non symmetric) simply integrate by parts

$$\mathbb{E}[X] = \int_0^{\infty} xe^{-\lambda x}\,dx \qquad = xe^{-\lambda x}\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\,dx$$
$$= xe^{-\lambda x}\Big|_0^{\infty} + \frac{e^{-\lambda x}}{\lambda}\Big|_0^{\infty} \qquad = \frac{1}{\lambda}$$

Expected value of a function of a RV

- Consider a function $g(X)$ of a RV X. Expected value of $g(X)$?
- g(X) is also a RV, then it also has a pmf $p_{g(X)}\big(g(\mathbf{X})\big)$

$$\mathbb{E}\left[g(X)\right] = \sum_{g(x):p_{g(X)}(g(x))>0} g(x)p_{g(X)}\big(g(x)\big)$$

- If possible values of X are x_i possible values of $g(X)$ are $g(x_i)$ and

$$p_{g(X)}\big(g(x_i)\big) = p_X(x_i)$$

- Then we can write $\mathbb{E}\left[g(X)\right]$ as

$$\mathbb{E}\left[g(X)\right] = \sum_{i=1}^{\infty} g(x_i)p_{g(X)}\big(g(x_i)\big) = \sum_{i=1}^{\infty} g(x_i)p_X(x_i)$$

- Weighted average of functional values. No need to find pmf of $g(X)$
- Same thing can be proved for a continuous RV

$$\mathbb{E}\left[g(X)\right] = \int_{-\infty}^{\infty} g(x)f_X(x)\,dx$$

Expected value of a linear transformation

- Consider a linear function (actually affine) $g(X) = aX + b$

$$\begin{aligned}
\mathbb{E}\left[aX + b\right] &= \sum_{i=1}^{\infty}(ax_i + b)p_X(x_i) \\
&= \sum_{i=1}^{\infty} ax_i p_X(x_i) + \sum_{i=1}^{\infty} bp_X(x_i) \\
&= a\sum_{i=1}^{\infty} x_i p_X(x_i) + b\sum_{i=1}^{\infty} p_X(x_i) \\
&= a\mathbb{E}\left[X\right] + b1
\end{aligned}$$

- Can interchange expectation with additive/multiplicative constants

  $\Rightarrow \mathbb{E}\left[aX + b\right] = a\mathbb{E}\left[X\right] + b$

Expected value of an indicator function

- Indicator function indicates an event by taking value 1 and 0 else
- Let $\mathcal{X}$ be a set $\Rightarrow \mathbb{I}\{x \in \mathcal{X}\} = 1$, if $x \in \mathcal{X}$
  $\Rightarrow \mathbb{I}\{x \in \mathcal{X}\} = 0$, if $x \notin \mathcal{X}$
- Expected value of $\mathbb{I}\{x \in \mathcal{X}\}$ (discrete case)

$$\mathbb{E}\left[\mathbb{I}\{x \in \mathcal{X}\}\right] = \sum_{x: p_X(x) > 0} \mathbb{I}\{x \in \mathcal{X}\} p_X(x) = \sum_{x \in \mathcal{X}} p_X(x) = \mathrm{P}\left[x \in \mathcal{X}\right]$$

- Likewise in the continuous case

$$\mathbb{E}\left[\mathbb{I}\{x \in \mathcal{X}\}\right] = \int_{-\infty}^{\infty} \mathbb{I}\{x \in \mathcal{X}\} f_X(x) = \int_{x \in \mathcal{X}} f_X(x) = \mathrm{P}\left[x \in \mathcal{X}\right]$$

- Expected value of indicator variable = Probability of indicated event
- Compare with expectation of Bernoulli RV (it "indicates success")

Moments, central moments & variance

- n-th moment of a RV is the expected value of its n-th power $\mathbb{E}[X^n]$

$$\mathbb{E}[X^n] = \sum_{i=1}^{\infty} x_i^n p(x_i)$$

- n-th central moment corrects for expected value $\mathbb{E}\left[\left(X - \mathbb{E}[X]\right)^n\right]$

$$\mathbb{E}\left[\left(X - \mathbb{E}[X]\right)^n\right] = \sum_{i=1}^{\infty} \left(x_i - \mathbb{E}[X]\right)^n p(x_i)$$

- 0-th order moment is $\mathbb{E}\left[X^0\right] = 1$; 1-st moment is the mean $\mathbb{E}[X]$
- Second central moment is the variance. Measures width of the pmf

$$\text{var}[X] = \mathbb{E}\left[\left(X - \mathbb{E}[X]\right)^2\right] = \mathbb{E}\left[X^2\right] - \mathbb{E}^2[X]$$

- 3-rd moment measures skewness (0 if pmf symmetric around mean)
- 4-th moment measures heaviness of tails (related to kurtosis)