Ranking of nodes in graphs

Alejandro Ribeiro
Dept. of Electrical and Systems Engineering
University of Pennsylvania
aribeiro@seas.upenn.edu
http://www.seas.upenn.edu/users/~aribeiro/

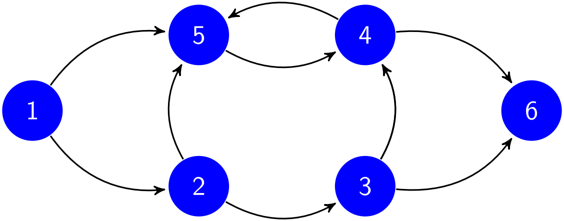
October 1, 2014

Ranking of nodes in graphs: Random walk

Ranking of nodes in graphs: Random walk

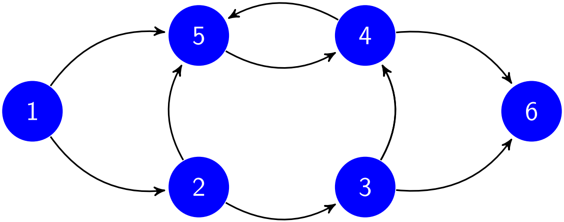
Ranking of nodes in graphs: Markov chain

Graphs



- Graph $\Rightarrow$ A set of J nodes $j = 1, \ldots, J$
  $\Rightarrow$ Connected by a set of edges E defined as ordered pairs (i, j)
- In figure $\Rightarrow$ nodes are $j = 1, 2, 3, 4, 5,$
  $\Rightarrow$ edges $E = \{(1, 2), (1, 5), (2, 3), (2, 5), (3, 4), \ldots$
  $(3, 6), (4, 5), (4, 6), (5, 4)\}$
- Websites and links $\Rightarrow$ "The" web.
- People and friendship $\Rightarrow$ Social network

How well connected nodes are?

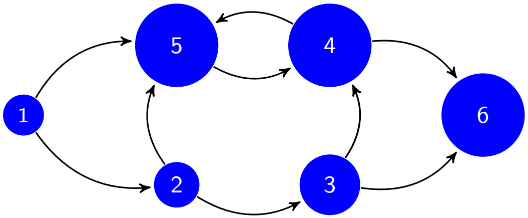


- Q: Which node is the most connected? A: Define most connected
- Can define "most connected" in different ways

- Node rankings to measure website quality, social influence
- There are two important connectivity indicators
  - ⇒ How many nodes point to a link (outgoing links irrelevant)
  - ⇒ How Important are the links that point to a node

Connectivity ranking

- Insight $\Rightarrow$ There is information in the structure of the network
- Knowledge is distributed through the network
  $\Rightarrow$ The network (not the nodes) knows the rankings
- Idea exploited by Google's PageRank© to rank webpages
- ... by social scientists to study trust & reputation in social networks
- ... by ISI to rank scientific papers, transactions & magazines ...



- No one points to 1
- Only 1 points to 2
- Only 2 points to 3, but 2 more important than 1
- 4 as high as 5 with less links
- Links to 5 have lower rank
- Same for 6

Preliminary definitions

- Graph $\mathcal{G} = (V, E)$ $\Rightarrow$ sets of vertices $V = \{1, 2, \ldots, J\}$ and edges E
- Edges (elements of E) are ordered pairs (i, j)
- We say there is a connection from i to j
- Outgoing neighborhood of i is the set of nodes j to which i points

$$n(i) := \{j : (i, j) \in E\}$$

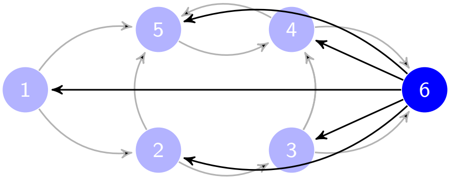
- Incoming neighborhood, $n^{-1}(i)$ is the set of nodes that point to i:

$$n^{-1}(i) := \{j : (j, i) \in E\}$$

- Connected graph
  $\Rightarrow$ There is a path from any node to any other node

Definition of rank

- Agent A chooses node i, e.g., web page, at random for initial visit
- Next visit randomly chosen between links in the neighborhood $n(i)$
  - $\Rightarrow$ All neighbors chosen with equal probability
- If reach a dead end because node i has no neighbors
  - $\Rightarrow$ Chose next visit at random equiprobably among all nodes
- Redefine graph $\mathcal{G} = (V, E)$ adding edges from dead ends to all nodes
- Restrict attention to connected (modified) graphs



- Rank of node i is the average number of visits of A to i

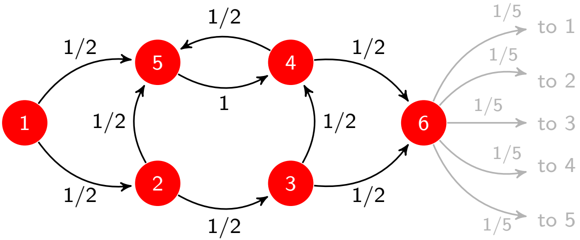
- Formally, let A_n be the node visited at time n
- Define transition probability P_{ij} from node i into node j

$$P_{ij} := \text{P}\left[A_{n+1} = j \,\middle|\, A_n = i\right]$$

- Next visit equiprobable among neighbors

$$P_{ij} = \frac{1}{\#[n(i)]} = \frac{1}{N_i}, \qquad \text{for all } j \in n(i)$$

- Defined number of neighbors $N_i = \#[n(i)]$



- Still have a graph
- But also a MC
- Red (not blue) circles

Formal definition of rank

- Consider variable $\mathbb{I}\{A_m = i\}$ to indicate visit to state i at time m.
- Rank r_i of i-th node defined as time average of number of visits, i.e.,

$$r_i := \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} \mathbb{I}\{A_m = i\}$$

- Define vector of ranks $\mathbf{r} := [r_1, r_2, \ldots, r_J]^T$
- Rank r_i can be approximated by average r_{ni} at time n

$$r_{ni} := \frac{1}{n}\sum_{m=1}^{n} \mathbb{I}\{A_m = i\}$$

- Since $\lim_{n\to\infty} r_{ni} = r_i$, it holds $r_{ni} \approx r_i$ for n sufficiently large
- Define vector of approximate ranks $\mathbf{r}_n := [r_{n1}, r_{n2}, \ldots, r_{nJ}]^T$
- If modified graph is connected, rank independent of initial visit

Algorithm

Output : Vector $\mathbf{r}(i)$ with ranking of node i

Input : Vector $N(i)$ containing number of neighbors of i

Input : Matrix $\mathbf{N}(i, k)$ containing indices j of neighbors of i

$m = 1$; $\mathbf{r}$=zeros(J,1); % Initialize time and ranks

$A_0 =$ random('unid',J); % Draw first visit uniformly at random

while $m < n$ **do**

  jump = random('unid',$N_{A_{m-1}}$); % Neighbor uniformly at random

  $A_m = \mathbf{N}(A_{m-1}, \text{jump})$; % Jump to selected neighbor

  $\mathbf{r}(A_m) = \mathbf{r}(A_m) + 1$; % Update ranking for A_m

  $m = m + 1$;

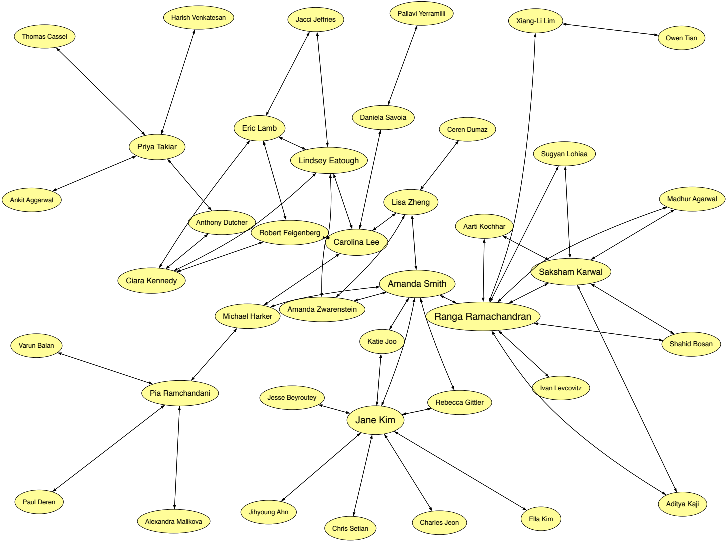
end

$\mathbf{r} = \mathbf{r}/n$; % Normalize by number of iterations n

Example: Social graph

- Asked students taking ESE303 about homework collaboration
- Created (crude) graph of the social network of students in this class
- Used ranking algorithm to understand connectedness
- E.g., If I want to know how well students are coping with the class it is best to ask people with higher connectivity ranking
- 2009 data

Ranked class graph



Convergence metrics

- Recall $\mathbf{r}$ is vector of ranks and $\mathbf{r}_n$ of rank iterates
- By definition $\lim_{n\to\infty} \mathbf{r}_n = \mathbf{r}$. How fast $\mathbf{r}_n$ converges to $\mathbf{r}$ ($\mathbf{r}$ given)?
- Can measure by distance between $\mathbf{r}$ and $\mathbf{r}_n$ $\Rightarrow$

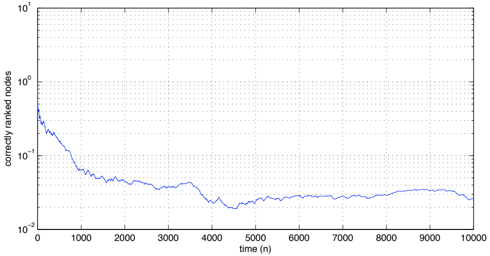
$$\zeta_n := \|\mathbf{r} - \mathbf{r}_n\|_2 = \left(\sum_{i=1}^{J} (r_{ni} - r_i)^2\right)^{1/2}$$

- If interest is only on largest ranked nodes, e.g., a web search
- Denote $r^{(i)}$ as the index of the i-th highest ranked node
- Similarly, $r_n^{(i)}$ is the index of the i-th highest ranked node at time n
- First element wrongly ranked at time n

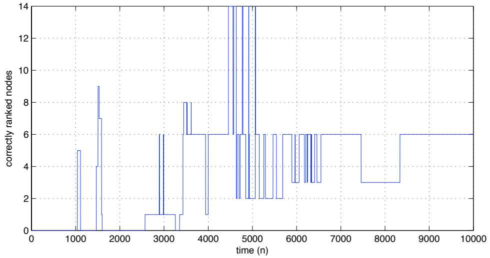
$$\xi_n := \min_i r^{(i)} \neq r_n^{(i)}$$

Evaluation of convergence metrics

Distance



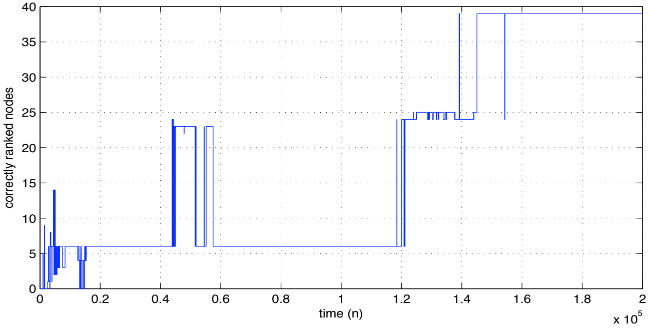
First element wrongly ranked



- Distance gets close to 10^{-2} in approx. 5×10^3 iterations
- Bad: Two largest ranks in 3×10^3 iterations
- Awful: Six best ranks in 8×10^3 iterations
- Convergence appears (very) slow

When does this algorithm converge?

- Can confidently claim convergence not until 10^5 iterations
- True for particular case. Slow convergence inherent to algorithm
- Example has 40 nodes, want to use in network with 10^9 nodes



- Use fact that this process is a MC to obtain faster algorithm

Ranking of nodes in graphs: Random walk

Ranking of nodes in graphs: Markov chain

Limit probabilities

- Recall definition of rank $\Rightarrow r_i := \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} \mathbb{I}\{A(u) = i\}$
- Rank is time average of number of state visits in a MC
  $\Rightarrow$ Can be equally obtained from limiting probabilities
- Recall transition probabilities $\Rightarrow P_{ij} = \frac{1}{N_i}$, for all $j \in n(i)$
- Stationary distribution $\boldsymbol{\pi} = [\pi_1, \pi_1, \ldots, \pi_J]^T$ solution of

$$\pi_i = \sum_{j\in n^{-1}(i)} P_{ji}\pi_j = \sum_{j\in n^{-1}(i)} \frac{\pi_j}{N_j} \qquad \text{for all } i$$

- Plus normalization equation $\sum_{i=1}^{J} \pi_i = 1$
- As per ergodicity $\Rightarrow \mathbf{r} = \boldsymbol{\pi}$

Matrix notation, eigenvalue problem

- As always, can define matrix $\mathbf{P}$ with elements P_{ij}

$$\pi_i = \sum_{j \in n^{-1}(i)} P_{ji}\pi_j = \sum_{j=1}^{J} P_{ji}\pi_j \qquad \text{for all } i$$

- Right hand side is just definition of a matrix product leading to

$$\boldsymbol{\pi} = \mathbf{P}^T\boldsymbol{\pi}, \qquad \boldsymbol{\pi}^T\mathbf{1} = 1$$

- Also added normalization equation
- Can solve as system of linear equations or eigenvalue problem on $\mathbf{P}^T$
- Non-iterative method $\Rightarrow$ Convergence not an issue
- But requires matrix $\mathbf{P}$ available at a central location
- Computationally costly (matrix $\mathbf{P}$ with 10^9 rows and columns)
  - All methods are costly to compute exact solution
  - This one is costly to find even approximate solution

What are limit probabilities?

- Let $p_i(n)$ denote probability of agent A visiting node i at time t

$$p_i(n) := \mathrm{P}\left[A_n = i\right]$$

- Probabilities at time $n+1$ and n can be related

$$\mathrm{P}\left[A_{n+1} = i\right] = \sum_{j \in n^{-1}(i)} \mathrm{P}\left[A_n = i \,\middle|\, A_n = j\right] \mathrm{P}\left[A_n = j\right]$$

- Which is, of course, probability propagation in a MC

$$p_i(n+1) = \sum_{j \in n^{-1}(i)} P_{ji} p_j(n)$$

- By definition limit probabilities are (let $\mathbf{p}(n) = [p_1(n), \ldots, p_J(n)]^T$)

$$\lim_{n \to \infty} \mathbf{p}(n) = \boldsymbol{\pi} = \mathbf{r}$$

- Compute ranks from limit of probability propagation

Probability propagation

- Can also write probability propagation in matrix form

$$p_i(n+1) = \sum_{j \in n^{-1}(i)} P_{ji} p_j(n) = \sum_{j=1}^{J} P_{ji} p_j(n) \qquad \text{for all } i$$

- Right hand side is just definition of a matrix product leading to

$$\mathbf{p}(n+1) = \mathbf{P}^T \mathbf{p}(n)$$

- Can approximate rank by probability distribution
  $\Rightarrow \mathbf{r} = \lim_{n\to\infty} \mathbf{p}(n) \approx \mathbf{p}(n)$ for n sufficiently large

Algorithm

- Algorithm is just a recursive matrix product

Output : Vector $\mathbf{r}(i)$ with ranking of node i
Input : Matrix $\mathbf{P}$ containing transition probabilities

```
m = 1; % Initialize time
r=(1/J)ones(J,1); % Initial distribution uniform across all nodes
while m < n do
    r = P^T r;  % Probability propagation
    m = m + 1;
end
```

Interpretation of probability propagation

- Why does the random walk converges so slow?
- What does it take to obtain a time average r_{ni} close to r_i?
- Need to register a large number of agent visits to every state
- Back of hand: 40 nodes, some 100 visits to each $\Rightarrow 4 \times 10^3$ iters.

- Idea: Unleash a large number of agents K

$$r_i = \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \frac{1}{K} \sum_{k=1}^{K} \mathbb{I}\left\{A_{km} = i\right\}$$

- Visits are now spread over time and space
  - $\Rightarrow$ Converges "K times faster" (depends agents' initial distribution)
  - $\Rightarrow$ But haven't changed computational cost

Interpretation of prob. propagation (continued)

- What happens if we unleash infinite number of agents K?

$$r_i = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \mathbb{I}\left\{A_{km} = i\right\}$$

- Using law of large numbers and expected value of indicator function

$$r_i = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{E}\left[\mathbb{I}\left\{A_m = i\right\}\right] = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathrm{P}\left[A_m = i\right]$$

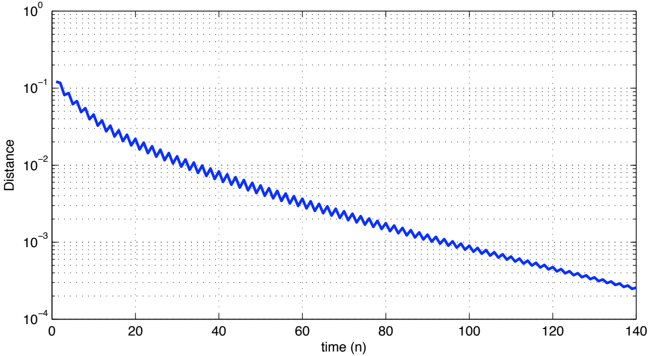
- Graph walk is a MC, then $\lim_{m\to\infty} \mathrm{P}\left[A_m = i\right] = \lim_{m\to\infty} p_i(m)$ exists, and

$$r_i = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} p_i(m) = \lim_{n\to\infty} p_i(n)$$

- Probability propagation ≈ Unleashing infinite number of agents
- Interpretation true for any MC

Distance to rank

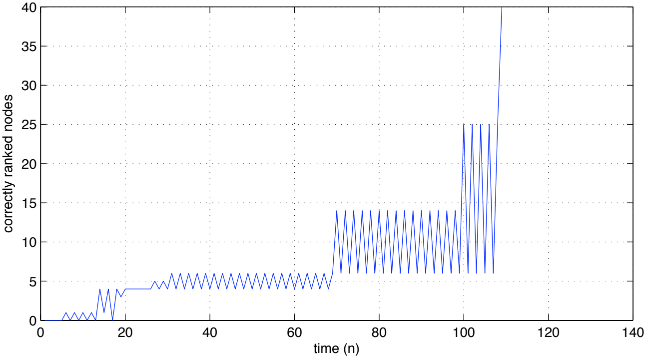
- Initialize with uniform probability distribution $\Rightarrow \mathbf{p}(0) = (1/J)\mathbf{1}$
- Distance between $\mathbf{p}(n)$ and $\mathbf{r}$



- Distance is 10^{-2} in approximately 30 iterations, 10^{-4} in 140 iterations
- Convergence is two orders of magnitude faster than random walk

Number of nodes correctly ranked

- Rank of highest ranked node that is wrongly ranked by time n



- Not bad: All nodes correctly ranked in 120 iterations
- Good: Ten best ranks in 80 iterations
- Great: Four best ranks in 20 iterations

Distributed algorithm to compute ranks

- Nodes want to compute their rank r_i
  - $\Rightarrow$ Can communicate with neighbors only (incoming + outgoing)
  - $\Rightarrow$ Access to neighborhood information only
- Recall probability update

$$p_i(n+1) = \sum_{j \in n^{-1}(i)} P_{ji} p_j(n) = \sum_{j \in n^{-1}(i)} \frac{1}{N_j} p_j(n)$$

- Uses local information only
- Algorithm. Nodes keep local rank estimates $p_i(n)$
- Receive rank (probability) estimates $p_j(n)$ from neighbors $j \in n^{-1}(i)$
- Update local rank estimate $p_i(n+1) = \sum_{j \in n^{-1}(i)} p_j(n)/N_j$
- Communicate rank estimate $p_i(n+1)$ to outgoing neighbors $j \in n(i)$
- Need only know number of neighbors of my neighbors

Distributed implementation of random walk

- Can communicate with neighbors only (incoming + outgoing)
- But cannot access neighborhood information
- Pass agent around

- Local rank estimates $r_i(n)$ and counter with number of visits V_i
- Algorithm run by node i at time n

if *Agent received from neighbor* **then**
    $V_i = V_i + 1$
    Choose random neighbor
    Send agent to chosen neighbor
end
$n = n + 1$; $r_i(n) = V_i/n$;

- Speed up convergence by generating many agents to pass around

Comparison of different algorithms

- Random walk implementation
  - $\Rightarrow$ Most secure & robust. No information shared with other nodes
  - $\Rightarrow$ Implementation can be distributed
  - $\Rightarrow$ Convergence exceedingly slow
- System of linear equations
  - $\Rightarrow$ Least security and robustness. Graph in central server
  - $\Rightarrow$ Distributed implementation not clear
  - $\Rightarrow$ Non-iterative method, convergence not a problem
  - $\Rightarrow$ But computationally costly to obtain approximate solutions
- Probability propagation, matrix powers
  - $\Rightarrow$ Somewhat secure/robust. Info. shared with neighbors only
  - $\Rightarrow$ Implementation can be distributed
  - $\Rightarrow$ Convergence rate acceptable (orders of magnitude faster than RW)