Introduction (week 1)

1 Lower bounded random walk. Consider a game in which players bet $1 to win $1 with probability $p$ and loose their bets with probability $q = 1 − p$. The wealth of a player as a function of time is a stochastic process.

If the player’s wealth at time $t$ is $w(t)$ the wealth at time $t + 1$ is either $w(t) + 1$ or $w(t) − 1$. Moreover, the probability of the wealth increasing to $w(t) + 1$ is $p$ and the probability of the wealth decreasing to $w(t) − 1$ is $q$. We write this as

\[
P \{ w(t + 1) = w(t) + 1 \mid w(t) \} = p,
\]
\[
P \{ w(t + 1) = w(t) − 1 \mid w(t) \} = q.
\]

The first equation, e.g., is read as “the probability of $w(t + 1)$ being $w(t) + 1$, given $w(t)$ is $p$.” The expression in $[1]$ is true as long as $w(t) ≠ 0$. When $w(t) = 0$ the gambler is ruined and $w(t + 1) = 0$. A rather sophisticated, yet sometimes useful way of expressing this fact is

\[
P \{ w(t + 1) = 0 \mid w(t) = 0 \} = 1.
\]

We saw in class that if $p > 1/2$ then it is likely that $w(t)$ diverges making this a rather good game to play. In this exercise $p$ can take any value. This process can be called a lower bounded random walk. Wealth can be reinterpreted as position on a line and wealth variations as steps taken randomly to left and right. The origin is home, in that if the walker reaches 0 it stays there. It is asked that:

A Simulation of a process realization. Write a function that accepts as parameters the probability $p$, the initial wealth $w(0) = w_0$ and a maximum number of runs $T$. The function returns a vector of length at most $T + 1$ containing the wealth’s history $w(0), \ldots, w(T)$ randomly computed according to the probabilities in $[1]$ and $[2]$. If the wealth is depleted at time $t < T$, that is, if $w(t) = 0$ for some $t < T$, the function returns a vector of length $t + 1$ with the wealth’s history up to time $t$, i.e., $w(0), \ldots, w(t)$. Optionally, you can also return a boolean variable to distinguish between a run that resulted in a broken player and one that did not. This might be useful for parts B-E. Show plots with simulated processes for $w_0 = 20$, $T = 10^3$ and $p = 0.25$, $p = 0.5$ and $p = 0.75$.

B Probability of reaching home. Fixing $p = 0.55$ and $w_0 = 10$ compute the probability $B(p, w_0)$ of eventually reaching home, that is the probability of having $w(t) = 0$ for some $t$. Notice that because once $w(t) = 0$ wealth stays at 0 this probability can be written as the limit

\[
B(p, w_0) = \lim_{t \to \infty} P \{ w(t) = 0 \mid w(0) = w_0 \}.
\]

Strictly speaking, you would need to run the simulation forever to make sure the gambler does not run out of money. However, you can truncate simulations at time $T = 100$ for this exercise. With this approximation you would be aiming to compute the probability of reaching home between times 0 and $T$, which we assume approximates the probability of reaching home between times 0 and $\infty$ reasonably well. Put differently, we are assuming that $P \{ w(T) = 0 \mid w(0) = w_0 \}$ for $T = 100$ is a good approximation of the limit in $[3]$. To estimate $P \{ w(T) = 0 \mid w(0) = w_0 \}$ we run the simulation code of part A multiple times. Each of these runs results in a wealth path $w_n(t)$, we then define the function $I \{ w_n(T) = 0 \}$ which equals 1 if wealth at time $T$ is $w_n(T) = 0$ and 0 if not. The probability of reaching home is then estimated as

\[
\hat{B}_N(p, w_0) = \frac{1}{N} \sum_{n=1}^{N} I \{ w_n(T) = 0 \}.
\]

The expression in $[4]$ is just the average number of times home was reached across all experiments. The function $I \{ w_n(T) = 0 \}$ is called the indicator function of the event $w_n(T) = 0$ because it “indicates” the event by taking the value 1.

To compute $\hat{B}_N(p, w_0)$ you need to decide on a number of experiments $N$. The more experiments $N$ you run the more accurate your estimation. Alas, the larger you need to wait. Report your probability estimate and the number of experiments $N$ used. Explain your criteria for selecting $N$.
C  Probability of reaching home as a function of initial wealth. We want to study the probability of reaching home as a function of initial wealth. Fix $p = 0.55$ and vary initial wealth between $w_0 = 1$ and $w_0 = 20$. Show a plot of your probability estimates $\hat{B}_N(p, w_0)$ as a function of initial wealth. The number of experiments $N$ run to compute probability estimates for different initial wealths need not be the same.

D  Probability of reaching home as a function of $p$. The goal is to understand the variation of the probability of reaching home for different values of the probability $p$. Fix $w_0 = 10$ and vary $p$ between 0.3 and 0.7 – increments 0.02 should do. Show a plot of your probability estimates $\hat{B}_N(p, w_0)$ as a function of $p$. You should observe a fundamentally different behavior for $p < 1/2$ and $p > 1/2$. Comment on that.

E  Time to reach home. Fix $p = 0.4$. With this value of $p$ it is possible to see that gamblers eventually deplete their wealth independently of their initial wealth $w_0$. This is something remarkable, despite the process being random it is possible to say that $w(t)$ eventually becomes 0. This needs to be qualified, though. Unlikely as it may be there is a chance of winning all hands. Of course, the probability of this happening becomes smaller as the gambler plays more hands. What we can say about a lower bounded random walk is that with probability 1, wealth $w(t)$ approaches 0 as $t$ grows. More formally, the limit $\lim_{t \to \infty} w(t)$ satisfies

$$P \left\{ \lim_{t \to \infty} w(t) = 0 \right\} = 1.$$  \hfill (5)

We say that $\lim_{t \to \infty} w(t) = 0$ almost surely. Different wealth paths are possible, but almost all of them result in a broken gambler. If we think of probabilities as measuring the likelihood of an event, the measure of the event $w(t) \neq 0$ is asymptotically null. An important quantity here is the time at which $w(t) = 0$ for the first time which we can write as

$$T_0 = \min_t \{ w(t) = 0 \} \hfill (6)$$

Estimate the probability distribution of $T_0$ and its average value.