Week 11: Gaussian processes
White Gaussian noise
Solutions

A Independence. Jointly normal random variables (RVs) have the property of being independent if and only if they are uncorrelated. This is *not* true in general: it is very specific of Gaussian RVs (the only other case I know is RVs that take on only two different values, e.g., the Bernoulli RV). Note also that the RVs must be *jointly* normal: it is possible for X and Y to be normally distributed and for (X, Y) to *not* be a bivariate Gaussian.

Nevertheless, the definition given in (1) from the exercise implies joint normality. Hence, suffices to show that $W(t_1)$ and $W(t_2)$ are not correlated for $t_1 \neq t_2$. From (5), the autocorrelation function of the Gaussian process is $R_W(t_1, t_2) = \sigma^2 \delta(t_1 - t_2)$. Intuitively, we have that $R_W(t_1, t_2) = 0$ for $t_1 \neq t_2$.

If you want to be formal, however, remember that we only defined $\delta(t)$ in terms of an integral [see (4)]. So we cannot say $\delta(t) = 0$ for $t \neq 0$ without proving it. This is actually fairly simple. Suppose $t \neq 0$ and write

$$\int_{t-\epsilon}^{t+\epsilon} \delta(\tau) d\tau = \int_{t-\epsilon}^{t+\epsilon} 1 \times \delta(\tau) d\tau = 0, \quad \text{for all } \epsilon > 0,$$

where we used $f(t) = 1$ for all t in (4). Since $f(\tau) > 0$ over the interval $[t - \epsilon, t + \epsilon]$, the integral vanishes if and only if $\delta(\tau) = 0$ for $\tau \in [t - \epsilon, t + \epsilon]$.

B The integral of WGN. [Refer to slides 13, 40, and 41 of `markov_gaussian_stationary_processes`]. Recall that integration is a linear functional. Thus, $X(t)$ is a Gaussian process since it is defined as the linear functional of a Gaussian process.

Given that $\mu_W(t) = 0$, the mean function of $X(t)$ is

$$\mu_X(t) = \mathbb{E}\left[\int_0^t W(u) du\right] = \int_0^t \mathbb{E}[W(u)] du = \int_0^t \mu_W(t) du = 0.$$

Switching the expected value and the integral like that should make you uneasy. After all, the expected value is an integral operator and it is not always the case that integrals can just be exchanged like that. In this case, however, we are justified (take a look at Fubini's theorem). We are going to use the same result to derive the autocorrelation function of $X(t)$:

$$\begin{aligned}
R_X(t_1,t_2) &= \mathbb{E}\left[\left(\int_0^{t_1} W(u_1)du_1\right)\left(\int_0^{t_2} W(u_2)du_2\right)\right] \\
&= \mathbb{E}\left[\int_0^{t_1}\int_0^{t_2} W(u_1)W(u_2)du_2du_1\right] \\
&= \int_0^{t_1}\int_0^{t_2} \mathbb{E}[W(u_1)W(u_2)]du_2du_1 \\
&= \int_0^{t_1}\int_0^{t_2} R_W(u_1,u_2)du_2du_1 \\
&= \int_0^{t_1}\int_0^{t_2} \sigma^2\delta(u_1-u_2)du_2du_1,
\end{aligned}$$

where we used the fact that $R_W(u_1,u_2)=\mathbb{E}[W(u_1)W(u_2)]=\sigma^2\delta(u_1-u_2)$. Now, from the definition of the δ distribution in (4), we obtain

$$\begin{aligned}
R_X(t_1,t_2) &= \begin{cases} \int_0^{t_1}\sigma^2 du_1 = \sigma^2 t_1, & \text{for } t_1 < t_2 \\ \int_0^{t_2}\sigma^2 du_2 = \sigma^2 t_2, & \text{for } t_1 > t_2 \end{cases} \\
&= \sigma^2 \min(t_1,t_2).
\end{aligned}$$

Given that $X(t)$ is a Gaussian process, $X(t)$ is normally distributed for all t, i.e., $X(t) \sim \mathcal{N}(\mu(t), \sqrt{R_X(t,t)})$. Hence,

$$\mathbb{P}\left[X(t) > a\right] = 1 - \mathbb{P}\left[X(t) \le a\right] = 1 - \Phi\left(\frac{a}{\sigma\sqrt{t}}\right), \tag{1}$$

where Φ is the cdf of a standard normal random variable.

C Discrete time representation of WGN. Proceeding as in Part B, we can obtain the mean value function of $W_h(n)$:

$$\mu_{W_h}(n) = \mathbb{E}\left[W_h(n)\right] = \mathbb{E}\left[\int_{nh}^{(n+1)h} W(\tau)d\tau\right] = \int_{nh}^{(n+1)h}\mathbb{E}\left[W(\tau)\right]d\tau = \int_{nh}^{(n+1)h} 0 d\tau = 0.$$

Its autocorrelation function, $R_{W_h}(n_1, n_2)$, is given by

$$\begin{aligned}
R_{W_h}(n_1, n_2) &= \mathbb{E}\left[W_h(n_1)W_h(n_2)\right] \\
&= \mathbb{E}\left[\left(\int_{n_1h}^{(n_1+1)h} W(u_1)du_1\right)\left(\int_{n_2h}^{(n_2+1)h} W(u_2)du_2\right)\right] \\
&= \mathbb{E}\left[\int_{n_1h}^{(n_1+1)h}\int_{n_2h}^{(n_2+1)h} W(u_1)W(u_2)du_2du_1\right] \\
&= \int_{n_1h}^{(n_1+1)h}\int_{n_2h}^{(n_2+1)h} \mathbb{E}\left[W(u_1)W(u_2)\right]du_2du_1 \\
&= \int_{n_1h}^{(n_1+1)h}\int_{n_2h}^{(n_2+1)h} \sigma^2\delta(u_1-u_2)du_2du_1 \\
&= \begin{cases} \sigma^2 h, & n_1 = n_2 \\ 0, & n_1 \neq n_2 \end{cases}.
\end{aligned}$$

D Simulating $X(t)$. The following MATLAB script simulates the process $X(t)$ using its discrete time version $X_h(n)$ obtained from $W_h(n)$ derived in Part C.

```
% Delete all variables and close figures
clear all
close all

h = 0.01;       % Discretization step size
sigma_sq = 1;   % Instantaneous variance
T = 10;         % Duration of simulation

% Simulation
W = sigma_sq*sqrt(h)*randn(floor(T/h) + 1, 1);
X = cumsum(W);

% Plot
figure();
plot(0:h:T, X, 'Linewidth', 2);
xlabel('Time (t)')
ylabel('X(t)');
xlim([0 T]);
grid;


%%% Export figure %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
set(gcf,'Color','w');
export_fig -q101 -pdf HW11_D.pdf
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

A result of this simulation is shown in Figure 1.

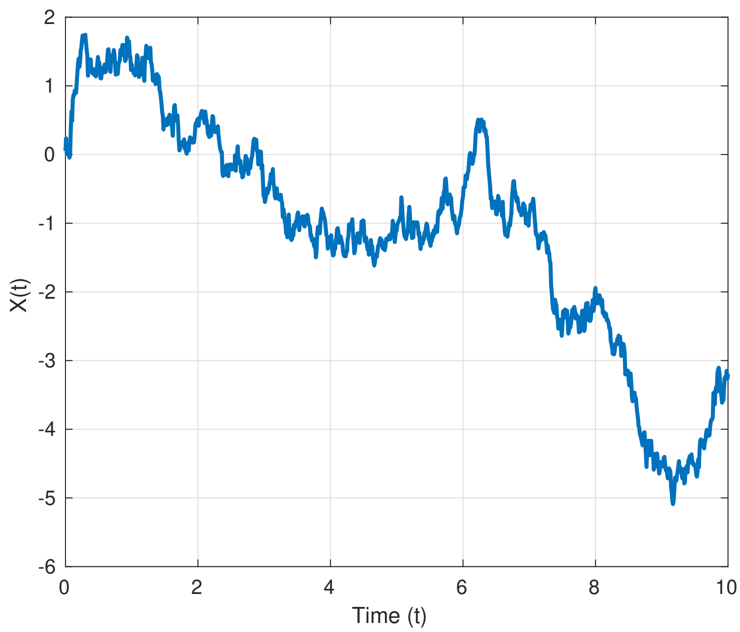

Figure 1: A sample path of the simulated Gaussian (Wiener) process $X(t)$ using a discrete approximation with step size $h = 0.01$ (part D).