Week 12: Gaussian processes
Arbitrages

In betting or investments, an arbitrage is an opportunity to profit regardless of the outcomes. Consider for example the 2010 Soccer World Cup in which the two favorite teams were Brazil (obviously!) and Spain. At the time, Prof. Ribeiro thought that Spain had about as high a chance of winning the cup as Uruguay (he did not mean that in a good way). He turned out to be right, since Spain won and Uruguay was fourth. He likes to say this sits high in his list of personal achievements, together with his 1998 prediction that short messages do not have a market because no normal person would write a short message if they could use their phones to make a call. He stands by the second part, saying he simply overestimated the number of normal people in the world. Fair enough. . .

Anyway, let us dial the calendar to November, 2009. You can place bets on Brazil, Spain, or "neither Brazil, nor Spain" (which we will call *other*). Booker 1 is offering the following odds for these three outcomes:

Brazil	Spain	Other
5.6:1	7.1:1	0.2:1

The odds on Brazil, for example, mean that for each dollar you bet on Brazil you are paid US$ 6.60 if Brazil wins the World Cup, i.e., your dollar (the stakes) plus US$ 5.60. Be careful here: we are using the British or fractional odds system which does not include the stakes in the odds. No one said gambling systems were simple! As another example, the odds on Spain mean that you get US$ 8.10 for each dollar bet on Spain, if Spain wins. And finally, the odds on "other" mean that you are paid US$ 1.20 dollars for each dollar you bet on the outcome "neither Brazil, nor Spain," if this indeed comes to happen (typically, booking agencies write 1:5 here).

Our goal is to look for arbitrage opportunities by placing mixed bets on all three outcomes. We bet x on Brazil, y on Spain, and z on other, so that the total money invested is a given constant c of how much money you have. In other words, $x + y + z = c$. The earnings associated with this investment strategy are

$$\begin{aligned} \text{Brazil wins:} \quad & 6.6x - c \\ \text{Spain wins:} \quad & 8.1y - c \\ \text{Other wins:} \quad & 1.2z - c \end{aligned}$$

A Single booker arbitrage. Since Booker 1 is not in the business of giving away money, it is unlikely that there is an arbitrage opportunity here. Show that indeed arbitrage is not possible for the odds given above.

B Many bookers arbitrage. When different bookers are accepting bets, it may be possible to find arbitrages by placing bets on different outcomes with different bookers. In particular, consider three different bookers offering the following odds:

Booker	Brazil	Spain	Other
1	5.6:1	7.1:1	0.2:1
2	5.4:1	6.8:1	0.3:1
3	4.9:1	7.4:1	0.1:1

Find if there exists an arbitrage opportunity by combining bets placed with different bookers. If an arbitrage opportunity exists, find a combination of bets that yields an arbitrage. Assuming the world cup final is 6 months away compute, the annualized rate of return for your investment strategy. Since it is patently clear that merit-based pay yields superb performance, the larger the rate of return of your investment allocation, the larger your grade for on homework.