Week 13: Gaussian processes
Option pricing
Solutions

A Deriving (3) and (4). Recall from (1) that $Y_t(s)$ was normally distributed with mean μs and variance $\sigma^2 s$, i.e., it is a Gaussian process. Immediately, since $Y_n = Y_{(n-1)h}(h)$ from (2), we have that it is normally distributed with mean μh and variance $\sigma^2 h$.

Moreover, observe that the relative price increments $Y_t(s)$ are independent for disjoint intervals, i.e., $(Y_t(s), Y_u(r))$ are independent for $(t, t+s)$ disjoint of $(u, u+r)$. Since Y_{n_1} and Y_{n_2}, $n_1 \neq n_2$, refer to the disjoint intervals $(n_1, n_1 + h)$ and $(n_2, n_2 + h)$ respectively, they are therefore independent.

B Determine drift and volatility for CSCO. Using the data provided, the following MATLAB script estimates the drift and volatility of the CSCO stock:

```
% Delete all variables and close figures
clear all
close all

% Load CSCO price history
cisco_stock_price;

Z = log(close_price);
Y = Z(2:end)-Z(1:end-1);

% Time steps = 1 day
h = 1;

% Number of data points in the year
N = length(Y);

% Drift estimate
mu_hat = sum(Y)/(N*h);

% Volatility estimate
var_hat = sum((Y-mu_hat*h).^2)/((N-1)*h);
```

We obtain $\hat{\mu} = 1.72 \times 10^{-3}$ and $\hat{\sigma}^2 = 5.96 \times 10^{-4}$.

C Is geometric Brownian motion a good model? The following MATLAB script compares the pdf and CDF of Y_n to the postulated normal model:

```
% Delete all variables and close figures
clear all
close all

% Estimate Y, mu_hat, var_hat
HW13_B;

% Define bins
```

```
x = -0.1:0.01:0.1;

% Compute hitogram
n_elements = histc(Y, x);

% Compare with Gaussian pdf
h1 = figure();
pdf = n_elements/N/0.01;
bar(x, pdf);
hold on;
plot(x, normpdf(x, mu_hat*h, sqrt(var_hat*h)), 'r', 'Linewidth', 2);
xlabel('x');
ylabel('pdf');
grid;
legend('Estimated', 'Gaussian pdf', 'Location', 'Best');

% Compare with Gaussian cdf
h2 = figure();
cdf = cumsum(n_elements)/N;
stairs(x, cdf);
hold on;
plot(x, normcdf(x, mu_hat*h, sqrt(var_hat*h)), 'r', 'Linewidth', 2);
xlabel('x');
ylabel('cdf');
grid;
legend('Estimated', 'Gaussian pdf', 'Location', 'Best');


%%% Export figure %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
set(h1,'Color','w');
export_fig(h1, '-q101', '-pdf', 'HW13_C1.pdf');

set(h2,'Color','w');
export_fig(h2, '-q101', '-pdf', 'HW13_C2.pdf');
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Fig. 1 displays the results of this code. The comparison shows that the geometric Brownian motion model is just barely acceptable.

D Expected return. Using the expected return expression that was introduced in the slide 27 of `arbitrage_stock_and_option_pricing`, the discounted expected return of the stock is given by

$$E_r = e^{-\alpha t} \frac{\mathbb{E}\left[X(t) \mid X(0)\right]}{X(0)} = e^{(\hat{\mu}+\hat{\sigma}^2/2-\alpha)t}.$$

For the parameters estimated in Part B, we obtain $E_r = 1.2921$, i.e., a net profit of almost 30%.

To evaluate the probability, recall that $\log\left[\frac{X(t)}{X(0)}\right] \sim \mathcal{N}(\mu t, \sigma^2 t)$. Hence,

$$\begin{aligned}
\mathbb{P}\left[\frac{e^{-\alpha t}X(t)}{X(0)} \geq 1.05\right] &= \mathbb{P}\left[\log\left(\frac{e^{-\alpha t}X(t)}{X(0)}\right) \geq \log(1.05)\right] \\
&= \mathbb{P}\left[\log\left(\frac{X(t)}{X(0)}\right) \geq \log(1.05) + \alpha t\right] \\
&= 1 - \Phi\left(\frac{\log(1.05) + \alpha t - \mu t}{\sqrt{\sigma^2 t}}\right),
\end{aligned}$$

where Φ is the cdf of the standard normal distribution. Using the parameters from Part B, this expression evaluates to 0.63 [you can compute this by typing `1-normcdf(0.3, mu_hat*252,` `sqrt(var_hat*252))` into MATLAB].

E Risk neutral measure. From slide 26 of `arbitrage_stock_and_option_pricing`, the risk neutral measure refers to a rescaled geometric Brownian motion whose drift parameter is $\alpha - \sigma^2/2$ and whose variance parameter is σ^2. Thus, the risk neutral measure for the CSCO stock is a geometric Brownian motion with mean $7.02 \times 10^{-4} \cdot t$ and variance $5.96 \times 10^{-4} \cdot t$.

F Expected return for the risk neutral measure. From the definition of risk neural measure, it is ready that the expected discounted rate of return in the alternative reality is one (its logarithm is 0). Thus, the non-discounted rate of return is α, i.e., the rate of return of the risk-free investment.

G Deriving the Black-Scholes formula. Notice that $\mathbb{E}_q\left[e^{-\alpha t}\left[X(t) - K\right]_+\right]$ is the discounted expected return with respect to the risk neutral measure. Since c is a deterministic quantities, (7) guarantees that the net gain with respect to the risk neutral measure is zero.

To obtain a closed-form expression for the price c, note that (7) implies that

$$c(t) = e^{-\alpha t}\,\mathbb{E}_q\left[\left[X(0)e^{Y(t)} - K\right]_+\right], \tag{1}$$

where we used the fact $X(t) = X(0)e^{Y(t)}$ is a geometric Brownian motion. Hence, $Y(t) \sim \mathcal{N}(\mu t, \sigma^2 t)$ and (1) can be written as

$$c(t) = e^{-\alpha t}\int_{\log[K/X(0)]}^{\infty} (X(0)e^y - K)\exp\left[-\frac{(y-\mu t)^2}{2\sigma^2 t}\right] dy,$$

where we used the fact that $[X(0)e^y - K]_+ = 0$ for $y \leq \log\left[K/X(0)\right]$ to constraint the limits of the integral. By performing the change of variables and manipulations in slides 36–39 of `arbitrage_stock_and_option_pricing` yields

$$c = X(0)Q(b) - e^{-\alpha t}KQ(a), \tag{2}$$

for $a = \frac{\log(K) - \log[X(0)] - (\alpha - \sigma^2/2)t}{\sqrt{\sigma^2 t}}$ and $b = a - \sqrt{\sigma^2 t}$.

H Determine option price. Using (2), the parameters estimated in Part B, and the expression for the (undiscounted) expected return from Part D, we can use the following MATLAB script to obtain the option prices $c(t)$ as a function of time:

```
% Delete all variables and close figures
clear all
close all

% Estimate Y, mu_hat, var_hat
HW13_B;

alpha = 0.1/100;        % Risk-free rate of return
X0 = close_price(1,1);  % Initial price
t = 1:251;              % Strike times (days)

% Expected return for each strike time
EX = X0 * exp((mu_hat + var_hat/2)*t);

```

```
% Variance of return for each strike time
varX = X0^2 * exp((2*mu_hat + var_hat)*t) .* (exp(var_hat*t) - 1);

% One standard deviation band
EX_upper = EX + sqrt(varX);
EX_lower = EX - sqrt(varX);

% Strike prices
K = [0.8 ; 1 ; 1.2]*EX;

% Compute option prices for each strike price
c = zeros(size(K,1), length(t));
for ii = 1:size(K,1)
    a = (log(K(ii,:)/X0) - (alpha - var_hat/2)*t) ./ sqrt(var_hat*t);
    b = a - sqrt(var_hat*t);

    Qa = 1 - normcdf(a, 0, 1);
    Qb = 1 - normcdf(b, 0, 1);
    c(ii,:) = X0 * Qb - exp(-alpha*t) .* K(ii,:) .* Qa;
end


% Plot results
h1 = figure();
plot(t, c, 'LineWidth', 2);
xlabel('Strike time (days)');
ylabel('Option price');
grid;
legend('K = 0.8 EX', 'K = EX', 'K = 1.2 EX', 'Location', 'Best');
xlim([1 250]);

h2 = figure();
fill([t fliplr(t)], [EX_upper fliplr(EX_lower)], 'r', 'Linestyle', 'None', 'FaceAlpha', 0.2);
hold on
plot(t, EX, 'k', 'LineWidth', 2);
xlabel('Strike time (days)');
ylabel('Expected stock price');
grid;
xlim([1 250]);


%%% Export figure %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
set(h1,'Color','w');
export_fig(h1, '-q101', '-pdf', 'HW13_H1.pdf');

set(h2,'Color','w');
export_fig(h2, '-q101', '-pdf', 'HW13_H2.pdf');
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

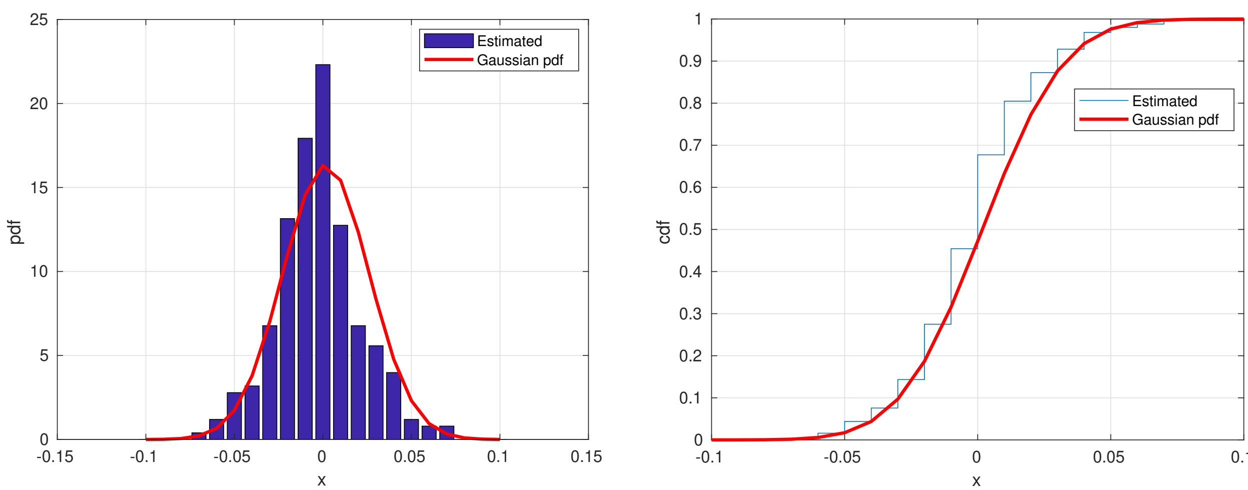
The results are shown in Fig. 2. Due to the law of large numbers, we expect the value of $X(t)$ to concentrate around its expected value. Hence, by buying an option with strike price lower than the expected value of the stock at time t, i.e., $K = 0.8\,\mathbb{E}\left[X(t)\right]$, we have a good chance of the strike price being lower than the value of the stock when the option's strike time comes. By exercising the option (buying the stock at price K) and reselling the stock at the current price, we therefore make a profit.

On the other hand, taking $K = 1.2\,\mathbb{E}\left[X(t)\right]$ means that exercising a buying option would make us lose money: we would be better off buying directly at the market price. But there is still volatility enough that, for a cheap enough option, it would be worth trying (see Fig. 2b). For instance, take a strike time of 250 days. The expected stock price at that date is approximately US\$ 25.00, so that our strike price is $K \approx$ US\$ 30.00. In other words, if the world were deterministic we would lose

US$ 5.00 every time. Notice, however, that the one standard deviation band goes up to US$ 35.00. Hence, there is a significant probability that we may still be able to make a profit in this condition. You can compute this probability as in Part D by doing

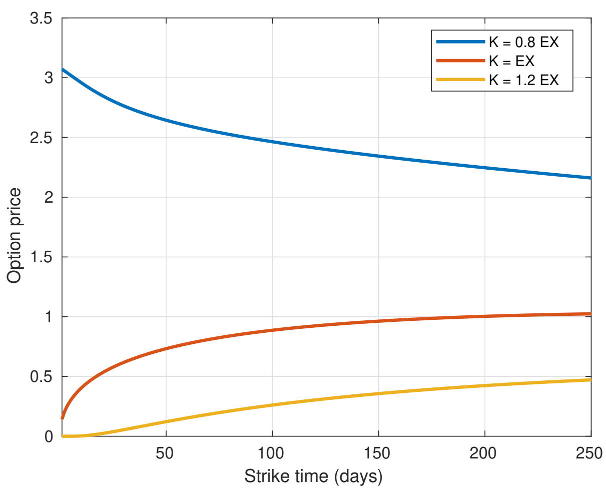
$$\mathbb{P}\left[X(t) \geq 1.2\,\mathbb{E}[X(t)]\right] = \mathbb{P}\left[\frac{X(t)}{X(0)} \geq \frac{1.2\,\mathbb{E}[X(t)]}{X(0)}\right] = 1 - \Phi\left(\frac{\log(1.2) + (\mu + \sigma^2/2)t}{\sqrt{\sigma^2 t}}\right),$$

where Φ is again the cdf of the standard normal distribution. This gives a probability of 0.25 of making a profit. If the cost of the option was 50 cents, we might consider buying some. Moreover, if we were to also considering selling options, i.e., options in which we agree to sell the stock for K at time t, we could then make a profit by buying the stock at market price and then selling it at a premium if $K > X(t)$.

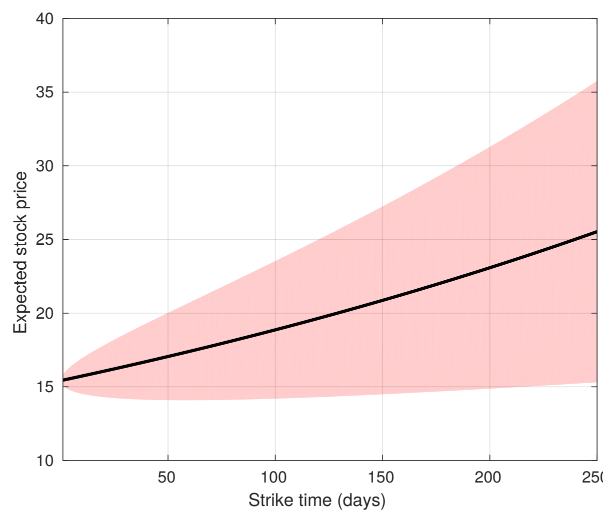


(a) (b)

Figure 1: Comparison between distribution of Y_n and normal distribution (part C): (a) pdf and (b) cdf.



(a)



(b)

Figure 2: Pricing option using the Black-Scholes equation for different strike price and time (part H): (a) option price and (b) expected stock price with one standard deviation band.