Week 3: Probability review
Decision making
Solutions

A Simulate an individual experiment. Below you find two versions of the function, each using a different programming technique.

```
% Version 1: for loop

function [accepted_rank, accepted_round] = HW3_A1(J,K,L)

% Generate random order of offers
offers = randperm(J);

% Find X0
rejected_offers = offers(1:K);
sorted_rejected_offers = sort(rejected_offers);
X0 = sorted_rejected_offers(L);

% Go through decision making procedure
for i = K+1:J-1
    if (offers(i) < X0)
        accepted_rank = offers(i);
        accepted_round = i;
        return;
    end
end

% If we get to the last offer, then take it
accepted_rank = offers(J);
accepted_round = J;

end
```

```
% Version 2: using "find" and no loop

function [accepted_rank, accepted_round] = HW3_A2(J,K,L)

% Generate random order of offers
offers = randperm(J);

% Find X0
rejected_offers = offers(1:K);
sorted_rejected_offers = sort(rejected_offers);
X0 = sorted_rejected_offers(L);

% Find first offer after K that is better than X0
i = find(offers(K+1:end) < X0, 1);

if isempty(i)
    % No offer was better then X0
    accepted_round=J;
```

```
    accepted_rank=offers(J);
else
    accepted_round = K+i;
    accepted_rank = offers(accepted_round);
end

end
```

B Probability distribution of the rank of accepted offers. The MATLAB code to compute the pmf of accepted ranks is shown below.

```
% Delete all variables and close figures
clear all
close all

J = 50;      % Total number of offers
K = 30;      % Number of offers to automatically reject
L = 1;       % Constant to choose X0
% L = 2;
% L = 5;
N = 1e4;     % Number of realizations

% Initialization of the vector of accepted ranks
accepted_ranks = zeros(1,N);

% Run experiments
for i = 1:N
    [accepted_ranks(i), ~] = HW3_A2(J, K, L);
end

% Compute frequencies and evaluate probabilities
[count, n] = hist(accepted_ranks, 1:J);
pmf_vector = count/N;

% Plot results
figure();
bar(1:J, pmf_vector);
xlabel('Xstar');
ylabel('pmf');
title(['L = ' num2str(L)]);
xlim([0, J+1]);



%%% Export figure %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
set(gcf,'Color','w');
export_fig('-q101', '-pdf', '-append', 'HW3_B.pdf');
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

The plots for different L are shown in Fig. 1. As you can see, the first L ranks are approximately equiprobable and the behavior of the probability of ranks $L+1$ to $J = 50$ is very similar. Through trial and error we found that $N = 10^4$ realizations (experiments) gave good results. Using $N = 10^5$ lead to a similar pmf (this should strike you as quite intriguing, given that there are $J! \approx 3 \times 10^{64}$ different possible rank permutations, all equiprobable…I'll let you ponder about that). We will keep using that number from now on.

Notice that the proposed policy for selling the car performs quite favorably. Indeed, the likelihood of selling the car to best offer is quite high (more than 0.3) and the probability of accepting one of the top three offers is above 0.4.

C Probability of accepting the best offer as a function of K. In this section we will create a function to determine the probability distribution for selecting the best offer, while varying the number of rejected offers, K, between 1 and $J-1$. As in previous sections, several versions of code are provided.

```
% Delete all variables and close figures
clear all
close all

J = 50;      % Total number of offers
K = 30;      % Number of offers to automatically reject
N = 1e4;     % Number of realizations

% Values of L to use
L_vector = [1 2 5];

% Initialization of the vector of accepted ranks
prob_of_rank_one = zeros(length(L_vector),J);


for ell = 1:length(L_vector)
    L = L_vector(ell);

    for K = L:J-1
        % Initialization of the vector of accepted ranks
        accepted_ranks = zeros(1,N);

        % Run experiments
        for i = 1:N
            [accepted_ranks(i), ~] = HW3_A2(J, K, L);
        end

        % Evaluate probabilities of accepting best offer
        prob_of_rank_one(ell,K) = sum(accepted_ranks == 1)/N;
    end
end


figure();
plot(1:J, prob_of_rank_one(1,:), 'x', 'LineWidth', 2);
hold on;
plot(1:J, prob_of_rank_one(2,:), '.', 'LineWidth', 2);
plot(1:J, prob_of_rank_one(3,:), 's', 'LineWidth', 2);
xlabel('K');
ylabel('Probability');
title('Probability of picking best offer');
xlim([0, J+1]);
grid
legend('L = 1', 'L = 2', 'L = 5');



%%% Export figure %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
set(gcf,'Color','w');
export_fig('-q101', '-pdf', 'HW3_C.pdf');
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

The results are depicted in Fig. 2. Notice that if we choose $L = 1$ and $K \approx 18$, the probability of accepting the best offer is as high as 0.37. Moreover, for a wide range of K, namely $10 \le K \le 30$, this probability is at least one-third. On the other hand, increasing L only deteriorates the probability of accepting the best offer. Moreover, if we increase L, the maximum probability is achieved for larger K.

D Probability of accepting the last offer. We only accept the last offer if one of two things happen: (i) if the best offer was one of the first K ones—since in this case, we end up setting $X_0 = 1$ and rejecting all other offers; (ii) if the last offer is the best offer ($X_J = 1$) and the second to best offer is one of the first K ones ($X_0 = 2$). Formally, we can write

$$\mathbb{P}\left[X^\star = X_J\right] = \mathbb{P}\left[\underbrace{\{\exists i \in \{1,\ldots,K\} : X_i = 1\}}_{\mathcal{A}} \text{ OR } \underbrace{\{X_J = 1 \text{ AND } \exists i \in \{1,\ldots,K\} : X_i = 2\}}_{\mathcal{B}}\right].$$

Notice that the events $\mathcal{A}$ and $\mathcal{B}$ are disjoint. Indeed, either the best offer is one of the first K ones or it is the last one (assuming $K < J$). Both of these conditions cannot be met simultaneously. We therefore have that

$$\mathbb{P}\left[X^\star = X_J\right] = \mathbb{P}\Big[\{\exists i \in \{1,\ldots,K\} : X_i = 1\}\Big] + \mathbb{P}\Big[\{X_J = 1 \text{ AND } \exists i \in \{1,\ldots,K\} : X_i = 2\}\Big].$$

Suffices now to evaluate these probabilities.

First, observe that the probability of the best offer being at a given position is $1/J$, since each permutation of ranks is equiprobable. Hence,

$$\mathbb{P}\Big[\{\exists i \in \{1,\ldots,K\} : X_i = 1\}\Big] = K \times \frac{1}{J} = \frac{K}{J}. \tag{1}$$

To compute the second probability, we can use Bayes rule to write

$$\mathbb{P}\Big[\{X_J = 1 \text{ AND } \exists i \in \{1,\ldots,K\} : X_i = 2\}\Big] = \mathbb{P}\left[X_J = 1\right] \times \mathbb{P}\left[\exists i \in \{1,\ldots,K\} : X_i = 2 \mid X_J = 1\right].$$

We have already seen that $\mathbb{P}[X_J = 1] = 1/J$. The second probability is readily obtained by the following argument: since the last position is fixed ($X_J = 1$), there are only $(J-1)!$ permutations left. Since these are equiprobable, the chance of getting a specific offer in the first K positions is exactly $K/(J-1)$. Thus,

$$\mathbb{P}\Big[\{X_J = 1 \text{ AND } \exists i \in \{1,\ldots,K\} : X_i = 2\}\Big] = \frac{1}{J} \times \frac{K}{J-1}.$$

So indeed the probability of picking the last offer is

$$\mathbb{P}\left[X^\star = X_J\right] = \frac{K}{J} + \frac{1}{J} \times \frac{K}{J-1}. \tag{2}$$

For those of you who did not like the heavy handwaving above, you can calculate that probability formally using combinatorics. The total number of permutations is $(J-1)!$. Out of those, we want to place offer 2 in the first K offers. Hence, we need to choose $K-1$ other offers out of the $J-2$ possibility left. Moreover, we should consider all possible permutations of the first K offers, since we do not care in which exact position offer 2 occurs (there are $K!$ such permutations). Finally, the other $J-K-1$ offers can also be permuted without affecting us. Explicitly, we obtain

$$\mathbb{P}\left[\exists i \in \{1,\ldots,K\} : X_i = 2 \mid X_J = 1\right] = \binom{J-2}{K-1} K!(J-K-1)! = \frac{K}{J-1}.$$

E Probability of accepting the best offer. As the hint suggests, we will use the total probability theorem (see eq. 3.4 in Ross's *Introduction to Probability Models*). By conditioning on the best offer being in each possible position, we obtain

$$\begin{aligned}\mathbb{P}\left[X^\star = 1\right] &= \sum_{i=1}^{J} \mathbb{P}\left[X^\star = 1 \mid X_i = 1\right] \mathbb{P}\left[X_i = 1\right] \\ &= \sum_{i=1}^{J} \mathbb{P}\left[X^\star = 1 \mid X_i = 1\right] \frac{1}{J} = \frac{1}{J} \sum_{i=1}^{J} \mathbb{P}\left[X^\star = 1 \mid X_i = 1\right],\end{aligned} \tag{3}$$

where we used the fact that the event that the best offer being in any specific position is equiprobable.

Now we have to examine two cases. First, if $i \leq K$, then we discard the best offer in our initial round of market probing and the probability of us accepting it is therefore zero. Immediately, (4) reduces to

$$\mathbb{P}\left[X^\star = 1\right] = \frac{1}{J} \sum_{i=K+1}^{J} \mathbb{P}\left[X^\star = 1 \mid X_i = 1\right] \tag{4}$$

Then, notice that the probabilities above are equivalent to saying that the best offer is chosen at time i. Indeed, if we see the best offer at any round $i > K$, we necessarily accept it (naturally, since $1 \leq X_0$). But for us to choose the offer at round i means that X_0 was not *beaten* up to round i. Formally, it must be that $X_j > X_0$ for all $j = K+1, \ldots, i$. Since X_0 is also the best offer of the first K rounds, we conclude that

$$\mathbb{P}\left[X^\star = 1 \mid X_i = 1\right] = \mathbb{P}\left[X_0 \text{ is the best offer up to round } i-1\right] = \frac{K}{i-1},$$

since all permutations are equiprobable. Rearranging the terms, we obtain the desired result

$$\mathbb{P}\left[X^\star = 1\right] = \frac{K}{J} \sum_{i=K+1}^{J} \frac{1}{i-1}.$$

F Optimal number of rejected offers K. The optimal number of offers to initially reject $K^\star$ is determined the approximation

$$\sum_{i=K+1}^{J} \frac{1}{i-1} \approx \int_{i=K}^{J-1} \frac{1}{x} dx \tag{5}$$

The exact bounds from which we obtain this approximation are actually:

$$\int_{i=K}^{J-1} \frac{1}{x} dx \leq \sum_{i=K+1}^{J} \frac{1}{i-1} \leq \int_{i=K}^{J-1} \frac{1}{x} dx + \frac{1}{K} - \frac{1}{J}.$$

This is just out of curiosity.

We can apply (5) directly to the result from part D to get

$$\begin{aligned}
\mathbb{P}\left[X^\star = 1\right] &= \frac{K}{J} \sum_{i=K+1}^{J} \frac{1}{i-1} \approx \frac{K}{J} \times \int_{i=K}^{J-1} \frac{1}{x}\, dx = \frac{K}{J} \times \ln\left(\frac{J-1}{K}\right) \\
&\approx \frac{K}{J} \times \ln\left(\frac{J}{K}\right). \qquad (6)
\end{aligned}$$

To find the K that maximizes $\mathbb{P}\left[X^\star = 1\right]$, we need to take the derivative of (6) with respect to K and set it equal to zero. Right? Actually, there's an important caveat here: K is an integer number and when taking derivatives, we are assuming $K \in \mathbb{R}$. This is not uncommon when solving such optimization problems (which are typically known as *integer programs*). This technique is often referred to as *relaxation* and we say that we have "relaxed the integer constraint on K." To conclude,

$$\begin{aligned}
\frac{d}{dK}\mathbb{P}\left[X^\star = 1\right] &\approx \frac{d}{dK}\left[\frac{K}{J} \times \ln\left(\frac{J}{K}\right)\right] \\
&= \frac{1}{J} \times \ln\left(\frac{J}{K}\right) + \frac{K}{J} \times \frac{K}{J} \times \left(-\frac{J}{K^2}\right) \\
&= \frac{1}{J} \times \ln\left(\frac{J}{K}\right) - \frac{1}{J}
\end{aligned}$$

Hence,

$$\frac{d}{dK}\mathbb{P}\left[X^\star = 1\right] = 0 \Leftrightarrow \ln\left(\frac{J}{K^\star}\right) = 1 \Leftrightarrow \frac{J}{K^\star} = e \Leftrightarrow K^\star = J/e. \qquad (7)$$

For $J = 50$, we have $K^\star \approx 18$, compatible with the result in Fig. 2. Using (7) in the result of part D we get

$$\mathbb{P}\left[X^\star = 1\right] \approx \frac{K^\star}{J} \times \ln\left(\frac{J}{K^\star}\right) = \frac{J/e}{J} \times \ln\left(\frac{J}{J/e}\right) = 1/e \approx 0.37,$$

which is quite high a probability considering we obtained it with almost no information on the offers. It also matches what we observed in Fig. 2.

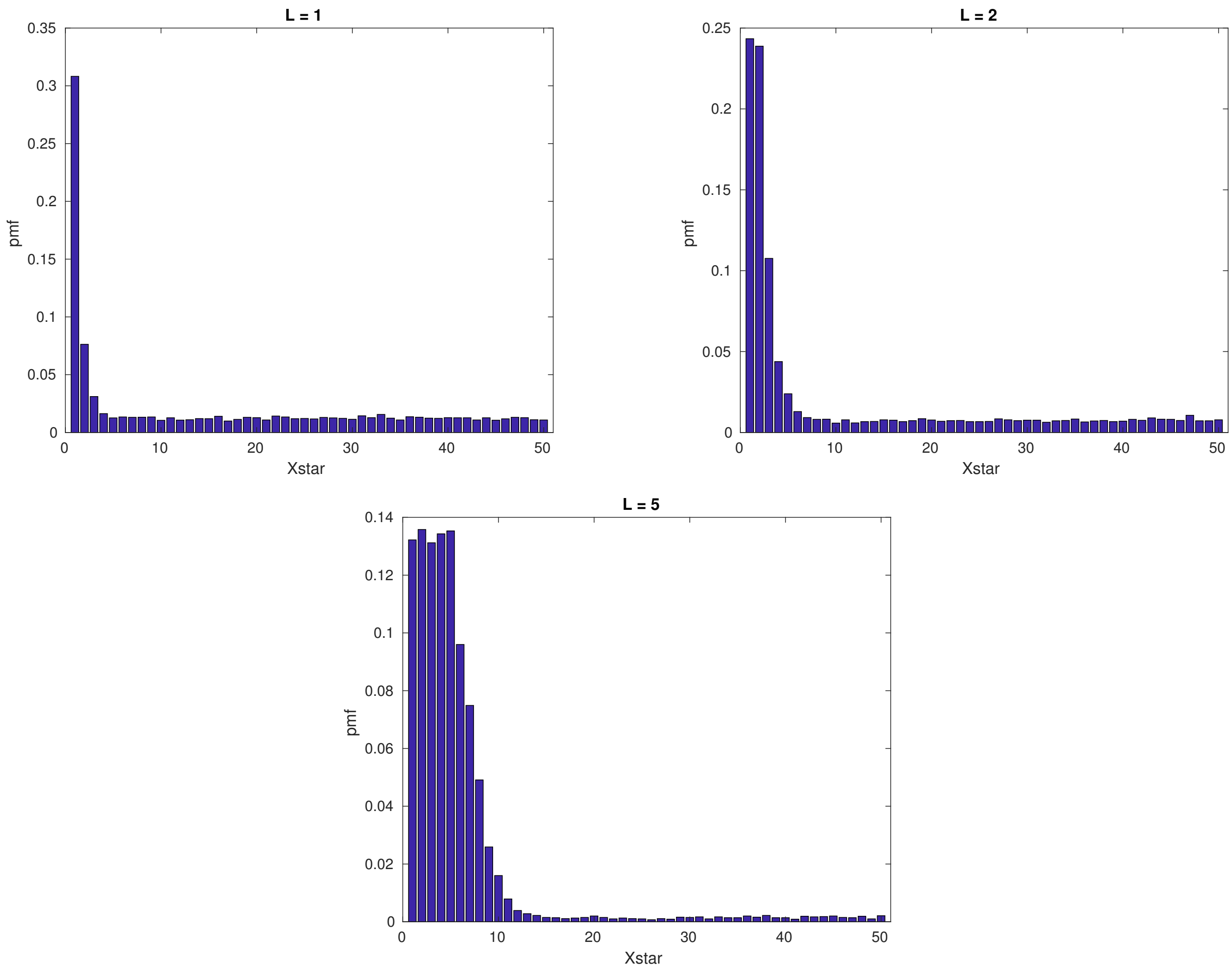

Figure 1: Estimated pmf of accepted ranks for $J = 50$, $K = 30$, and $L = 1, 2, 5$ with 10^4 realizations (part B).

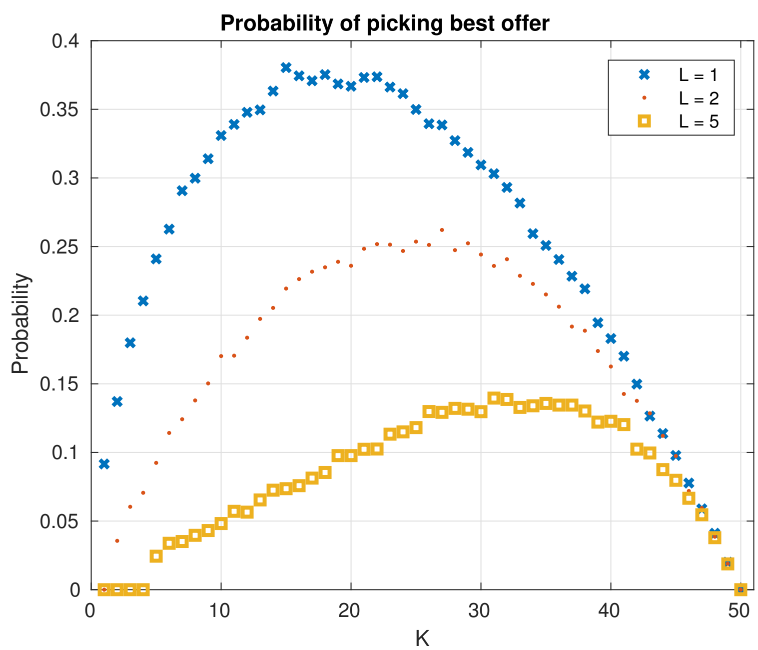

Figure 2: Probability of accepting the best offer for different K for $J = 50$ and $L = 1, 2, 5$. Probabilities estimated using 10^4 realizations (part C).