Week 7: Continuous-time Markov chains
Poisson process
Solutions

A Simulating $N(t)$. The MATLAB script to simulate the arrival process is given below.

```
function [ arrivals ] = passenger_arrivals( T, lambda, n )
%PASSENGER_ARRIVALS Simulate passenger arrivals using an approximate Poisson process

h = T/n;          % Subinterval length
p = lambda*h;     % Probability of arrival in each subinterval

% Generate arrivals by sampling from a Bernoulli(p)
arrivals = binornd(1, p, [n, 1]);

end
```

We then run the experiments to compare with the Poisson pmf as follows.

```
% Delete all variables and close figures
clear all
close all

T = 10;           % Observation interval (minutes)
lambda= 1;        % Rate of passengers (passenger per minute)
n = 1000;         % Number of subintervals
real = 10^4;      % Number of realizations

% Simulate arrivals
N_T = zeros(real,1);
N_T2 = zeros(real,1);
for i = 1:real
    arrivals = passenger_arrivals(T, lambda, n);
    N_T(i) = sum(arrivals);
    N_T2(i) = sum(arrivals(1:n/2));
end

% Approximate pmfs
x = 0:30;
pmf_N_T = histc(N_T, x)/real;
pmf_N_T2 = hist(N_T2,x)/real;


% Compare with Poisson pmfs
h1 = figure();
stem(x, pmf_N_T, 'o', 'Linewidth', 2);
hold on;
plot(x, poisspdf(x, lambda*T) ,'rx', 'Linewidth', 2);
xlabel('N(T)');
ylabel('Probability');
grid;
legend('Simulated', 'Poisson pmf', 'Location', 'Best');

```

```
h2 = figure();
stem(x, pmf_N_T2, 'o', 'Linewidth', 2);
hold on;
plot(x, poisspdf(x, lambda*T/2) ,'rx', 'Linewidth', 2);
xlabel('N(T)');
ylabel('Probability');
grid;
legend('Simulated', 'Poisson pmf', 'Location', 'Best');


%%% Export figure %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
set(h1,'Color','w');
export_fig(h1, '-q101', '-pdf', 'HW7_A1.pdf');

set(h2,'Color','w');
export_fig(h2, '-q101', '-pdf', 'HW7_A2.pdf');
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Results are shown in Figures 1 and 2.

B The distribution of $N(t)$. Notice that $N(T)$ can be written as

$$N(T) = \sum_{i=1}^{n} N_i(h)$$

and that each $N_i(h)$ is in fact a Bernoulli with probability of success λh (recall that we assumed that h is small enough). Since $h = T/n$, we obtain that $N(T)$ is a binomial random variable with parameters $(n, \lambda T/n)$. We know that for large n this binomial will converge to a Poisson, which explains why in part A the good fit of the Poisson pmf to the histograms.

To see how this implies that the distribution of $N(t)$ is Poisson for all t, define an approximate version of $N(t)$, namely

$$\tilde{N}(t) = \sum_{i=1}^{\lfloor t/h \rfloor} N_i(h) = \sum_{i=1}^{\lfloor nt/T \rfloor} N_i(T/n), \tag{1}$$

where we simply used the fact that $h = T/n$ to obtain the second equation. Two facts are important about the approximation in (1): (i) $\tilde{N}(t)$ is a binomial RV with parameters $(\lfloor nt/T \rfloor, \lambda T/n)$ and (ii) $\tilde{N}(t) \to N(t)$ as $n \to \infty$. Although (ii) is intuitive it can actually be made precise using a continuity argument by noticing that $N(t) = \tilde{N}(t)$ whenever $t = kT/n$, for $k = 1, \ldots, n$. For our purposes, however, these facts imply that

$$\mathbb{P}\left[N(t) = k\right] = \lim_{n \to \infty} \binom{\lfloor nt/T \rfloor}{k} \left(\frac{\lambda T}{n}\right)^k \left(1 - \frac{\lambda T}{n}\right)^{(\lfloor nt/T \rfloor - k)} \tag{2}$$

Suffices now for us to show that this limit leads to the expression of the Poisson pmf. To do so, start by rearranging (2) to read

$$
\begin{aligned}
\mathbb{P}\left[N(t)=k\right] &= \lim_{n\to\infty}\binom{\lfloor nt/T\rfloor}{k}\left(\frac{\lambda T}{n}\right)^k\left(1-\frac{\lambda T}{n}\right)^{(\lfloor nt/T\rfloor-k)} \\
&= \lim_{n\to\infty}\frac{\lfloor nt/T\rfloor!}{k!(\lfloor nt/T\rfloor-k)!}\frac{(\lambda T)^k}{n^k}\left(1-\frac{\lambda T}{n}\right)^{(\lfloor nt/T\rfloor-k)} \\
&= \frac{(\lambda T)^k}{k!}\lim_{n\to\infty}\frac{\lfloor nt/T\rfloor!}{n^k(\lfloor nt/T\rfloor-k)!}\left(1-\frac{\lambda T}{n}\right)^{(\lfloor nt/T\rfloor-k)} \\
&= \frac{(\lambda T)^k}{k!}\lim_{n\to\infty}\frac{\lfloor nt/T\rfloor!}{n^k(\lfloor nt/T\rfloor-k)!}\times\lim_{n\to\infty}\left(1-\frac{\lambda T}{n}\right)^{(\lfloor nt/T\rfloor-k)},
\end{aligned}
$$

where we used the fact that the limit of the product is the product of the limits *as long as* both limits exist. Now, we can use a common trick to deal with limits and the floor function: simply notice that for very large n, the difference between nt/T and $\lfloor nt/T\rfloor$ is negligible. But most importantly, it is always true that $\lfloor nt/T\rfloor = nt/T-\delta$, where $0\le\delta<1$. So we can write

$$
\mathbb{P}\left[N(t)=k\right] = \frac{(\lambda T)^k}{k!}\lim_{n\to\infty}\frac{(nt/T-\delta)!}{n^k(nt/T-\delta-k)!}\times\lim_{n\to\infty}\left(1-\frac{\lambda T}{n}\right)^{(nt/T-\delta-k)}.
$$

To evaluate the first limit, we start by doing the change of variable $u=nt/T$ to obtain

$$
\begin{aligned}
\mathbb{P}\left[N(t)=k\right] &= \frac{(\lambda T)^k}{k!}\lim_{n\to\infty}\frac{(u-\delta)!}{(uT/t)^k(u-\delta-k)!}\times\lim_{n\to\infty}\left(1-\frac{\lambda T}{n}\right)^{(nt/T-\delta-k)} \\
&= \frac{(\lambda t)^k}{k!}\lim_{n\to\infty}\frac{(u-\delta)!}{u^k(u-\delta-k)!}\times\lim_{n\to\infty}\left(1-\frac{\lambda T}{n}\right)^{(nt/T-\delta-k)}.
\end{aligned}
$$

Now, it is ready that the first limit goes to one using the fact that δ is bounded and that factorials grow much faster than polynomials (in fact, it grows much faster than exponentials! $\lim_{u\to\infty}x^u/u!=0$). For the second limit, we can rearrange it to obtain an exponential as in

$$
\begin{aligned}
\mathbb{P}\left[N(t)=k\right] &= \frac{(\lambda t)^k}{k!}\lim_{n\to\infty}\left(1-\frac{\lambda T}{n}\right)^{(nt/T-\delta-k)} \\
&= \frac{(\lambda t)^k}{k!}\lim_{n\to\infty}\left[\left(1-\frac{\lambda T}{n}\right)^{\frac{n}{\lambda T}}\right]^{\lambda T\times\frac{nt/T-\delta-k}{n}} \\
&= \frac{(\lambda t)^k}{k!}\left[\lim_{n\to\infty}\left(1-\frac{1}{\frac{n}{\lambda T}}\right)^{\frac{n}{\lambda T}}\right]^{\lambda T\times\frac{t}{T}} \\
&= \frac{(\lambda t)^k}{k!}e^{-\lambda t},
\end{aligned}
$$

which is indeed the expression for the Poisson pmf.

C Simulating T_1. The MATLAB script to compute statistics of T_1 is as follows.

```
% Delete all variables and close figures
clear all
close all
```

```
T = 10;          % Observation interval (minutes)
lambda= 1;       % Rate of passengers (passenger per minute)
n = 1000;        % Number of subintervals
real = 10^4;     % Number of realizations

% Simulate arrivals
T_1 = zeros(real,1);
for i = 1:real
    arrivals = passenger_arrivals(T, lambda, n);
    T_1(i) = find(arrivals == 1, 1, 'first');
end

% Approximate pmfs
h = T/n;
pdf_T_1 = hist(T_1, 1:n)/real/h;


% Compare with exponential RV
h1 = figure();
plot((1:n)*h, pdf_T_1, 'Linewidth', 2);
hold on;
plot((1:n)*h, exppdf((1:n)*h, lambda) ,'--', 'Linewidth', 2);
xlabel('N(T)');
ylabel('Probability');
grid;
legend('Simulated', 'Poisson pmf', 'Location', 'Best');

h2 = figure();
plot((1:n)*h, cumsum(pdf_T_1*h), 'Linewidth', 2);
hold on;
plot((1:n)*h, expcdf((1:n)*h, lambda) ,'--', 'Linewidth', 2);
xlabel('N(T)');
ylabel('Probability');
grid;
legend('Simulated', 'Poisson pmf', 'Location', 'Best');



%%% Export figure %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
set(h1,'Color','w');
export_fig(h1, '-q101', '-pdf', 'HW7_C1.pdf');

set(h2,'Color','w');
export_fig(h2, '-q101', '-pdf', 'HW7_C2.pdf');
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Figures 3 and 4 show a close fit with the exponential distribution.

D The distribution of T_1. From part B, we have that $\mathbb{P}[N(t) = 0] = e^{-\lambda t}$. Notice, however, that having no arrival by time t is the same event as the first arrival occurring after time t. Hence,

$$\mathbb{P}[N(t) = 0] = \mathbb{P}[T_1 > t] \Leftrightarrow \mathbb{P}[T_1 \leq t] = 1 - \mathbb{P}[N(t) = 0] = 1 - e^{-\lambda t}.$$

Observe that this is the cdf of an exponential random variable with parameter λ. Hence, T_1 is indeed exponentially distributed with parameter λ.

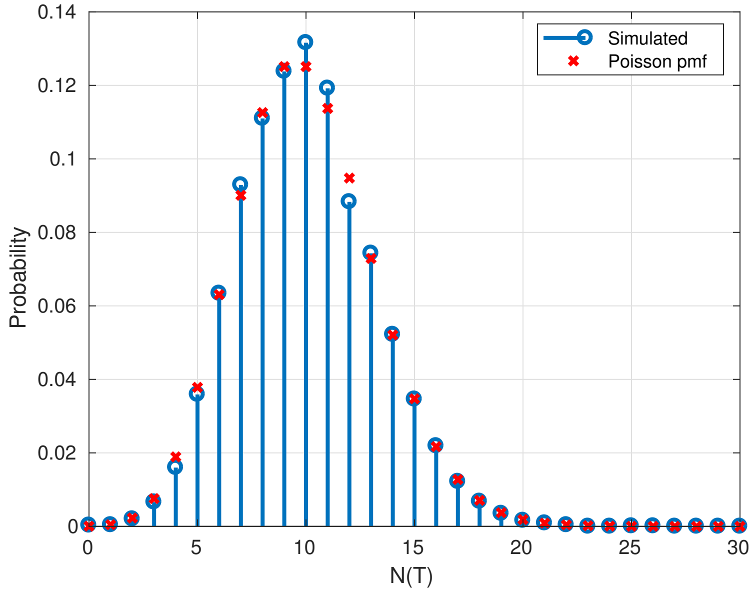

Figure 1: Comparison between the estimated pmf of $N(T)$ obtained from 10^4 experiments and the Poisson pmf with parameter λT (part A).

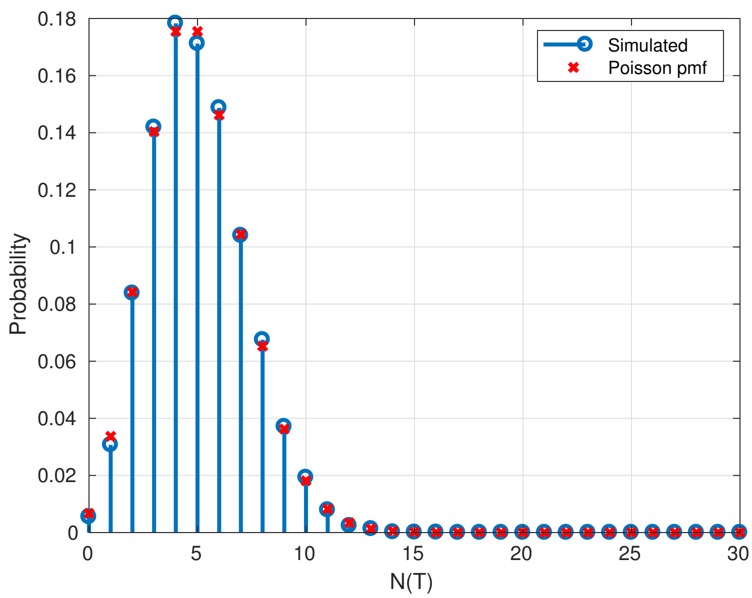

Figure 2: Comparison between the estimated pmf of $N(T/2)$ obtained from 10^4 experiments and the Poisson pmf with parameter $\lambda T/2$ (part A).

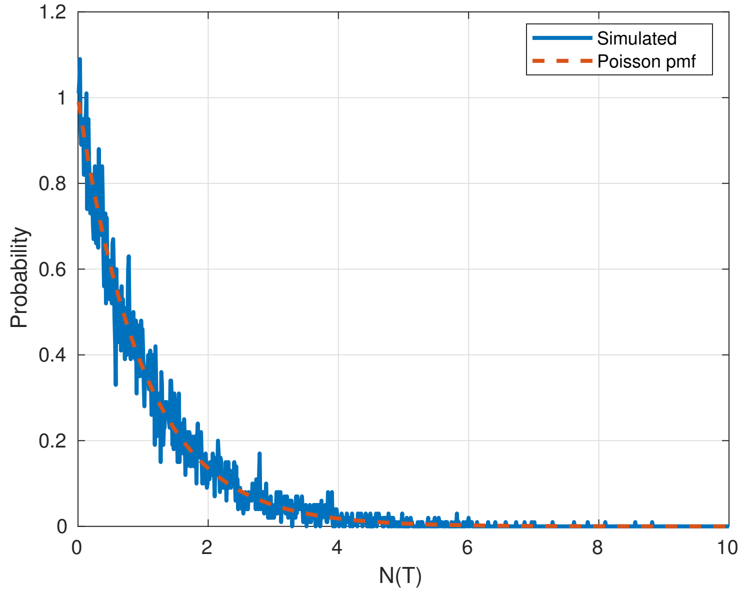

Figure 3: Comparison between the estimated pdf of T_1 obtained from 10^4 experiments and the exponential pdf with parameter λ (part C).

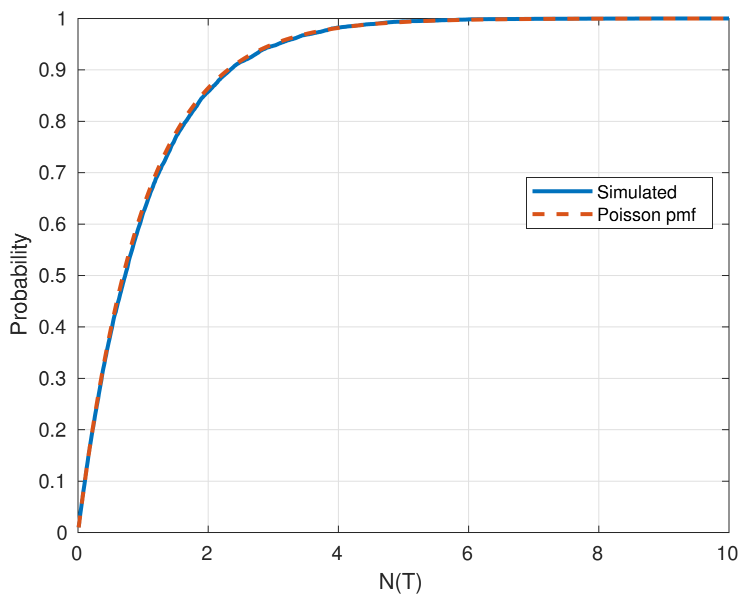

Figure 4: Comparison between the estimated cdf of T_1 obtained from 10^4 experiments and the exponential cdf with parameter λ (part C).