Continuous time Markov chains (week 8)
Solutions

1 Insurance cash flow.

A CTMC states.
Because c and d are assumed to be integers, and the premiums are each 1, the cash flow $X(t)$ of the insurance company can be any integer between 0 and X_{max}.

B Transition times out of given state When $X(t) = x$ and x is in range (D), the transition probability out of state x, T_x, is exponential because it is the probability of a dividend, claim, or premium being paid, which are all exponentially distributed:

$$T_p \sim \text{exp}(\lambda) = \text{exp}(N)$$

$$T_c \sim \text{exp}(\alpha) = \text{exp}(rN)$$

$$T_d \sim \text{exp}(\beta)$$

The transition will occur whenever one of these happens (whichever is first), therefore:

$$T_x = min(T_p, T_c, T_d)$$

The transition has not occurred when $(T_x > t)$, so it follows that none of the transitions have occurred. Because each event is independent:

$$P\{T_X > t\} = P\{T_p > t\}P\{T_c > t\}P\{T_d > t\}$$
$$P\{T_X > t\} = e^{-\lambda t}e^{-\alpha t}e^{-\beta t}$$
$$P\{T_X > t\} = e^{-(\lambda+\alpha+\beta)t}$$

The cdf of T_x is $e^{-(\lambda+\alpha+\beta)t}$, therefore it is exponentially distributed with parameter $\nu_x = (\lambda+\alpha+\beta)$. By the same logic, the parameters for T_x in the other ranges are:

Range	Possible events	Parameter ν_x
A	premium	λ
B	premium, claim paid at $X(t)$	$\lambda + \alpha$
C	premium, claim	$\lambda + \alpha$
E	claim, dividend	$\alpha + \beta$

C Possible states going out of $X(t) = x$ Given that $X(t) = x$, where x is in range (D), the possible states after a transition out of x are:

$x \to x + 1$ (a premium is paid)

$x \to x - c$ (a claim is paid)

$x \to x - d$ (a dividend is paid)

The possible states for each other range are:

Range	Possible events	Possible states out of $X(t) = x$
A	premium	1
B	premium, claim paid at $X(t)$	$x + 1$, 0
C	premium, claim	$x + 1$, $x - c$
E	claim, dividend	$x - c$, $x - d$

D Transition probabilities

(Reference: slide 11 of continuous_time_markov_chains.) When x is in range (D), the transition probabilities from state x to j, P_{xj}, for each possible state out of x, j, are:

$$P_{x,x+1} = \frac{\lambda}{\lambda + \alpha + \beta}$$
$$P_{x,x-c} = \frac{\alpha}{\lambda + \alpha + \beta}$$
$$P_{x,x-d} = \frac{\beta}{\lambda + \alpha + \beta}$$

This is because we know that a transition happens at time t, and the three possible events that could have caused it: a premium payment, a claim, or a dividend payment. The probability that one of these events happens (for example a premium payment), given a transition happens is:

$$P\{\text{premium payment}|\text{ a transition happens}\} = \frac{P\{\text{premium payment}\} \cap P\{\text{transition happens}\}}{P\{\text{transition happens}\}} = \frac{\lambda}{\lambda + \alpha + \beta}$$

The same reasoning applies to each other possible event.
The transition probabilities into each possible state when x is in each other range are:

Range	State j at time $t + 1$	Transition probabilities, $P_{x,j}$
A	1	$\frac{\lambda}{\lambda} = 1$
B	$x + 1$	$\frac{\lambda}{\lambda+\alpha}$
B	0	$\frac{\alpha}{\lambda+\alpha}$
C	$x + 1$	$\frac{\lambda}{\lambda+\alpha}$
C	$x - c$	$\frac{\alpha}{\lambda+\alpha}$
E	$x - c$	$\frac{\alpha}{\alpha+\beta}$
E	$x - d$	$\frac{\beta}{\alpha+\beta}$

E System simulation. Two methods are provided below. Please notice the nuances, as each of them is a different interpretation of the CTMC:

```
% following the method explained in the HW-8
function [X,T]=cashflow1(X_0,lambda,alpha,beta,c,d,X_r,X_max,T_max)
index=1;
X(index)=X_0;
T(index)=0;
while T(index)<T_max
    x=X(index);
```

```
    if x==0 %only premium is possible
        tau=exprnd(1/lambda);
        T(index+1)=T(index)+tau;
        X(index+1)=x+1;
    elseif 0<x && x<c %premium, claim payed at X(t) not c
        tau=exprnd(1/(lambda+alpha));
        T(index+1)=T(index)+tau;
        u=rand;
        if u<(lambda/(lambda+alpha))   % premium
            X(index+1)=x+1;
        else            % claim
            X(index+1)=0;
        end
    elseif c<=x && x<X_r %premium, claim
        tau=exprnd(1/(lambda+alpha));
        T(index+1)=T(index)+tau;
        u=rand;
        if u<(lambda/(lambda+alpha))   %
            X(index+1)=x+1;
        else
            X(index+1)=x-c;
        end
    elseif X_r<=x && x<X_max %premium, claim, dividend
        tau=exprnd(1/(lambda+alpha+beta));
        T(index+1)=T(index)+tau;
        u=rand;
        if u<(lambda/(lambda+alpha+beta))  % premium
            X(index+1)=X(index)+1;
        elseif u<((lambda+alpha)/(lambda+alpha+beta))  % claim
            X(index+1)=X(index)-c;
        else    % dividend
            X(index+1)=X(index)-d;
        end
    elseif x==X_max % claim, dividend
        tau=exprnd(1/(alpha+beta));
        T(index+1)=T(index)+tau;
        u=rand;
        if u<(alpha/(lambda+alpha))   % claim
            X(index+1)=x-c;
        else          % dividend
            X(index+1)=x-d;
        end
    else
        disp('Out Of Range')
        break
    end
    index=index+1;
end
```

```
end
```

Method 2:

```
% An alternative method, alarm clock interpretation
function [X,T]=cashflow2(X_0,lambda,alpha,beta,c,d,X_r,X_max,T_max)
index=1;
X(index)=X_0;
T(index)=0;
while T(index)<T_max
    x=X(index);
    if x==0 %only premium is possible
        t_premium=exprnd(1/lambda);
        T(index+1)=T(index)+t_premium;
        X(index+1)=x+1;
    elseif 0<x && x<c %premium, claim payed at X(t) not c
        t_premium=exprnd(1/lambda);
        t_claim=exprnd(1/alpha);
        T(index+1)=T(index)+min(t_premium,t_claim);
        X(index+1)=x+1*(t_premium<t_claim)-x*(t_premium>t_claim);
    elseif c<=x && x<X_r %premium, claim
        t_premium=exprnd(1/lambda);
        t_claim=exprnd(1/alpha);
        T(index+1)=T(index)+min(t_premium,t_claim);
        X(index+1)=x+1*(t_premium<t_claim)-c*(t_premium>t_claim);
    elseif X_r<=x && x<X_max %premium, claim, dividend
        t_premium=exprnd(1/lambda);
        t_claim=exprnd(1/alpha);
        t_dividend=exprnd(1/beta);
        [t_min,I]=min([t_premium,t_claim,t_dividend]);
        T(index+1)=T(index)+t_min;
        X(index+1)=x+(1:3==I)*[1;-c;-d];
    elseif x==X_max % claim, dividend
        t_claim=exprnd(1/alpha);
        t_dividend=exprnd(1/beta);
        [t_min,I]=min([t_claim,t_dividend]);
        T(index+1)=T(index)+t_min;
        X(index+1)=x+(1:2==I)*[-c;-d];
    else
        disp('Out Of Range')
        break
    end
    index=index+1;
end

end
```

Here, we run the code (call one of the functions) for the following values of the parameters: $X_0 = 200$, number of clients $N = 200$, risk $r = 4\%$, dividends payed quarterly; claim and dividend costs

$c = 20$ and $d = 30$; capital thresholds, $X_r = 200,$ and $X_{\max} = 300;$ and maximum amount of time $T_{\max} = 5$ years. The result is depicted in fig. (1(a)) and a zoomed one in fig. (1(b)).

```
clc; clear all; close all;
X_0=200;
N=200;
r=0.04;

lambda=N;
alpha=r*N;
beta=4;

X_r=200;
X_max=300;

T_max=5;
d=30;
c=20;

[X,t]=cashflow2(X_0,lambda,alpha,beta,c,d,X_r,X_max,T_max);

% Plotting the results
hold on
grid on
xlabel('time','Fontsize',14)
ylabel('Cash Level','Fontsize',14)
title('(a sample) Evolution of Cash Level over 5 Years','Fontsize',14)
axis([0 5 0 310])
stairs(t,X,'Linewidth',2,'Color','r');
```

F Kolmogorovs forward equation.
(Reference: slide 54 of continuous_time_markov_chains.) The transition rate q_{xy} is found by the equation $q_{xy} = \nu_x P_{xy}$, where ν_x is the rate of transition out of state x, and P_{xy} is the probability of transitioning from state x into state y. Therefore, the transition rates for each scenario are:

Range (A), state x at time t = 0:

state y at $t+1$	$\nu_x * P_{xy} =$	q_{xy}
1	$\lambda * 1 =$	λ

Range (B), $0 < x < c$:

state y at $t+1$	$\nu_x * P_{xy} =$	q_{xy}
$x+1$	$(\lambda+\alpha) * \frac{\lambda}{\lambda+\alpha} =$	λ
0	$(\lambda+\alpha) * \frac{\alpha}{\lambda+\alpha} =$	α

Range (C), $c \leq x < X_r$:

state y at $t+1$	$\nu_x * P_{xy} =$	q_{xy}
$x+1$	$(\lambda+\alpha) * \frac{\lambda}{\lambda+\alpha} =$	λ
$x-c$	$(\lambda+\alpha) * \frac{\alpha}{\lambda+\alpha} =$	α

Range (D), $X_r \le x < X_{max}$:

state y at $t+1$	$\nu_x * P_{xy} =$	q_{xy}
$x+1$	$(\lambda+\alpha+\beta) * \frac{\lambda}{\lambda+\alpha+\beta} =$	λ
$x-c$	$(\lambda+\alpha+\beta) * \frac{\alpha}{\lambda+\alpha+\beta} =$	α
$x-d$	$(\lambda+\alpha+\beta) * \frac{\beta}{\lambda+\alpha+\beta} =$	β

Range (E), $x = X_{max}$:

state y at $t+1$	$\nu_x * P_{xy} =$	q_{xy}
$x-c$	$(\alpha+\beta) * \frac{\alpha}{\alpha+\beta} =$	α
$x-d$	$(\alpha+\beta) * \frac{\beta}{\alpha+\beta} =$	β

Using the above transition rates, we can write Kolmogorov's forward equations for each range. The equation takes the form:

$$\frac{\partial P_{xy}(t)}{\partial t} = P'_{xy}(t) = \sum_{k=0, k\neq y}^{\infty} q_{ky} P_{xk}(t) - \nu_y P_{xy}(t)$$

For Range (A), $Y = 0$:

$$P'_{xY} = \alpha \sum_{k=1}^{c} P_{xk} - \lambda P_{xY}$$

For Range (B), $0 < Y < c$:

$$P'_{xY} = \lambda P_{x,Y-1} + \alpha P_{x,Y+c} - (\lambda+\alpha) P_{xY}$$

For Range (C), $c \le Y < X_r$:

- When $Y < X_r - d$ (a dividend was not paid to move from state x to state y because the initial x was not greater than X_r):

$$P'_{xY} = \lambda P_{x,Y-1} + \alpha P_{x,Y+c} - (\lambda+\alpha) P_{xY}$$

- When $Y \ge X_r - d$ (Y is close enough to the threshold X_r that the transition into Y could have occured due to a dividend payment):

$$P'_{xY} = \lambda P_{x,Y-1} + \alpha P_{x,Y+c} + \beta P_{x,Y+d} - (\lambda+\alpha) P_{xY}$$

For Range (D), $X_r \le Y \le X_{max}$

- When $Y \le min(c, d)$ (the transition could have been a result of a claim payment, premium payment, or dividend payment):

$$P'_{xY} = \lambda P_{x,Y-1} + \alpha P_{x,Y+c} + \beta P_{x,Y+d} - (\lambda+\alpha+\beta) P_{xY}$$

- When $Y > min(c, d)$ (the transition could only have been a result of a premium payment):

$$P'_{xY} = \lambda P_{x,Y-1} - (\lambda+\alpha+\beta) P_{xY}$$

- When $c < d$ and $d < Y \leq c$ (the transition could have been a result of a claim payment or a premium payment, but not a dividend payment):

$$P'_{xY} = \lambda P_{x,Y-1} + \alpha P_{x,Y+c} - (\lambda + \alpha + \beta) P_{xY}$$

- When $d < c$ and $c < Y \leq d$(the transition could have been a result of a dividend payment or a premium payment, but not a claim payment):

$$P'_{xY} = \lambda P_{x,Y-1} + \beta P_{x,Y+d} - (\lambda + \alpha + \beta) P_{xY}$$

For Range (E), $Y = X_{max}$

$$P'_{xY} = \lambda P_{x,Y-1} - (\alpha + \beta) P_{xY}$$

G Kolmogorovs backward equation.
(Reference: slide 56 of continuous_time_markov_chains.) Kolmogorov's backward equation is:

$$\frac{\partial P_{xy}(t)}{\partial t} = P'_{xy}(t) = \sum_{k=0,k\neq x}^{\infty} q_{xk} P_{ky}(t) - \nu_x P_{xy}(t)$$

For Range (A), $X = 0$ (the only posible state y is $y = 1$):

$$P'_{Xy} = \lambda P_{X+1,y} - P_{X,y}$$

For Range (B), $0 < X < c$:

$$P'_{Xy} = \lambda P_{X+1,y} + \alpha P_{0,y} - (\lambda + \alpha) P_{X,y}$$

For Range (C), $c \leq X < X_r$:

$$P'_{Xy} = \lambda P_{X+1,y} + \alpha P_{X-c,y} - (\lambda + \alpha) P_{X,y}$$

For Range (D), $X_r \leq X < X_{max}$:

$$P'_{Xy} = \lambda P_{X+1,y} + \alpha P_{X-c,y} + \beta P_{X-d,y} - (\lambda + \alpha + \beta) P_{X,y}$$

For Range (E), $X = X_{max}$:

$$P'_{Xy} = \alpha P_{X-c,y} + \beta P_{X-d,y} - (\alpha + \beta) P_{X,y}$$

H Solution of Kolmogorovs equations A function is written to construct the matrix $\mathbf{R}$ such that the forward equation is represented by $\dot{\mathbf{P}} = R\mathbf{P}$. Then using matlab's exponential of a matrix, find the solution. The code and the result follows (fig. (2)).

```
function [R]=Kolmogrov_F(lambda,alpha,beta,c,d,X_r,X_max)

R=zeros(X_max+1); % initialization

% Range A:
R(1,1)=-lambda;
```

```
R(1,2:c+1)=alpha;

% Range B:
for i=2:c
    R(i,i-1)=lambda;
    R(i,i+c)=alpha;
    R(i,i)=-(lambda+alpha);
end

% Range C-1:
for i=c+1:X_r-d
    R(i,i-1)=lambda;
    R(i,i+c)=alpha;
    R(i,i)=-(lambda+alpha);
end

% Range C_2:
for i=X_r-d+1:X_r
    R(i,i-1)=lambda;
    R(i,i+c)=alpha;
    R(i,i+d)=beta;
    R(i,i)=-(lambda+alpha);
end

% Range D_1:
for i=X_r+1:X_max-d+1
    R(i,i-1)=lambda;
    R(i,i+c)=alpha;
    R(i,i+d)=beta;
    R(i,i)=-(lambda+alpha+beta);
end

% Range D_2:
for i=X_max-d+2:X_max-c+1
    R(i,i-1)=lambda;
    R(i,i+c)=alpha;
    R(i,i)=-(lambda+alpha+beta);
end

% Range D_3:
for i=X_max-c+2:X_max
    R(i,i-1)=lambda;
    R(i,i)=-(lambda+alpha+beta);
end

% Range E:
R(X_max+1, X_max)=lambda;
R(X_max+1, X_max+1)=-(alpha+beta);
```

The execution code:

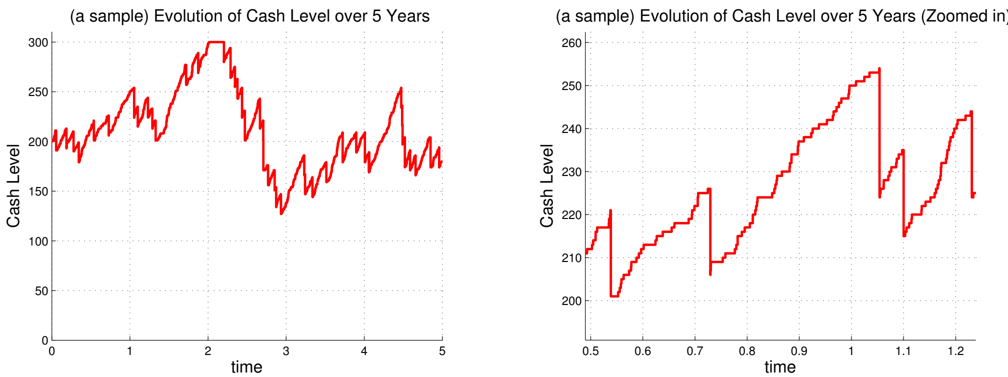
```
R=Kolmogrov_F(lambda,alpha,beta,c,d,X_r,X_max);
p0=zeros(X_max+1,1);
p0(X_0+1,1)=1;
T=0:0.25:5;
figure
hold on
xlabel('X','Fontsize',14)
ylabel('pmf','Fontsize',14)
title('pmf of the states between 0 and 5 over quarterly intervals','Fontsize',14)
axis([0 300 0 0.016])
for t=T
    pmf=expm(R.*t)*p0;
    plot(0:X_max,pmf,'r')
end
```

I Probability of paying dividends.

A dividend can be paid only when $X_r \leq x \leq X_{max}$ (when x is in range (D) or (E)). While in this interval the expected number of occurrences per year is given by the parameter ν_x because number of occurences is Poisson with ν_x. This is approximately $(\lambda+\alpha+\beta) = 212$ (note: approximation ignores that the parameter changes for $x = X_{max}$). We can divide this number by four to find the expected number of events for each quarter, which is 53. For a given quarter, the probability that at least one dividend is paid is:

$$\begin{aligned} P(\text{at least one dividend paid}) &= 1 - P(\text{no dividends paid}) \\ &= 1 - P(\text{all events are premium or claim payments}) \\ &= 1 - P(\text{single event is premium or claim payment})^{53} \\ &= 1 - (\frac{\lambda+\alpha}{\lambda+\alpha+\beta})^{53} \\ &= 1 - (\frac{208}{212})^{53} \\ &\approx 0.64 \end{aligned}$$

Thus, the probability of a dividend is approximately 0.6 times the probability that $X(t) \geq X_r$, which is achieved by summing up the pmf's from X_r to $X_{\max}$. The result is depicted in fig. 3.



(a) pdf (b) cdf

Fig. 1. Evolution of the Cash Flow in part E for 5 years(subfig. (a)). Subfig. (b) shows a zoomed portion of the path to clarify the details: small steps are due to premiums.

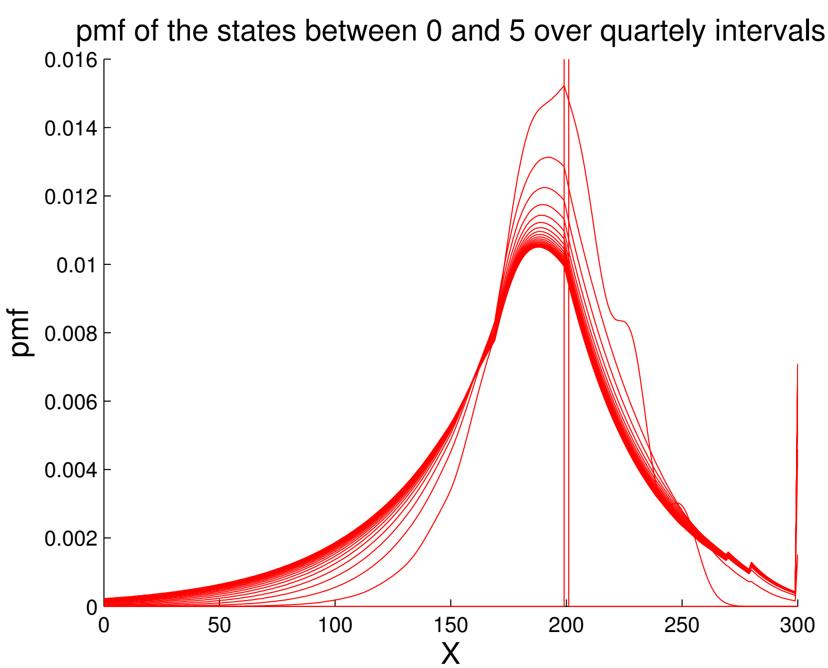


Fig. 2. Part H: pmf of the states between 0 and 5 over quarterly intervals using solution of Kolmogorov's Forward equation.

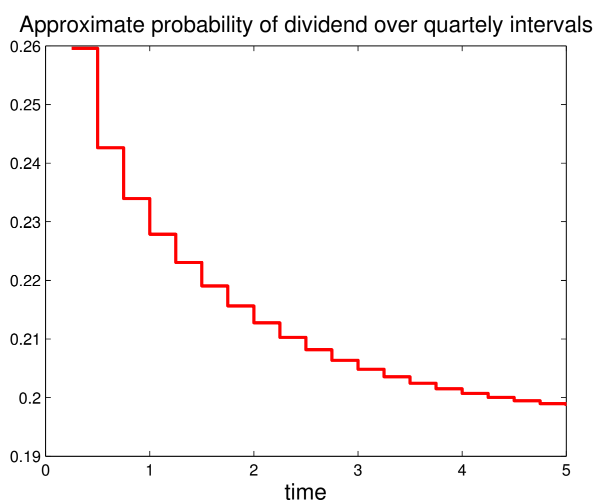


Fig. 3. Part I: Approximate probability of giving a dividend in each of the quarters.