Bode Plot Review
High Frequency BJT Model

Logarithmic Frequency Response Plots (Bode Plots)

Generic form of frequency response *rational polynomial*, where we substitute jω for *s*:

$$H(s)=K\frac{s^m+a_{m-1}s^{m-1}+\cdots+a_1 s+a_0}{s^n+b_{n-1}s^{n-1}+\cdots+b_1 s+b_0}$$

$H(j\omega)$can represent an impedance, an admittance, or a gain transfer function. If we are lucky, we can factor each of the polynomials as a product of first degree terms:

$$H(s)=K\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

Logarithmic Frequency Response Plots (Bode Plots)

Determination of a frequency response requires evaluating the complex number expression:

$$H(j\omega)=K\frac{(j\omega+z_1)(j\omega+z_2)\cdots(j\omega+z_m)}{(j\omega+p_1)(j\omega+p_2)\cdots(j\omega+p_n)}$$

Bode's approach was to simplify the calculations, using polar representation of the factors:

$$H(j\omega)=K\frac{z_1*z_2*..*z_m}{p_1*p_2*..*p_n}\frac{|j\frac{\omega}{z_1}+1|e^{j\phi_1}|j\frac{\omega}{z_2}+1|e^{j\phi_2}\cdots|j\frac{\omega}{z_m}+1|e^{j\phi_m}}{|j\frac{\omega}{p_1}+1|e^{j\theta_1}|j\frac{\omega}{p_2}+1|e^{j\theta_2}\cdots|j\frac{\omega}{p_n}+1|e^{j\theta_n}}$$

where $|j\frac{\omega}{z_k}+1|=\sqrt{[(j\frac{\omega}{z_k}+1)(-j\frac{\omega}{z_k}+1)]}=\sqrt{(\frac{\omega}{z_k})^2 1+1}$ $\qquad \phi_k=\tan^{-1}\frac{\omega}{z_k}$

Bode Plot cont.

$$|j\frac{\omega}{z_k}+1|=\sqrt{[(j\frac{\omega}{z_k}+1)(-j\frac{\omega}{z_k}+1)]}=\sqrt{(\frac{\omega}{z_k})^2+1}$$

$$\omega \ll z_k \Rightarrow |j\frac{\omega}{z_k}+1| \approx 1$$

$$\omega = z_k \Rightarrow |j\frac{\omega}{z_k}+1| = \sqrt{2}$$

$$\omega \gg z_k \Rightarrow |j\frac{\omega}{z_k}+1| \approx \frac{\omega}{z_k}$$

$$\phi_k=\tan^{-1}\frac{\omega}{z_k}$$

$$\omega \ll z_k \Rightarrow \tan^{-1}\frac{\omega}{z_k} \rightarrow 0^o$$

$$\omega = z_k \Rightarrow \tan^{-1}\frac{\omega}{z_k} = 45^o$$

$$\omega \gg z_k \Rightarrow \tan^{-1}\frac{\omega}{z_k} \rightarrow 90^o$$

Bode Plots cont.

$$G(\omega)e^{j\rho(\omega)}=K\frac{|j\omega+z_1|e^{j\phi_1}|j\omega+z_2|e^{j\phi_2}\cdots|j\omega+z_m|e^{j\phi_m}}{|j\omega+p_1|e^{j\theta_1}|j\omega+p_2|e^{j\theta_2}\cdots|j\omega+p_n|e^{j\theta_n}}$$

We can separate magnitude and phase angle calculations:

$$G(\omega)=|H(j\omega)|=K\frac{\prod_1^m z_i}{\prod_1^n p_i}\frac{|j\frac{\omega}{z_1}+1||j\frac{\omega}{z_2}+1|\cdots|j\frac{\omega}{z_m}+1|}{|j\frac{\omega}{p_1}+1||j\frac{\omega}{p_2}+1|\cdots|j\frac{\omega}{p_n}+1|}$$

$$\rho(\omega)=(\phi_1+\phi_2+\cdots\phi_m)-(\theta_1+\theta_2+\cdots+\theta_n)$$

where $\phi_k=\tan^{-1}\frac{\omega}{z_k}$ $\theta_k=\tan^{-1}\frac{\omega}{p_k}$

Bode Plots

$$G(\omega)=|H(j\omega)|=K_{dc}\frac{|j\frac{\omega}{z_1}+1||j\frac{\omega}{z_2}+1|\cdots|j\frac{\omega}{z_m}+1|}{|j\frac{\omega}{p_1}+1||j\frac{\omega}{p_2}+1|\cdots|j\frac{\omega}{p_n}+1|}$$

Where we define the *dc gain*, the value of the magnitude of H at $\omega=0$, as:

$$K_{dc}=K\frac{\prod_1^m z_i}{\prod_1^n p_i}$$

Logarithmic Frequency Response Plots (Bode Plots)

$$G(\omega) = |H(j\omega)| = K_{dc} \frac{|j\frac{\omega}{z_1}+1||j\frac{\omega}{z_2}+1|\cdots|j\frac{\omega}{z_m}+1|}{|j\frac{\omega}{p_1}+1||j\frac{\omega}{p_2}+1|\cdots|j\frac{\omega}{p_n}+1|}$$

Another “simplification” converts the magnitude computation from multiplication to addition by working with its logarithm (in decibels):

$$G(\omega)_{dB} = \sum_{1}^{m} 20\log_{10}|j\frac{\omega}{z_i}+1| - \sum_{1}^{n} 20\log_{10}|j\frac{\omega}{p_i}+1|$$

Logarithmic Frequency Response Plots (Bode Plots)

$$G(\omega)_{dB} = \sum_{1}^{m} 20\log_{10}\left| j\frac{\omega}{z_i} + 1 \right| - \sum_{1}^{n} 20\log_{10}\left| j\frac{\omega}{p_i} + 1 \right|$$

Let's calculate the frequency response for a simple transfer function and make some observations:

$$H(j\omega) = 10\,\frac{j\dfrac{\omega}{1} + 1}{j\dfrac{\omega}{10} + 1}$$

Simple Example

$$H(j\omega) = 10\frac{j\frac{\omega}{1}+1}{j\frac{\omega}{10}+1} = 10\frac{\sqrt{(\frac{\omega}{1})^2+1}}{\sqrt{(\frac{\omega}{10})^2+1}}$$

Working with logs:

$$|H_{dB}| = 20\log_{10}|10| + 20\log_{10}|j\frac{\omega}{1}+1| - 20\log_{10}|j\frac{\omega}{10}+1|$$

Note:

1. if the coefficient of j, *i.e.* $\frac{\omega}{\omega_a} \leq 0.1$, then $|j\frac{\omega}{\omega_a}+1| \approx 1$ and $\log_{10}|j\frac{\omega}{\omega_a}+1| = 0$.
2. If the coefficient of j, i.e. $\frac{\omega}{\omega_a} \geq 10$, then $|j\frac{\omega}{\omega_a}+1| \approx \frac{\omega}{\omega_a}$ and $\log_{10}|j\frac{\omega}{\omega_a}+1| = \log_{10}\frac{\omega}{\omega_a}$.
3. When $\omega = \omega_a$, $|j\frac{\omega}{\omega_a}+1| = \sqrt{2}$ and $\log_{10}|j\frac{\omega}{\omega_a}+1| = 0.15$.

Simple Example

$$|H_{dB}| = 20\log_{10}|K_{dc}| + 20\log_{10}|j\frac{\omega}{1}+1| - 20\log|j\frac{\omega}{10}+1|$$

Applying the approximation on the previous slide:

$$\omega < 1 \Rightarrow |H_{dB}| = 20\log_{10} K_{dc}$$

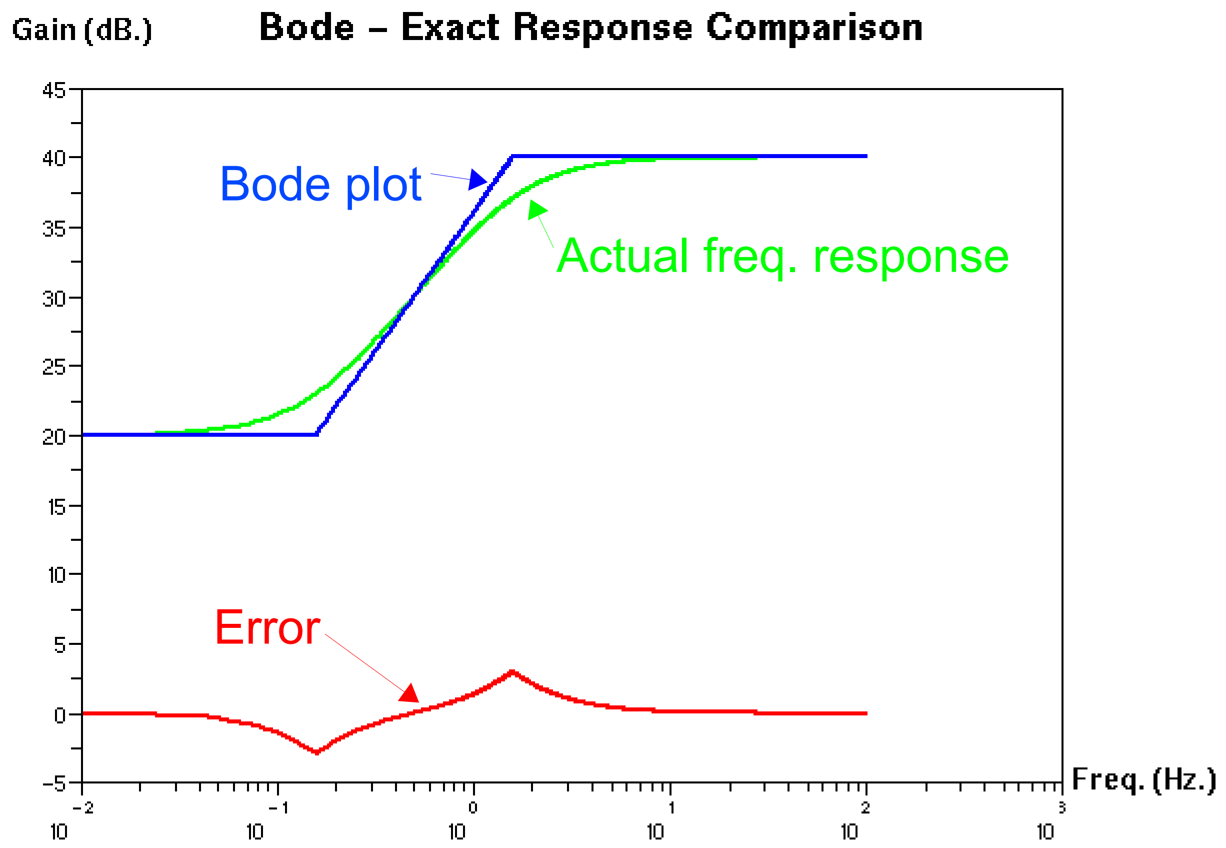
$$1 < \omega < 10 \Rightarrow |H_{dB}| = 20\log_{10} K_{dc} + 20\log_{10}\frac{\omega}{1}$$

$$\omega > 10 \Rightarrow |H_{dB}| = 20\log_{10} K_{dc} + 20\log_{10}\frac{\omega}{1} - 20\log_{10}\frac{\omega}{10}$$

Scilab Simulation

```
Kdc=10; //Example Bode Plot
KdB= 20*log10(Kdc);
omegaz=1;
fz=omegaz/(2*%pi);
omegap=10;
fp=omegap/(2*%pi);
f=0.01:0.01:100;
magnum=sqrt((f/fz)^2 + 1);
magden=sqrt((f/fp)^2 + 1);
FreqResp=KdB+20*(log10(magnum)-log10(magden));
plot(f,FreqResp)
term1=KdB*sign(f);  //Create constant array of length len(f)
term2=max(0,20*log10(f/fz));  //Zero for f < fz;
term3=min(0,-20*log10(f/fp)); //Zero for f < fp;
BodePlot=term1+term2+term3;
plot(f,BodePlot);
err=BodePlot-FreqResp;
plot(f,err);
```

Scilab Results



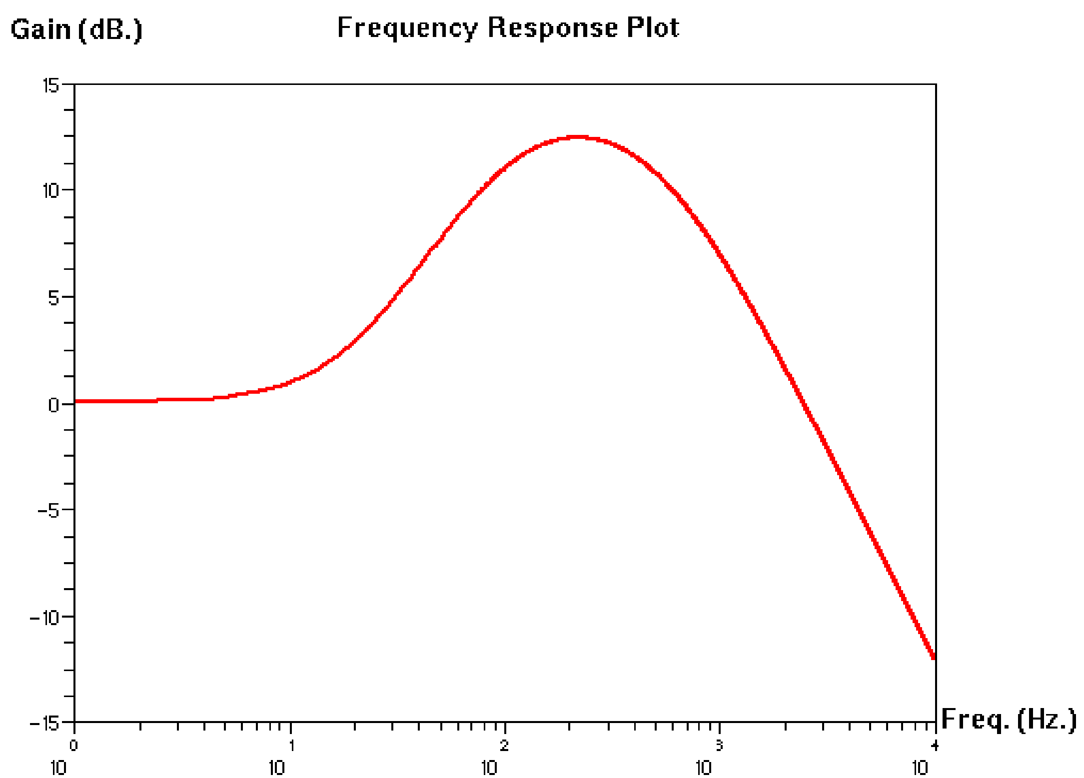
Observations

The ω/ω_a ratio changes by 20 dB for each order of magnitude change in frequency ($20\log_{10}(10) = 20$).

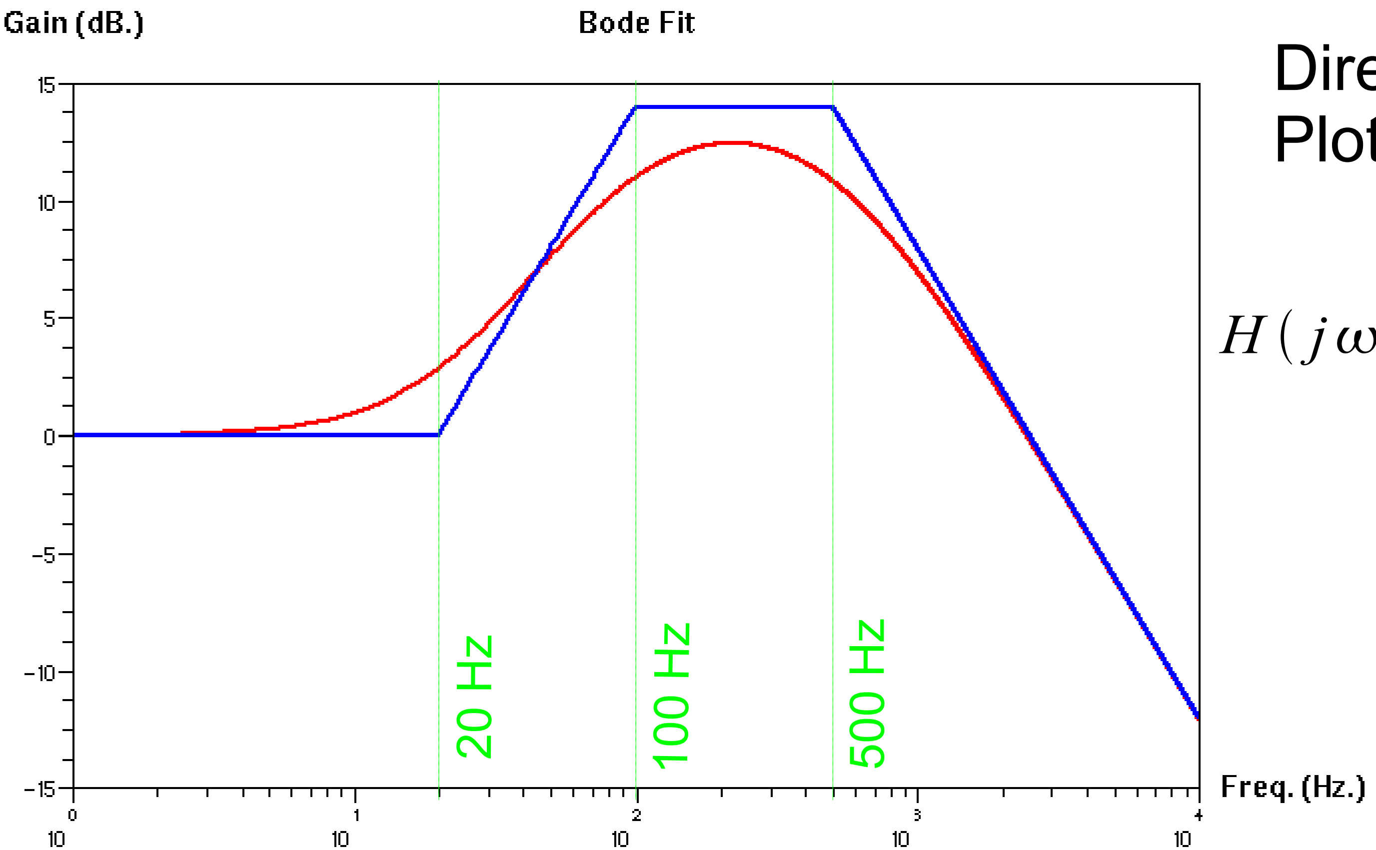
Our "rule of 10" scheme for using either 1, or ω/ω_a for magnitude estimation is quite accurate. This why the Bode plot is called an asymptotic plot.

We can plot a system transfer function, then position straight line segments of $\pm x * 20\, dB/decade$ on the Bode plot. The intersection of the lines occurs at the break frequencies.

Bode Plot Used to Estimate Zeros & Poles



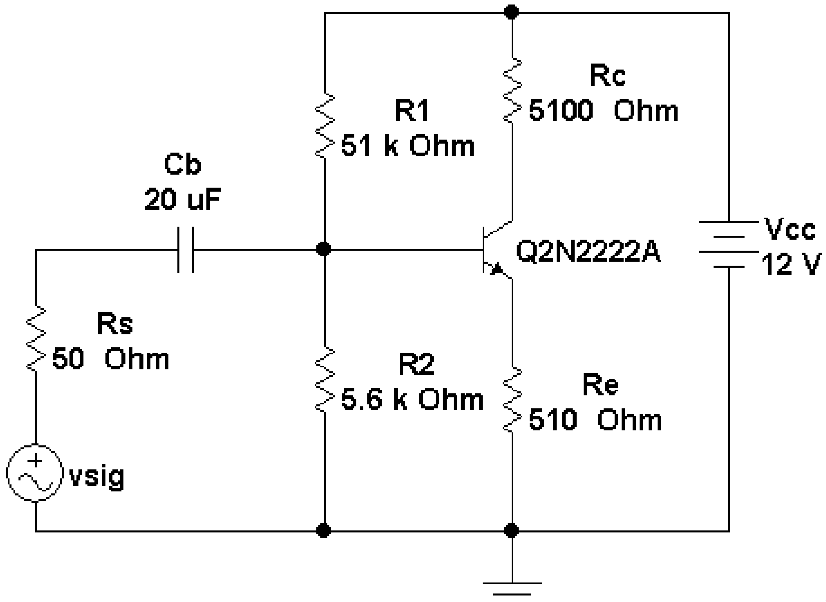
Bode Plot Superposition

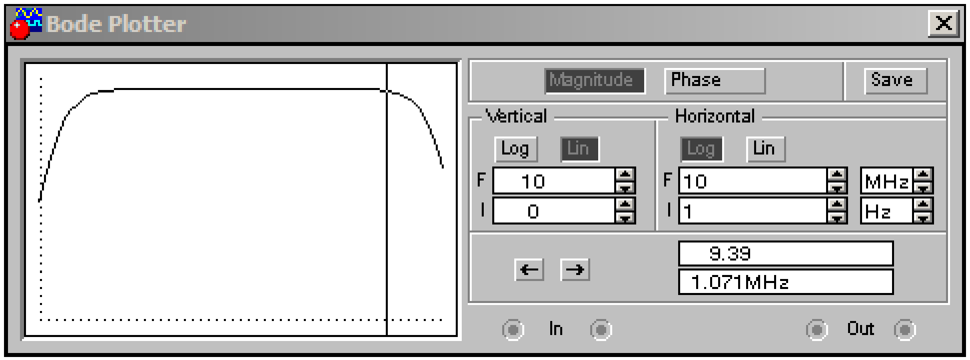


Directly from the Bode Plot!

$$H(j\omega)=\frac{\frac{j\omega}{20}+1}{\left(\frac{j\omega}{100}+1\right)\left(\frac{j\omega}{500}+1\right)}$$

Gain of 10 Amplifier – Non-ideal Transistor

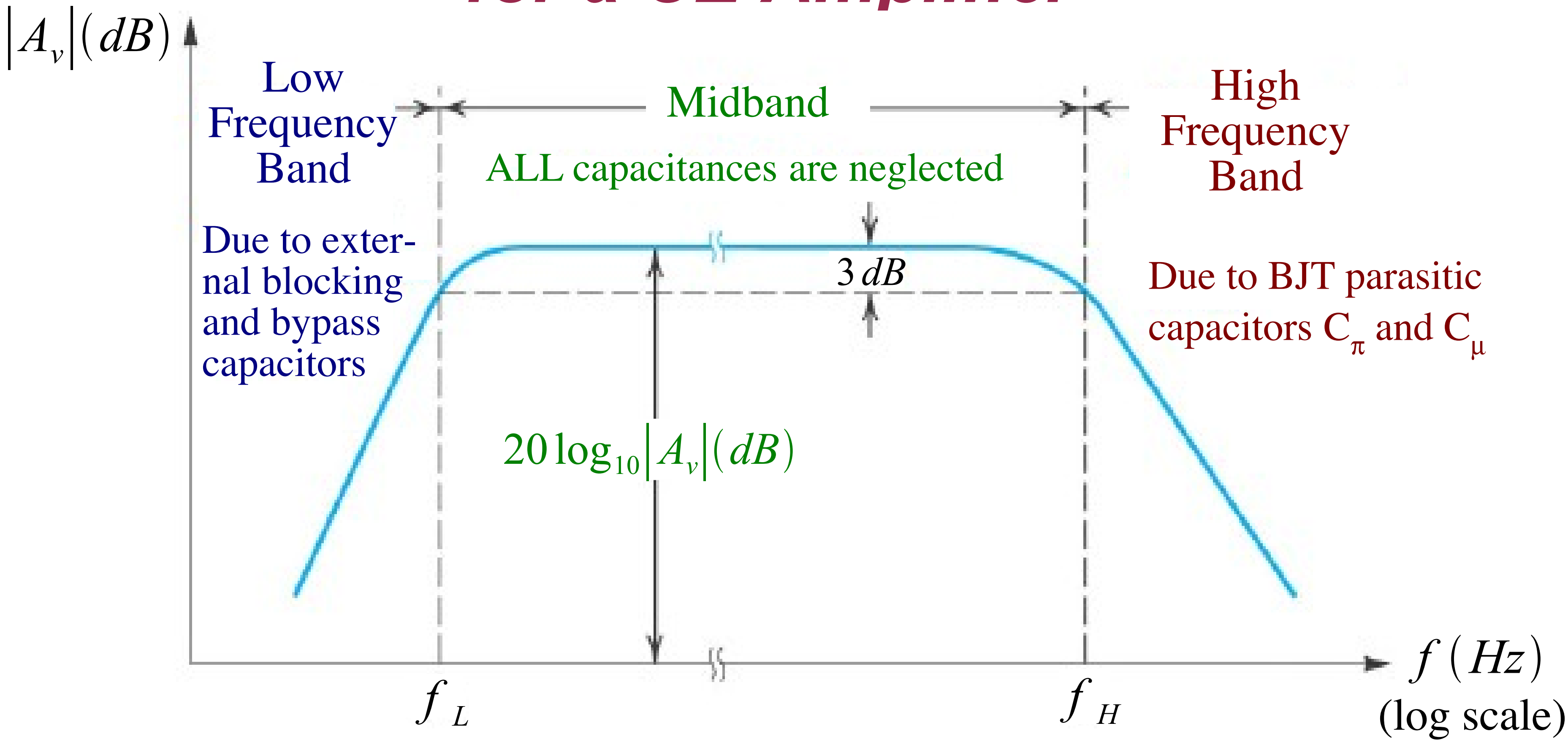




Gain starts dropping at about 1MHz.

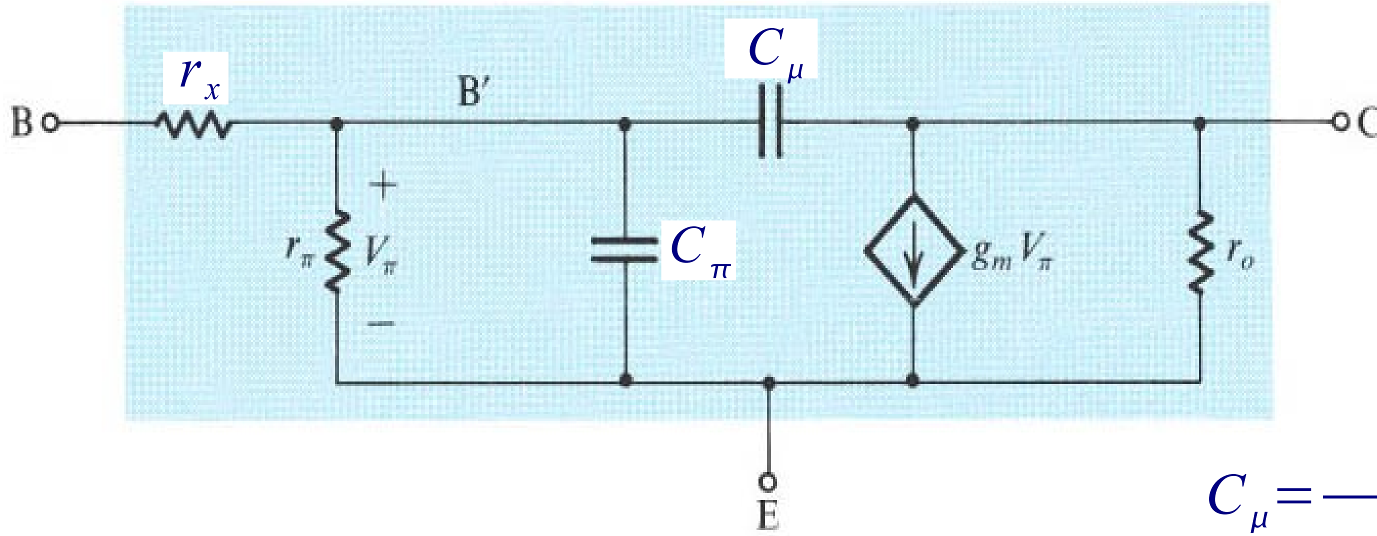
Why!
Because of internal transistor capacitances that we have ignored in our models.

Sketch of Typical Voltage Gain Response for a CE Amplifier



$BW = f_H - f_L \approx f_H$ $\quad$ $GBP = |A_v| BW$

High Frequency Small-signal Model



$$C_\mu = \frac{C_{\mu 0}}{\left(1+\frac{V_{CB}}{V_{0c}}\right)^m}$$

$$C_\pi = C_{de} + \frac{C_{je0}}{\left(1+\frac{V_{BE}}{V_{0e}}\right)^m}$$

Two capacitors and a resistor added.

A base to emitter capacitor, C_π

A base to collector capacitor, C_μ

A resistor, r_x, representing the base terminal resistance ($r_x << r_\pi$)

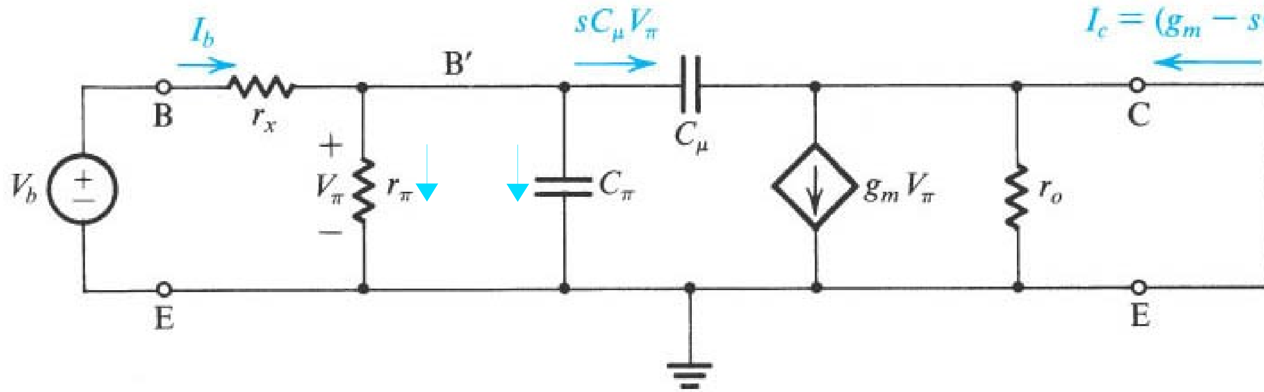
High Frequency Small-signal Model

The internal capacitors on the transistor have a strong effect on circuit high frequency performance! They attenuate base signals, decreasing v_{be} since their reactance approaches zero (short circuit) as frequency increases.

As we will see later C_μ is the principal cause of this gain loss at high frequencies. At the base C_μ looks like a capacitor of value $k\,C_\mu$ connected between base and emitter, where $k > 1$ and may be $>> 1$.

This phenomenon is called the *Miller Effect*.

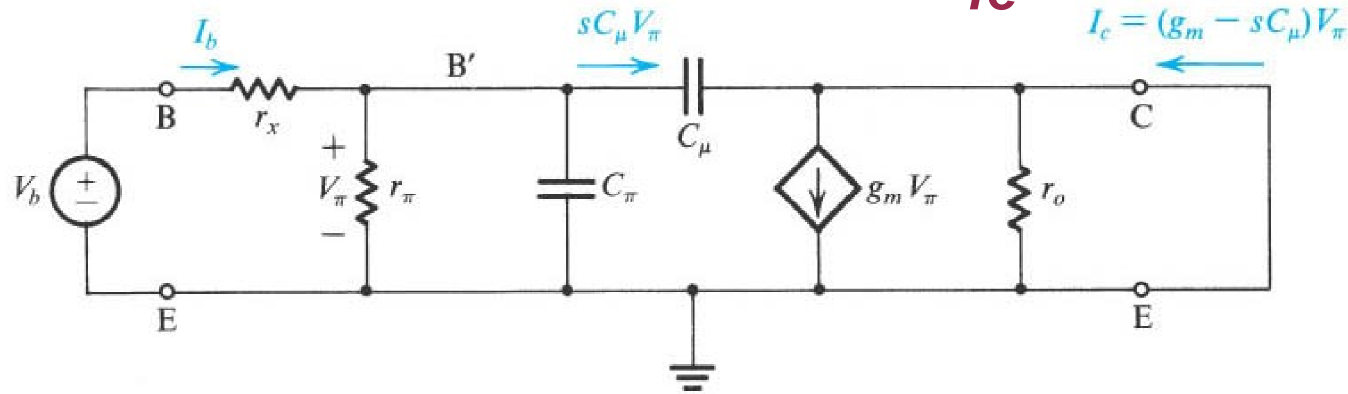
Frequency-dependent "beta" h_{fe}



The relationship $i_c = \beta i_b$ does not apply at high frequencies $f > f_H$!
Using the relationship – $i_c = f(V_\pi)$ – find the new relationship between i_b and i_c. For i_b (using *phasor notation (I_x & V_x) for frequency domain analysis*):

@ node B': $I_b = \left(\frac{1}{r_\pi} + sC_\pi + sC_\mu\right) V_\pi$ where $r_x \approx 0$ (ignore r_x)

Frequency-dependent h_{fe} or "beta"



The ratio of the two equations:

$$I_b = \left(\frac{1}{r_\pi} + sC_\pi + sC_\mu \right) V_\pi \quad \text{@ node C:} \quad I_c = (g_m - sC_\mu) V_\pi \quad \text{(ignore } r_o\text{)}$$

Leads to a new relationship between the I_b and I_c:

$$\boxed{h_{fe} = \frac{I_c}{I_b} = \frac{g_m - sC_\mu}{\frac{1}{r_\pi} + sC_\pi + sC_\mu}}$$

Frequency Response of h_{fe}

$$h_{fe}=\frac{g_m-sC_\mu}{\frac{1}{r_\pi}+sC_\pi+sC_\mu}$$

multiply N&D by r_π

$$h_{fe}=\frac{(g_m-j\omega C_\mu)r_\pi}{1+j\omega(C_\pi+C_\mu)r_\pi}$$

factor N to isolate g_m

$$h_{fe}=\frac{\left(1-j\omega\frac{C_\mu}{g_m}\right)\overbrace{g_m r_\pi}^{\beta}}{1+j\omega(C_\pi+C_\mu)r_\pi}$$

$$\boxed{g_m=\frac{I_C}{V_T} \qquad r_\pi=\beta\frac{V_T}{I_C}}$$

For small $\omega=\omega_{low}$: $\omega_{low}\frac{C_\mu}{g_m}\ll 1<\frac{1}{10}$

and: $\omega_{low}(C_\pi+C_\mu)r_\pi\ll 1<\frac{1}{10}$

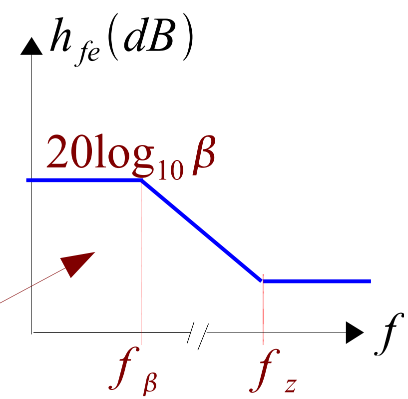
Note: $\omega_{low}(C_\pi+C_\mu)r_\pi=\omega_{low}(C_\pi+C_\mu)\frac{\beta}{g_m}\gg\omega_{low}\frac{C_\mu}{g_m}$

We have: $\boxed{h_{fe}=g_m r_\pi=\beta}$

Frequency Response of h_{fe} cont.

$$h_{fe}=\frac{\left(1-j\omega\frac{C_\mu}{g_m}\right)g_m r_\pi}{1+j\omega(C_\pi+C_\mu)r_\pi}=\frac{\left(1-j\frac{\omega}{\omega_z}\right)}{\left(1+j\frac{\omega}{\omega_\beta}\right)}g_m r_\pi=\frac{\left(1-j\frac{f}{f_z}\right)}{\left(1+j\frac{f}{f_\beta}\right)}\beta$$

ω_β



$$(C_\pi+C_\mu)r_\pi=(C_\pi+C_\mu)\frac{\beta}{g_m}\gg\frac{C_\mu}{g_m} \text{ => } f_z\gg f_\beta$$

Hence, the lower break frequency or $-3dB$ frequency is f_β

$$f_\beta=\frac{1}{2\pi(C_\pi+C_\mu)r_\pi}=\frac{g_m}{2\pi(C_\pi+C_\mu)\beta}$$ the upper: $$f_z=\frac{1}{2\pi C_\mu/g_m}=\frac{g_m}{2\pi C_\mu}$$

where $f_z>10 f_\beta$

Frequency Response of h_{fe} cont.

Using Bode plot concepts, for the range where: $\omega < \omega_\beta$

$$h_{fe} = g_m r_\pi = \beta$$

For the range where: $f_\beta < f < f_z$ s.t. $|1 - j f / f_z| \approx 1$

We consider the frequency-dependent numerator term to be 1 and focus on the response of the denominator:

$$h_{fe} = \frac{g_m r_\pi}{\left(1 + j\frac{f}{f_\beta}\right)} = \frac{\beta}{\left(1 + j\frac{f}{f_\beta}\right)}$$

Frequency Response of h_{fe} cont.

Neglecting numerator term: $$h_{fe}=\frac{g_m r_\pi}{\left(1+j\frac{f}{f_\beta}\right)}=\frac{\beta}{\left(1+j\frac{f}{f_\beta}\right)}$$

And for $f/f_\beta >> 1$ (but $< f/f_z$): $$\left|h_{fe}\right|\approx\frac{\beta}{\left(\frac{f}{f_\beta}\right)}=\beta\frac{f_\beta}{f}$$

Unity gain bandwidth: $$\left|h_{fe}\right|=1\Rightarrow\beta\frac{f_\beta}{f}=1\Rightarrow f_T=\beta f_\beta$$

$$\boxed{f_T=\frac{\omega_T}{2\pi}=\beta f_\beta}$$

BJT unity-gain frequency or GBP

Frequency Response of h_{fe} cont.

$$\beta = 100 \quad r_\pi = 2500\,\Omega \quad C_\pi = 12\ pF \quad C_\mu = 2\ pF \quad g_m = 40 \cdot 10^{-3}\ S$$

$$\omega_\beta = \frac{1}{\left(C_\pi + C_\mu\right) r_\pi} = \frac{10^{12} \cdot 10^{-3}}{(12+2) \cdot 2.5} = 28.57 \cdot 10^6\ rps$$

$$f_\beta = \frac{\omega_\beta}{2\pi} = \frac{28.57}{6.28} 10^6\ Hz = 4.55\ MHz$$

$$f_T = \beta f_\beta = 455\ MHz$$

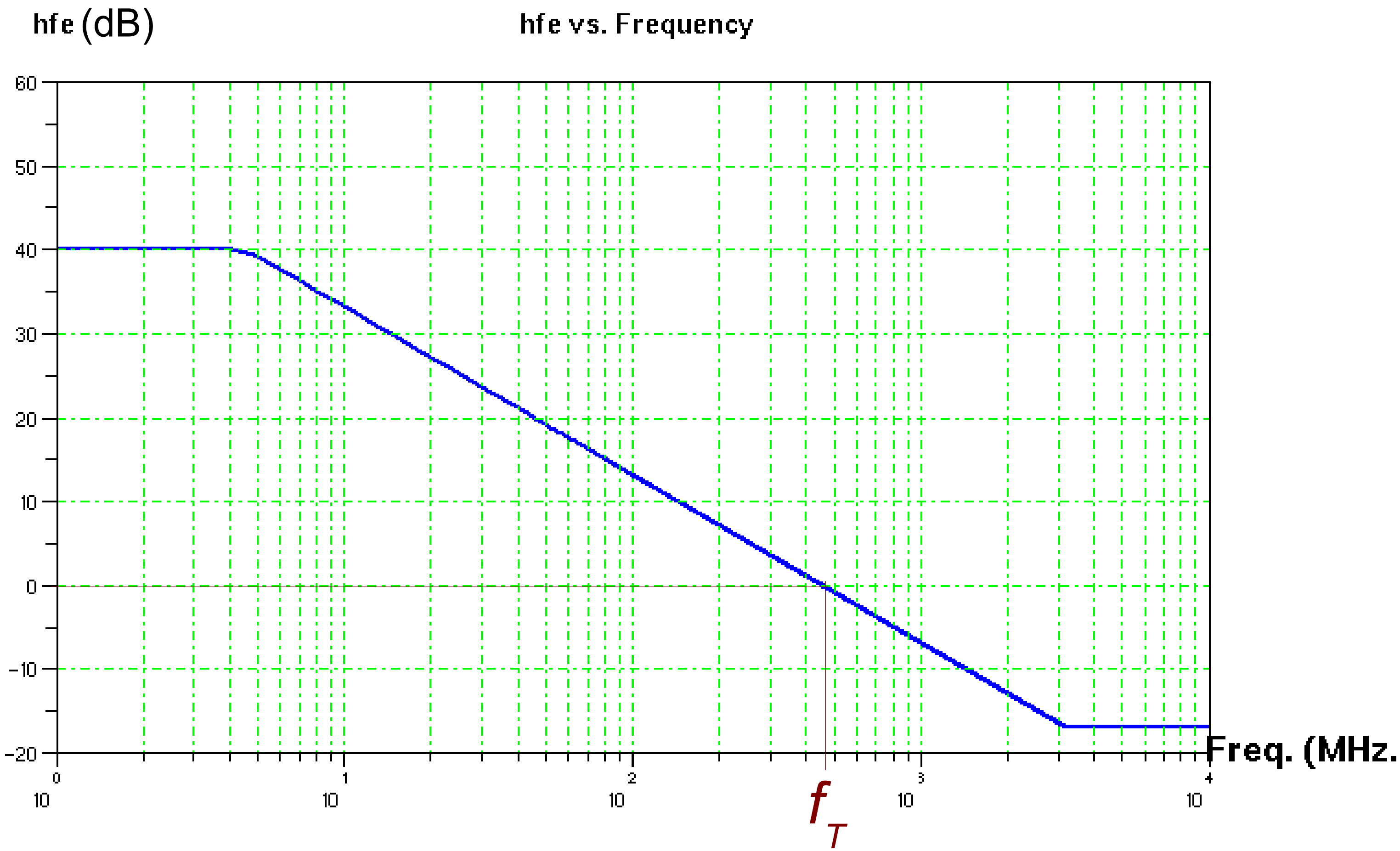
$$\omega_z = \frac{g_m}{C_\mu} = \frac{40 \cdot 10^{-3} \cdot 10^{12}}{2}\ Hz = 20 \cdot 10^9\ rps$$

$$f_z = \frac{\omega_z}{2\pi} = 3.18 \cdot 10^9\ Hz = 3180\ MHz$$

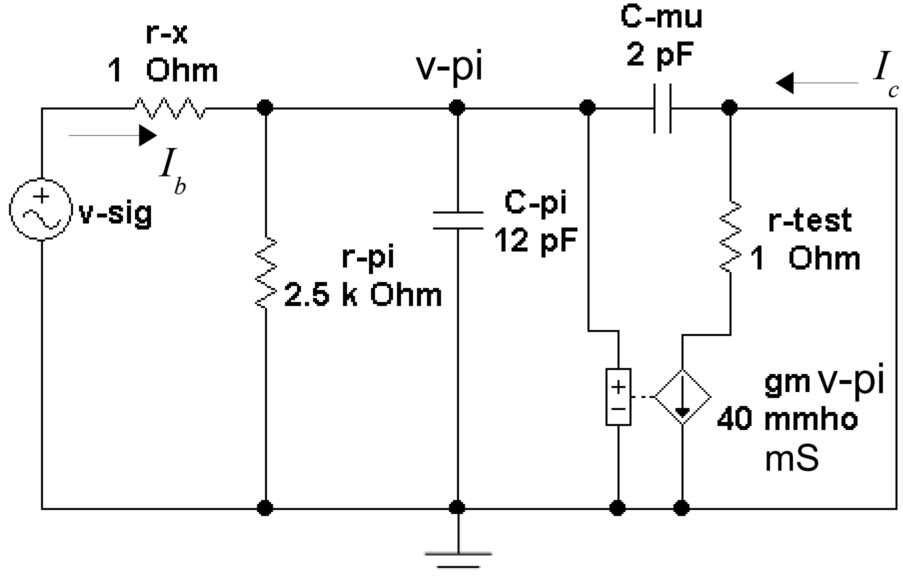
Scilab f_T Plot

```
//fT Bode Plot
Beta=100;
KdB= 20*log10(Beta);
fz=3180;
fp=4.55;
f= 1:1:10000;
term1=KdB*sign(f); //Constant array of len(f)
term2=max(0,20*log10(f/fz));   //Zero for f < fz;
term3=min(0,-20*log10(f/fp));  //Zero for f < fp;
BodePlot=term1+term2+term3;
plot(f,BodePlot);
```

h_{fe} Bode Plot



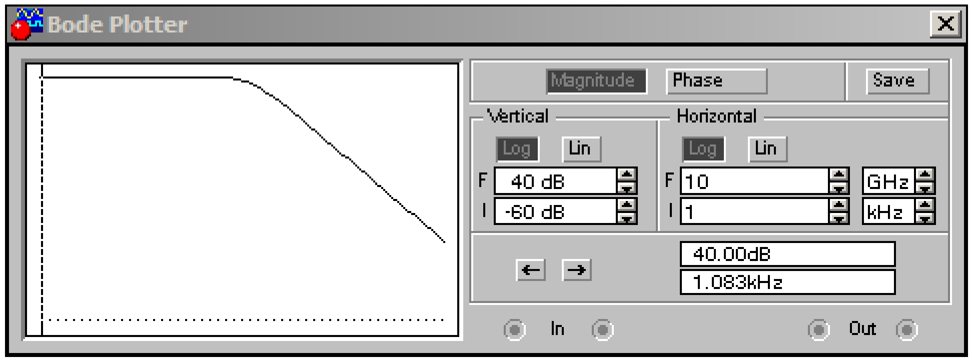
Multisim Simulation



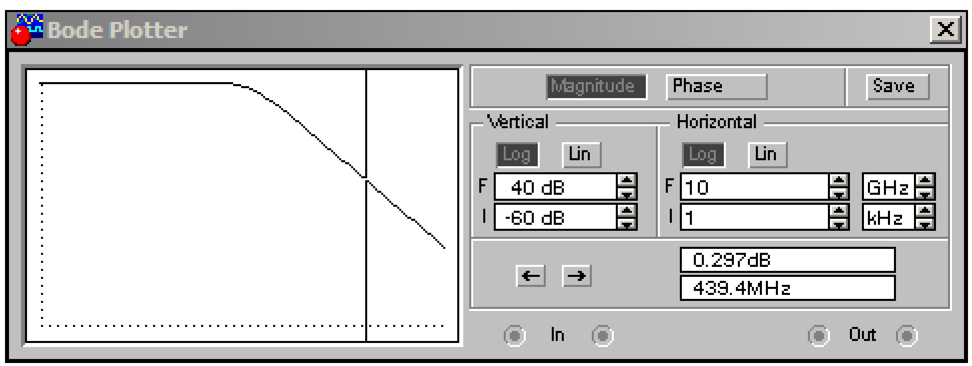
Insert 1 ohm resistors – we want to measure a current ratio.

$$h_{fe}=\frac{I_c}{I_b}=\frac{g_m-s\,C_\mu}{\frac{1}{r_\pi}+s\,(C_\pi+C_\mu)}$$

Simulation Results



Low frequency $|h_{fe}|$



Unity Gain frequency about 440 MHz.