Single-Amplifier-Biquad (SAB) Filter Stages

- **Sallen and Key Lowpass (LP) Stage**
  - $R_1C_1 = R_2C_2 = RC$ strategy
  - $R_1C_1 \neq R_2C_2$ strategy
- **Sallen and Key Highpass (HP) Stage**
  - LP to HP Transformation
- **Delyannis-Friend Bandpass (BP) Stage**
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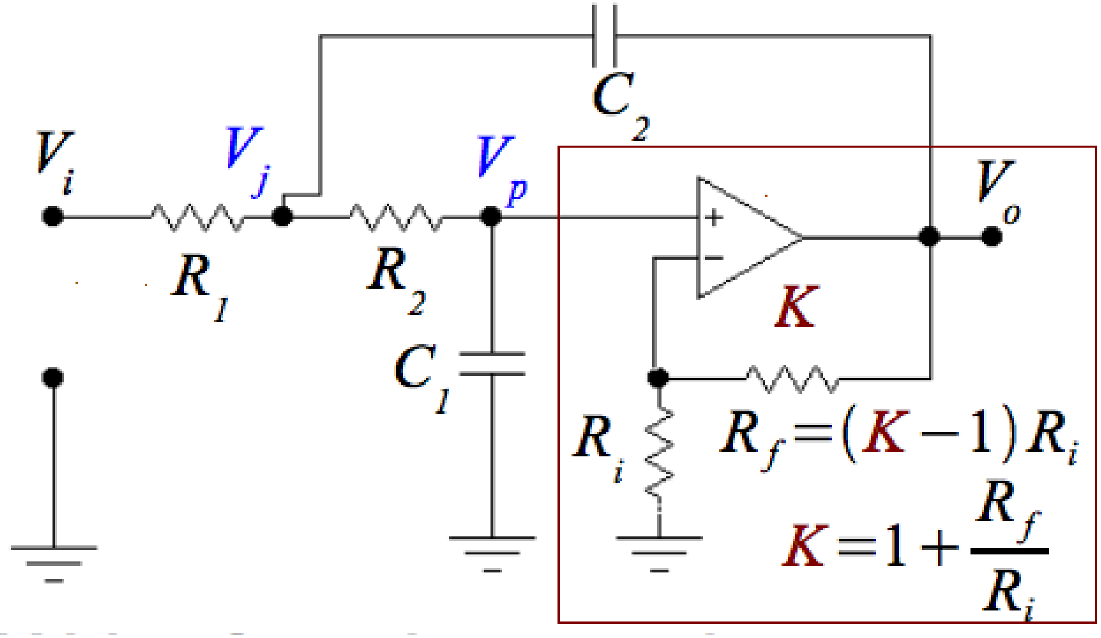
Sallen-Key LP Stage



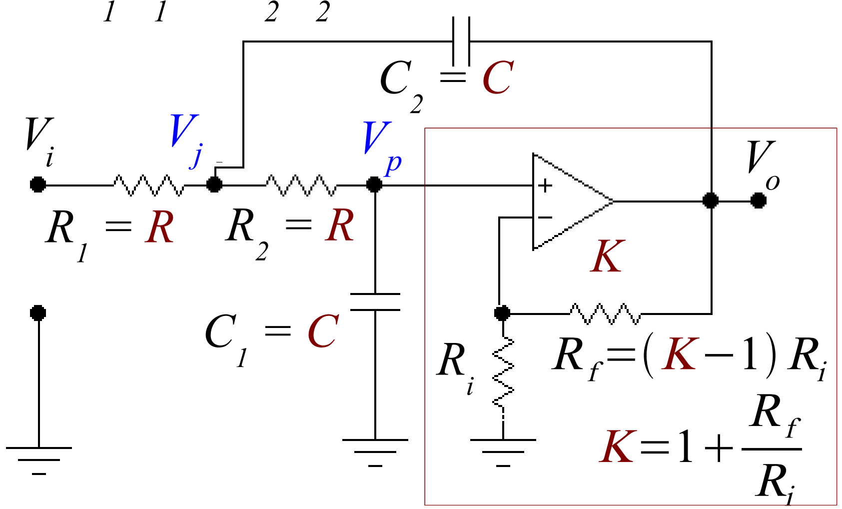
$$G(s)=\frac{V_o}{V_i}=\frac{\omega_0^2 K_0}{s^2+\frac{\omega_0}{Q}s+\omega_0^2}$$

Choice of dependent element variables is important!

1. Choose $R_1C_1 = R_2C_2 = RC$ => RC -> $1/\omega_0$ & K -> $1/Q$, K_0
2. Choose K, say $K = 1$ => $K_0 = 1$, R_1, R_2, C_1, C_2 -> $1/\omega_0$, $1/Q$

Sallen-Key LP Stage Analysis

LET $R_1C_1 = R_2C_2 = RC$



Write 2 node equations:

V_j: $$\frac{V_j - V_i}{R} + sC(V_j - V_o) + \frac{V_j - V_p}{R} = 0$$

V_p: $$\frac{V_p - V_j}{R} + sCV_p = 0$$

and $V_o = K V_p$

Multiplying by R:

V_j: $$V_j + sCR(V_j - V_o) + V_j - V_p = V_i$$

V_p: $$V_p - V_j + sCRV_p = 0$$

and $V_o = K V_p$

Eliminate V_p using $V_o = K V_p$, and combine terms:

V_j: $$(2 + sCR)V_j - \left(\frac{1}{K} + sCR\right)V_o = V_i$$

V_o: $$-V_j + (1 + sCR)\frac{V_o}{K} = 0$$

Sallen-Key LP Stage Analysis - cont.

$$V_j\text{:}\quad (2+sCR)\,V_j-\left(\frac{1}{K}+sCR\right)V_o=V_i$$

$$V_o\text{:}\quad -V_j+(1+sCR)\frac{V_o}{K}=0$$

Eliminate: V_j (multiply V_o Eq by $(2+sCR)$)

$$V_j\text{:}\quad (2+sCR)\,V_j-\left(\frac{1}{K}+sCR\right)V_o=V_i$$

$$V_o\text{:}\quad -(2+sCR)\,V_j+(2+sCR)(1+sCR)\frac{V_o}{K}=0$$

Add and solve for V_o and multiply by K:

$$\left[\frac{1}{K}(2+sCR)(1+sCR)-\left(\frac{1}{K}+sCR\right)\right]V_o=V_i$$

$$\left[(2+sCR)(1+sCR)-(1+sKCR)\right]V_o=K\,V_i$$

Sallen-Key LP Stage Analysis - cont.

$$\left[(2+sCR)(1+sCR)-(1+sKCR)\right]V_o=KV_i$$

Multiplying to expand into a polynomial:

$$\left[(sCR)^2+3\mathrm{s}\,CR+2-1-sKCR\right]V_o=KV_i$$

Collecting terms:

$$\left[s^2(CR)^2+(3-K)\,s\,CR+1\right]V_o=KV_i$$

Dividing by $(CR)^2$:

$$\left[s^2+\frac{(3-K)}{CR}s+\left(\frac{1}{CR}\right)^2\right]V_o=K\left(\frac{1}{CR}\right)^2V_i$$

Sallen-Key LP Stage Analysis - cont.

$$\left[s^2+\frac{(3-K)}{CR}s+\left(\frac{1}{CR}\right)^2\right]V_o=K\left(\frac{1}{CR}\right)^2V_i$$

For $R_1C_1=R_2C_2=RC$

Form $G(s)=V_o/V_i$:

$$G(s)=\frac{V_o}{V_i}=\frac{\left(\frac{1}{CR}\right)^2K}{s^2+\frac{(3-K)}{CR}s+\left(\frac{1}{CR}\right)^2}=\frac{\omega_0^2K_0}{s^2+\frac{\omega_0}{Q}s+\omega_0^2}$$

$$\omega_0=\frac{1}{CR}\qquad Q=\frac{1}{3-K}$$

Note: $K=1\Rightarrow Q=\frac{1}{2}$

sensitivity & potential stability problem, i.e. $K\geq 3$ => $G(s)$ unstable

Sallen-Key LP Stage Analysis - cont.

Design Equations:

$$G(s)=\frac{V_o}{V_i}=\frac{\left(\frac{1}{CR}\right)^2 K}{s^2+\frac{(3-K)}{CR}s+\left(\frac{1}{CR}\right)^2}=\frac{\omega_0^2 K_0}{s^2+\frac{\omega_0}{Q}s+\omega_0^2}$$

$\omega_0=\frac{1}{CR}$: choose C => $R=\frac{1}{C\omega_0}$

$Q=\frac{1}{3-K}$: => $K=3-\frac{1}{Q}$

$Q \geq 1/2$ to be realizable, i.e. $K \geq 1$

$Q \geq 1/2$ to be realizable, i.e. $R_f \geq 0$

$K=1+\frac{R_f}{R_i}$: choose R_i => $R_f=R_i(2-\frac{1}{Q})$

Sallen-Key 20 kHz MFM LP Stage

1. Let's do the initial design with normalized R and C.

$$\Omega_0 = \omega_0 = \frac{1}{CR} = 2\pi\, 20\cdot 10^3 \quad \text{and} \quad R_0 = 1\, k\Omega$$

$$G(s_n) = \frac{K/(R_n C_n)^2}{s_n^2 + \frac{(3-K)}{R_n C_n} s_n + \frac{1}{(R_n C_n)^2}} = \frac{K_0}{s_n^2 + \boxed{\frac{1}{Q}}\, s_n + 1} = \frac{K_0}{s_n^2 + \boxed{\sqrt{2}}\, s_n + 1}$$

$$\boxed{Q = \frac{1}{\sqrt{2}}}$$

Normalized design equations:

$$\frac{1}{R_n C_n} = 1 \quad \text{and} \quad K = 1 + \frac{R_{fn}}{R_{in}} = 3 - \frac{1}{Q} = 3 - \sqrt{2} = 1.586$$

where $R_n = \frac{R}{R_0}$ and $C_n = \Omega_0 R_0 C$

Choose: $\boxed{C_n = 1 \Rightarrow R_n = 1}$ Choose: $\boxed{R_{in} = 10 \Rightarrow R_{fn} = 5.86}$

Sallen-Key 20 kHz MFM LP Stage - cont.

$C_n=1 \Rightarrow R_n=1$ where $\Omega_0=\omega_0=2\pi 20\cdot 10^3$ and $R_0=1\,k\Omega$

$$C=\frac{C_n}{\Omega_0 R_0}F=\frac{1}{2\pi 20\cdot 10^3}\frac{1}{R_0}F=\frac{25}{\pi}\frac{1}{R_0}\mu F=\frac{7.96}{R_0}\mu F$$

$R=R_n R_0\,\Omega$

$R_0=1\,k\Omega$ => $\boxed{R=1\,k\Omega \text{ and } C=7.96\,nF}$

$R_{in}=10 \Rightarrow R_{fn}=5.86$

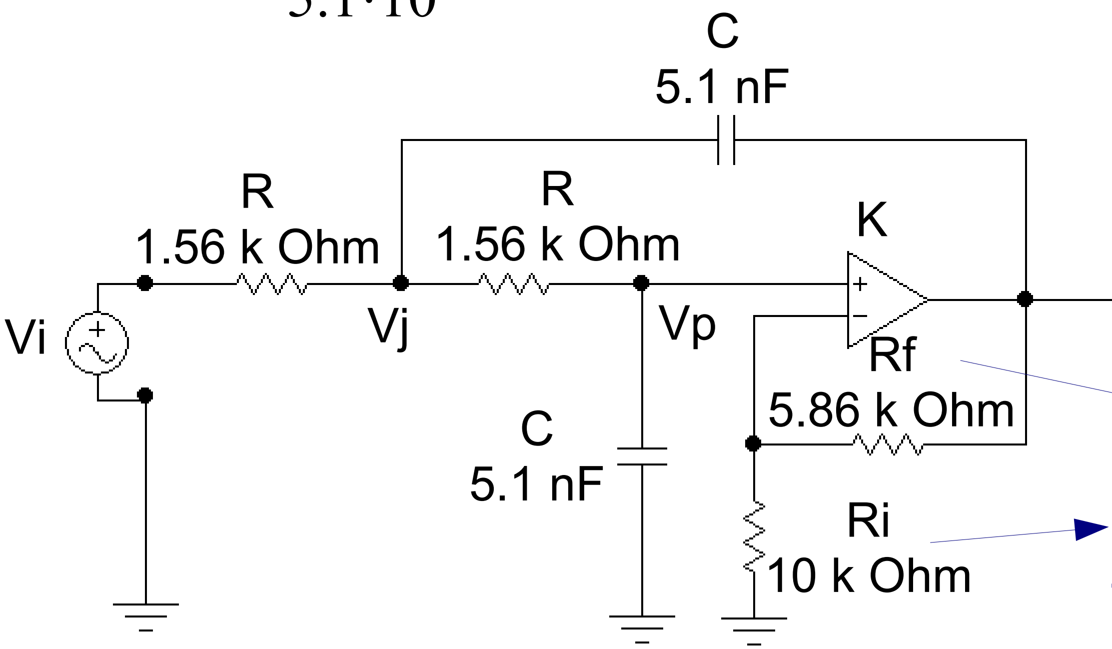
$R_0=1\,k\Omega$ => $\boxed{R_i=10\,k\Omega \text{ and } R_f=5.86\,k\Omega}$

Sallen-Key 20 kHz MFM LP Stage - cont.

Suppose we wanted to choose $C = 5.1\ nF$ from the RCA Lab.

$$5.1\,nF = \frac{7.96}{R_0}\,\mu\,F \Rightarrow 5.1\cdot 10^{-9} = \frac{7.96\cdot 10^{-6}}{R_0}$$

Solving for R_0: $$R_0 = \frac{7.96\cdot 10^{-6}}{5.1\cdot 10^{-9}} = 1.56\,k\Omega \Rightarrow \boxed{R = 1.56\,k\Omega}$$



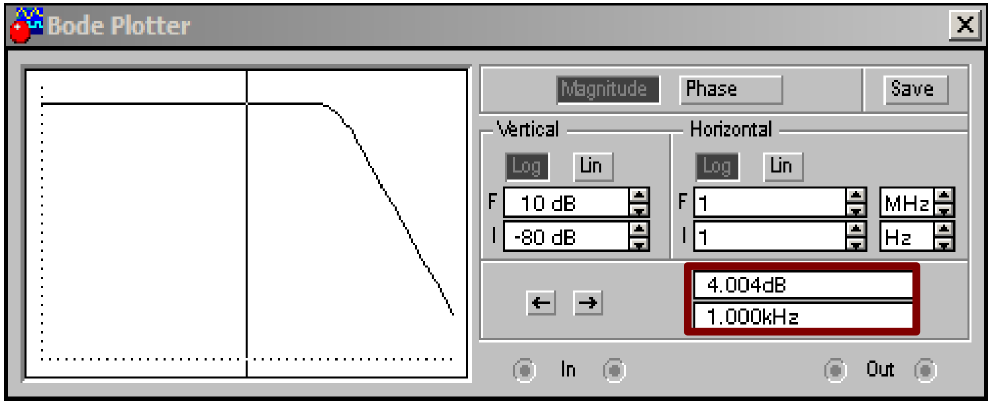
Note: R_f & R_i do not need to be rescaled to new R_0.

Simulation Results

$$K = 1.586 \Rightarrow 20\log_{10}\left|T(j0)\right| = 20\log_{10} K = 4\ dB$$

4 dB

Pass-band Gain



Vertical: Log
F 10 dB
I -80 dB

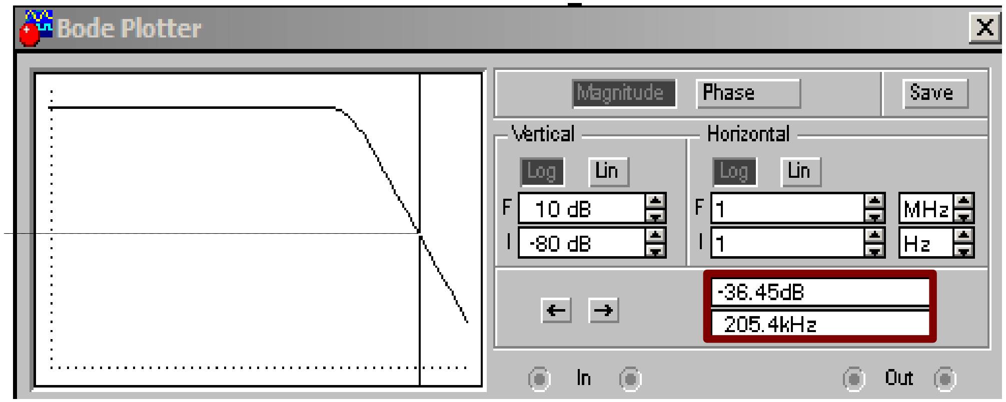
Horizontal: Log
F 1 MHz
I 1 Hz

4.004dB
1.000kHz

1 kHz

Stop-band Gain
at 10x cutoff ($20\ k\ Hz$)

-36.45 dB



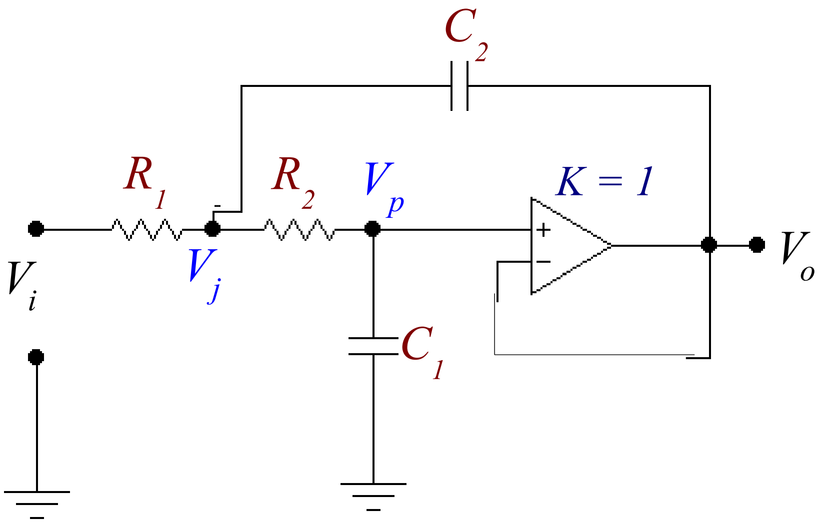
Vertical: Log
F 10 dB
I -80 dB

Horizontal: Log
F 1 MHz
I 1 Hz

-36.45dB
205.4kHz

205.4 kHz

Sallen and Key LP Stage with K = 1



Recall : $R_1C_1 = R_2C_2 = RC$

$\frac{1}{2} \leq Q < \infty \Rightarrow 1 \leq K < 3$

When: $R_1C_1 \neq R_2C_2$

we can set $K = 1$

Note $K = 1 \Rightarrow K_0 = 1$

$$T(s) = \frac{V_o}{V_i} = \frac{\dfrac{1}{C_1 C_2 R_1 R_2}}{s^2 + \left(\dfrac{1}{C_1 R_1} + \dfrac{1}{C_1 R_2}\right) s + \dfrac{1}{C_1 C_2 R_1 R_2}} = \frac{K_0 \omega_0^2}{s^2 + \dfrac{\omega_0}{Q} s + \omega_0^2}$$

Sallen and Key LP Stage with K = 1 - cont.

Normalize frequency and impedance, i.e. $s_n = \frac{s}{\omega_0}$ and $Z_n = \frac{Z}{R_0}$

$$G(s_n) = \frac{V_o}{V_i} = \frac{\frac{1}{C_{1n} C_{2n} R_{1n} R_{2n}}}{s_n^2 + \left(\frac{1}{C_{1n} R_{1n}} + \frac{1}{C_{1n} R_{2n}}\right) s_n + \frac{1}{C_{1n} C_{2n} R_{1n} R_{2n}}} = \frac{1}{s_n^2 + \frac{1}{Q} s_n + 1}$$

Design Formulas

$$\frac{1}{C_{1n} R_{1n}} + \frac{1}{C_{1n} R_{2n}} = \frac{1}{Q}$$

$$\frac{1}{C_{1n} C_{2n} R_{1n} R_{2n}} = 1$$

Design Formulas (MFM)

$$\frac{1}{C_{1n} R_{1n}} + \frac{1}{C_{1n} R_{2n}} = \sqrt{2}$$

$$\frac{1}{C_{1n} C_{2n} R_{1n} R_{2n}} = 1$$

Sallen and Key LP Stage with K = 1 - cont.

$$\frac{1}{C_{1n}R_{1n}}+\frac{1}{C_{1n}R_{2n}}=\sqrt{2} \qquad \frac{1}{C_{1n}C_{2n}R_{1n}R_{2n}}=1$$

Choose $C_{1n}=1$ and $R_{2n}=R_{1n}=R_n$:

$$\frac{2}{R_n}=\sqrt{2}\Rightarrow R_n=\frac{2}{\sqrt{2}}=\sqrt{2} \qquad \frac{1}{C_{2n}R_n^2}=1\Rightarrow C_{2n}=\frac{1}{R_n^2}=\frac{1}{2}$$

Normalized Design:

$$\boxed{\begin{aligned} C_{1n}&=1\\ C_{2n}&=\frac{1}{2}\\ R_{1n}&=R_{2n}=\sqrt{2}\end{aligned}}$$

Denormalized Design:

$$\boxed{\begin{aligned} C_1&=\frac{C_{1n}}{\Omega_0 R_0}=\frac{1}{\omega_0 R_0}\\ C_2&=\frac{1}{2}\frac{1}{\omega_0 R_0}\\ R_1&=R_2=\sqrt{2}\,R_0\end{aligned}}$$

Comparison

- $R_1C_1 = R_2C_2 = RC$ strategy $$G(s_n)=\frac{K/(R_nC_n)^2}{s_n^2+\frac{(3-K)}{R_nC_n}s_n+\frac{1}{(R_nC_n)^2}}=\frac{K_0}{s_n^2+\frac{1}{Q}s_n+1}$$

$$\frac{1}{R_nC_n}=1 \qquad \boxed{K=1+\frac{R_{fn}}{R_{in}}=3-\frac{1}{Q}}$$

- $R_1C_1 \neq R_2C_2$ strategy

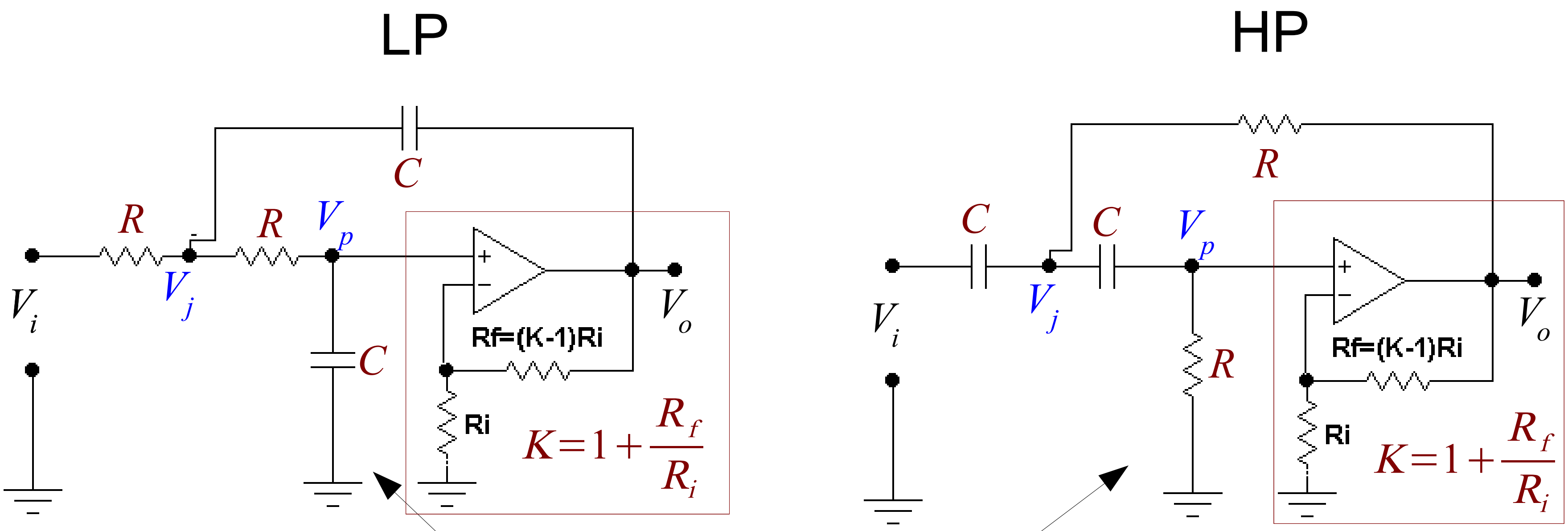
$$G(s_n)=\frac{V_o}{V_i}=\frac{\frac{1}{C_{1n}C_{2n}R_{1n}R_{2n}}}{s_n^2+\left(\frac{1}{C_{1n}R_{1n}}+\frac{1}{C_{1n}R_{2n}}\right)s_n+\frac{1}{C_{1n}C_{2n}R_{1n}R_{2n}}}=\frac{1}{s_n^2+\frac{1}{Q}s_n+1}$$

$$\frac{1}{C_{1n}R_{1n}}+\frac{1}{C_{1n}R_{2n}}=\frac{1}{Q} \qquad \frac{1}{C_{1n}C_{2n}R_{1n}R_{2n}}=1$$

$C_{1n} = 1$ and $R_{2n} = R_{1n} = R_n$ => $\frac{2}{R_n}=\frac{1}{Q}\Rightarrow\boxed{R_n=2Q}$ $\quad \frac{1}{C_{2n}R_n^2}=1\Rightarrow\boxed{C_{2n}=\frac{1}{R_n^2}=\frac{1}{4Q^2}}$

Sallen and Key HP Stage

By interchanging $2\ R's$ and $2\ C's$ on the LP schematic, we convert an LP circuit to an HP circuit



LET $R_1C_1 = R_2C_2 = RC$

Sallen-Key HP Function

Interchanging sC and $1/R$ in the LP nodal equations gives:

$$V_j: \quad \frac{V_j - V_i}{R} + sC(V_j - V_o) + \frac{V_j - V_p}{R} = 0 \rightarrow sC(V_j - V_i) + \frac{V_j - V_o}{R} + sC(V_j - V_p) = 0$$

$$V_p: \quad \frac{V_p - V_j}{R} + sC\,V_p = 0 \longrightarrow sC(V_p - V_j) + \frac{V_p}{R} = 0$$

$$V_o = K\,V_p \longrightarrow V_o = K\,V_p$$

$$\boxed{G(s) = \frac{V_o}{V_i} = \frac{K s^2}{s^2 + \frac{(3-K)}{CR} s + \left(\frac{1}{CR}\right)^2}}$$

Sallen-Key HP Function - cont.

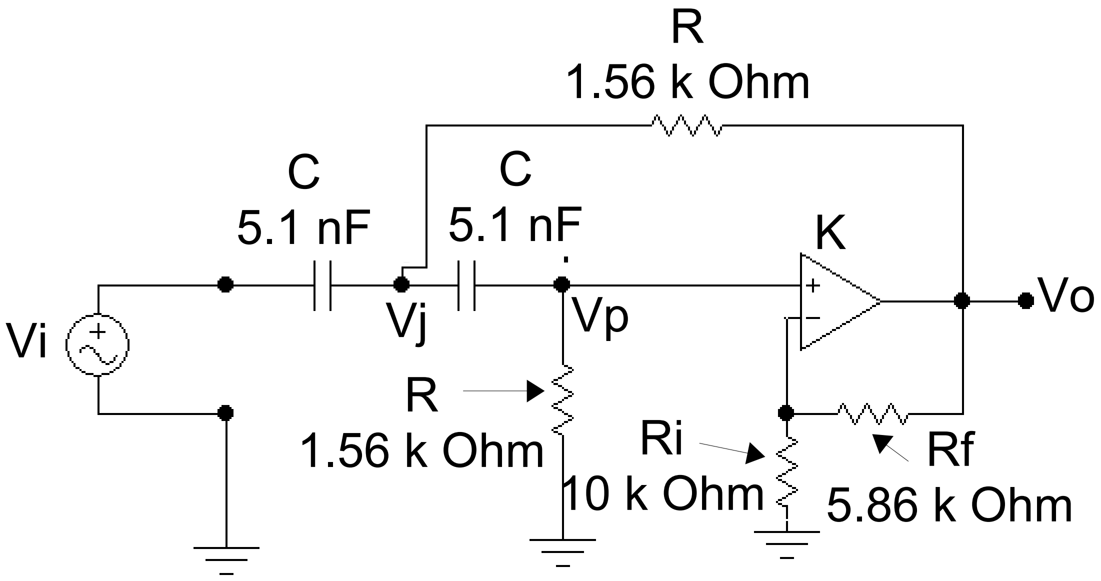
$$G(s)=\frac{V_o}{V_i}=\frac{K\,s^2}{s^2+\frac{(3-K)}{CR}s+\left(\frac{1}{CR}\right)^2}=\frac{K_\infty\,s^2}{s^2+\frac{\omega_0}{Q}s+\omega_0^2}$$

Note that the HP denominator polynomial (or poles) is the same as for the LP section and the same design equations apply:

$$\omega_0=\frac{1}{CR} \qquad Q=\frac{1}{3-K} \qquad K<3\,!$$

$$\left|G(j\infty)\right|=K_\infty=K$$

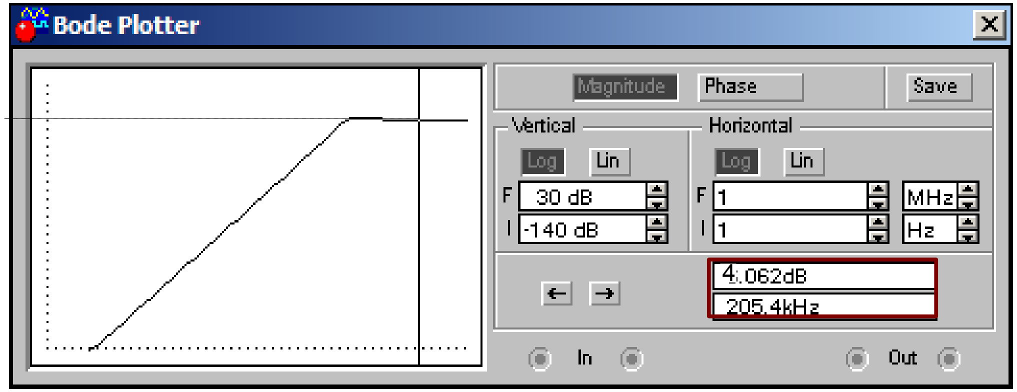
20 Khz MFM HP Stage



HP Stage Response

Passband response

4.06 dB



Vertical: Log
F 30 dB
I -140 dB

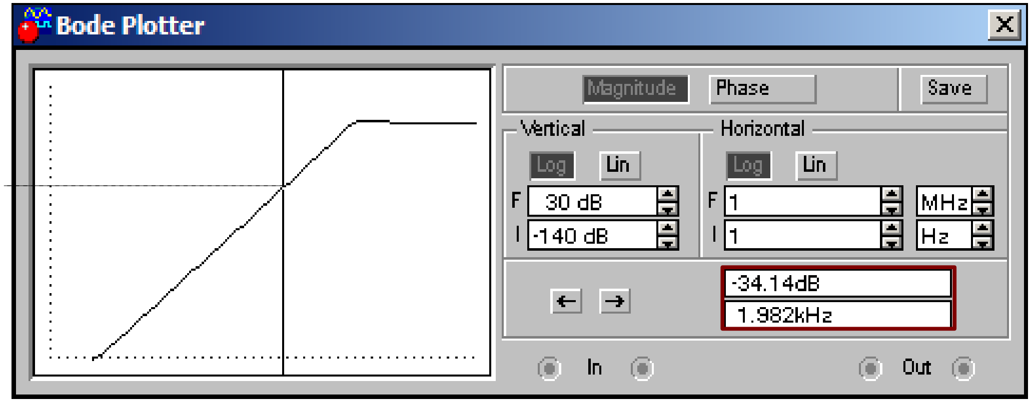
Horizontal: Log
F 1 MHz
I 1 Hz

4.062dB
205.4kHz

205.4 kHz

Stopband response

-34.14 dB



Vertical: Log
F 30 dB
I -140 dB

Horizontal: Log
F 1 MHz
I 1 Hz

-34.14dB
1.982kHz

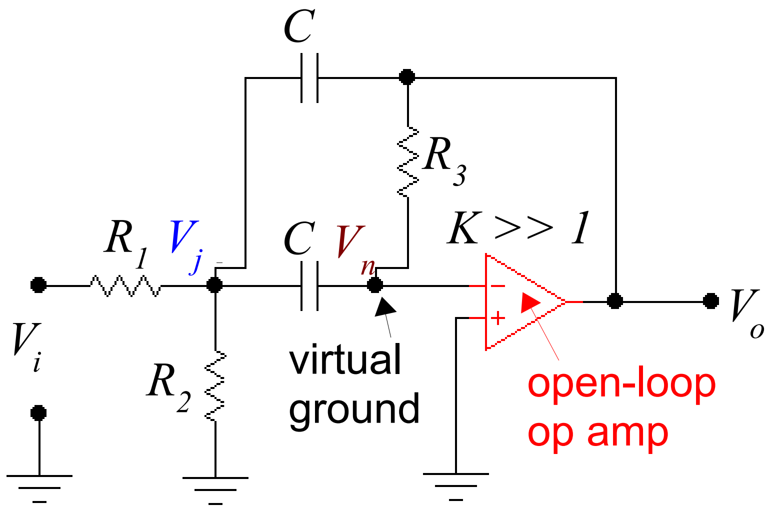
1.98 kHz

Delyannis-Friend BP Stage

Node Equations:

$$V_j:\ \frac{V_j - V_i}{R_1} + sC(V_j - V_o) + sC(V_j - \cancelto{0}{V_n}) + \frac{V_j}{R_2} = 0$$

Choose $C1 = C2 = C$



$$V_n:\ sC(V_n - V_j) + \frac{\cancelto{0}{V_n} - V_o}{R_3} = 0$$

$$V_o = -K\,V_n$$

Ideal op amp: $K \rightarrow \infty \Rightarrow V_n \rightarrow 0$

$$V_o = -K\,V_n \neq 0$$

Delyannis-Friend BP Stage - cont.

$$G(s)=\frac{V_o}{V_i}=\frac{-\frac{s}{C R_1}}{s^2+\frac{2}{C R_3}s+\frac{1}{C^2 R_3\cdot R_1||R_2}}=\frac{-s(\frac{\omega_0}{Q})K_{\omega_0}}{s^2+(\frac{\omega_0}{Q})s+\omega_0^2}$$

Identify:

$$\omega_0^2=\frac{1}{C^2 R_3\cdot R_1||R_2}$$

$$\frac{\omega_0}{Q}=\frac{2}{C R_3}$$

$$K_{\omega_0}=|G(j\omega_0)|=\frac{\cancel{\omega_0}}{\cancel{C} R_1}\frac{\cancel{C} R_3}{2\cancel{\omega_0}}=\frac{R_3}{2R_1}$$

Delyannis-Friend BP Stage - cont.

$$\omega_0^2 = \frac{1}{C^2 R_3 \cdot R_1 \| R_2}$$

$$\frac{\omega_0}{Q} = \frac{2}{C R_3}$$

$$K_{\omega_0} = \frac{R_3}{2 R_1}$$

Given $_{0}$, Q and $K_{\omega 0}$, choose C and solve for the resistors:

$$R_3 = \frac{2Q}{C\omega_0} \qquad R_1 = \frac{R_3}{2K_{\omega_0}} = \frac{Q}{C\omega_0 K_{\omega_0}}$$

$$R_1 \| R_2 = \frac{1}{\omega_0^2 C^2 R_3} = \frac{1}{2\omega_0 C Q}$$

$$R_2 = \frac{R_1}{2\omega_0 C Q R_1 - 1}$$

Normalize $\omega_{0n} = 1$ and choose $C_n = 1$:

$$\boxed{\begin{array}{l} R_{3n} = 2Q \qquad R_{1n} = \dfrac{Q}{K_{\omega_0}} \qquad R_{1n} \| R_{2n} = \dfrac{1}{2Q} \\ R_{2n} = \dfrac{R_{1n}}{2QR_{1n} - 1} \end{array}}$$

Delyannis-Friend BP Stage

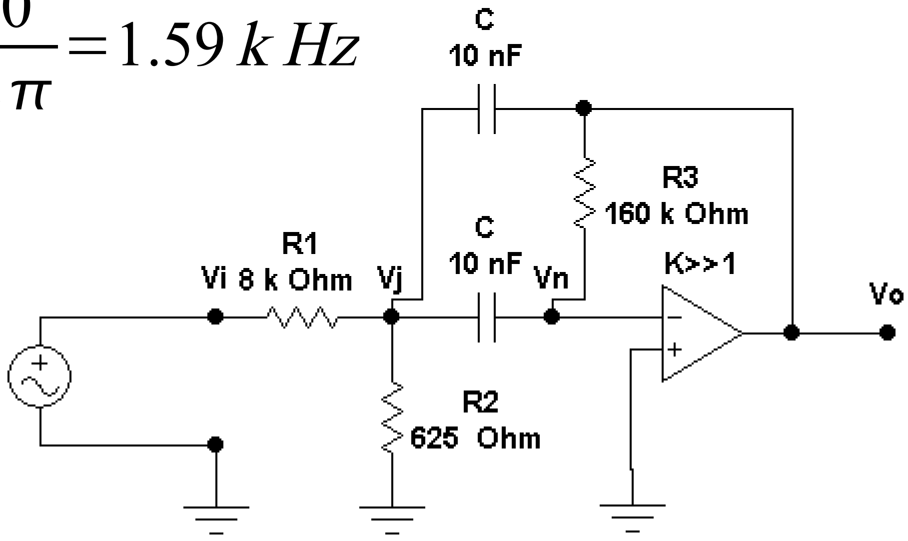
$$\omega_0 = 10^4\, rps \Rightarrow f_0 = \frac{10^4}{2\pi} = 1.59\, k\, Hz$$

$$Q = 8$$

$$K_{\omega_0} = 10$$

Choose:

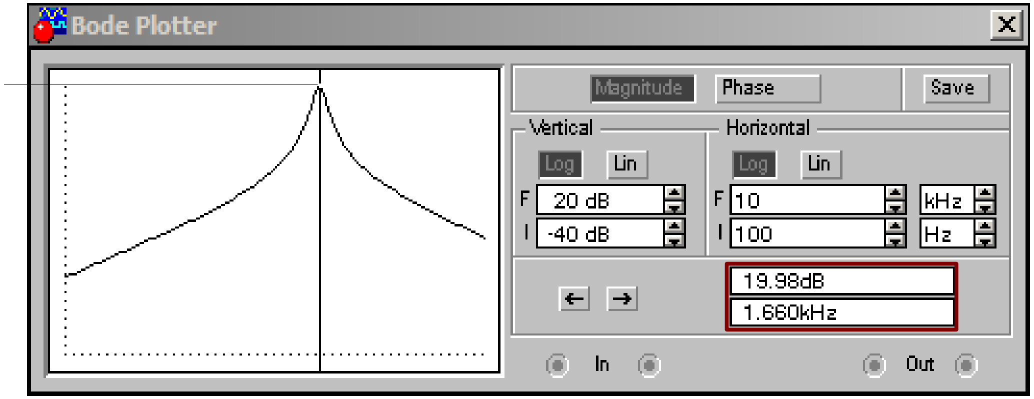
$$C = 10\, nF$$



Simulation Results

Peak gain

19.98 dB



Vertical: Log

F 20 dB

I -40 dB

Horizontal: Log

F 10 kHz

I 100 Hz

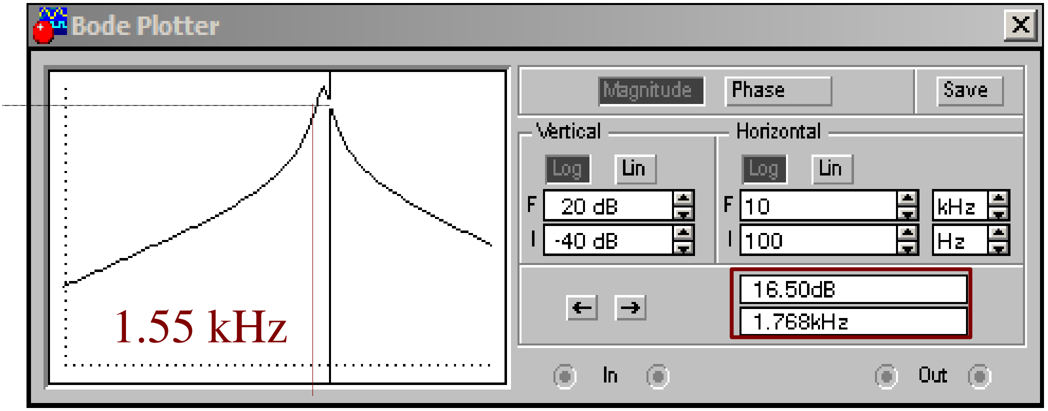
19.98dB

1.660kHz

1.66 kHz

High -3 dB point

16.50 dB



Vertical: Log

F 20 dB

I -40 dB

Horizontal: Log

F 10 kHz

I 100 Hz

16.50dB

1.768kHz

1.77 kHz

$$Q = \frac{1.77 - 1.55}{1.66} = 7.54$$