ESE 3400: Medical Devices Lab

Lec 10: October 17, 2022

Discrete Time Signals and Systems, DTFT

Lecture Outline

- Discrete Time Signals
  - Aliasing and periodicity
- Discrete Time Systems
  - LTI Systems
  - Convolution
  - Frequency Response

Discrete Time Signals

Signals

DEFINITION

Signal (n): A detectable physical quantity . . . by which messages or information can be transmitted (Merriam-Webster)

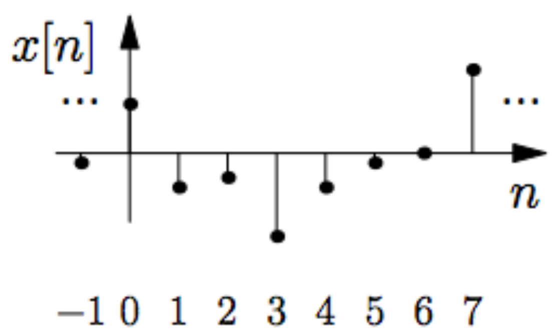
- Signals carry information
- Examples:
  - Speech signals transmit language via acoustic waves
  - Radar signals transmit the position and velocity of targets via electromagnetic waves
  - Electrophysiology signals transmit information about processes inside the body
  - Financial signals transmit information about events in the economy
- Signal processing systems manipulate the information carried by signals

Signals are Functions

DEFINITION

A **signal** is a function that maps an independent variable to a dependent variable.

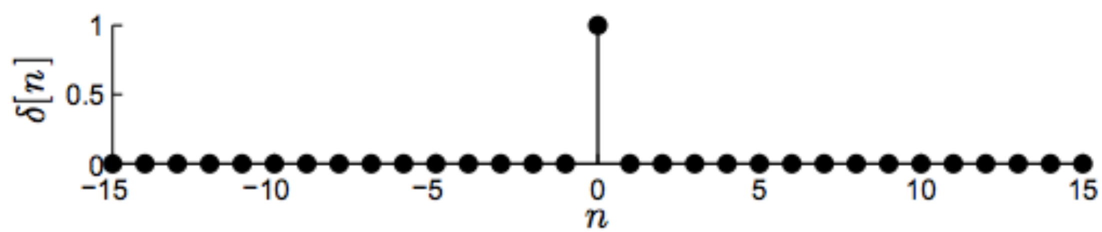
- Signal $x[n]$: each value of n produces the value $x[n]$
- In this course we will focus on **discrete-time** signals:
  - Independent variable is an **integer**: $n \in Z$ (will refer to n as time)
  - Dependent variable is a real or complex number: $x[n] \in R$



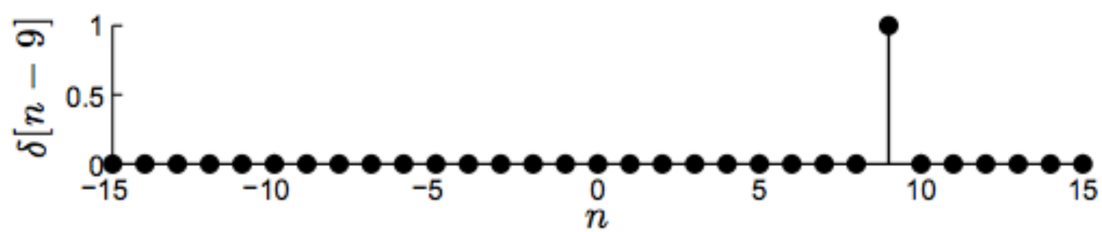
Unit Sample

DEFINITION

The **delta function** (aka unit impulse) $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$



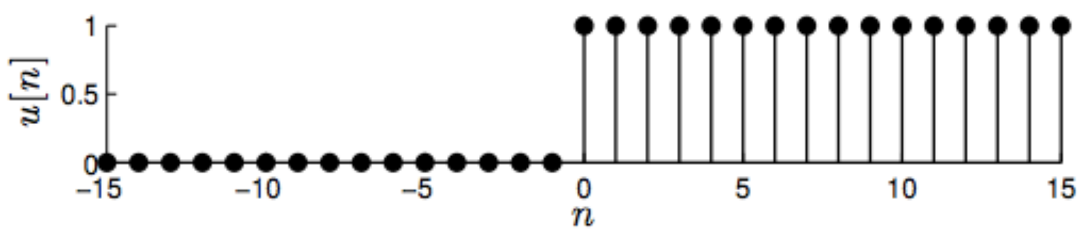
- The shifted delta function $\delta[n-m]$ peaks up at $n = m$; here $m = 9$



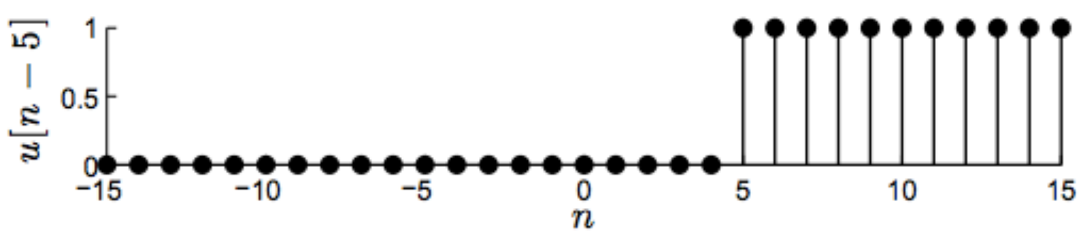
Unit Step

DEFINITION

The **unit step** $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$



- The shifted unit step $u[n-m]$ jumps from 0 to 1 at $n=m$; here, m=5

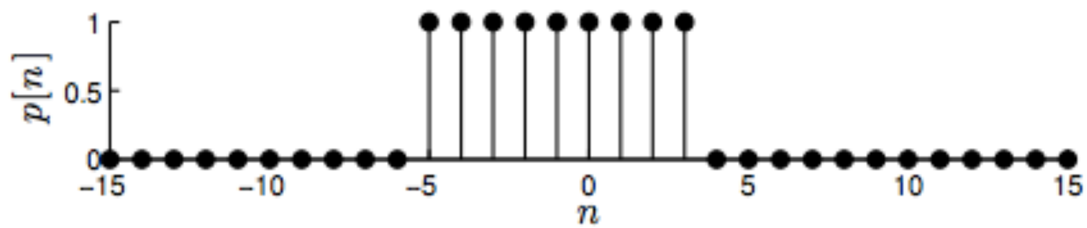


Unit Pulse

DEFINITION

The **unit pulse** (aka boxcar) $p[n] = \begin{cases} 0 & n < N_1 \\ 1 & N_1 \le n \le N_2 \\ 0 & n > N_2 \end{cases}$

- Ex: $p[n]$ for $N_1 = -5$ and $N_2 = 3$



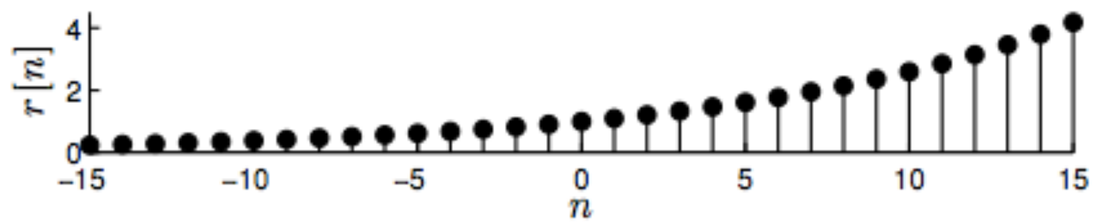
- One of many different formulas for the unit pulse

$$p[n] = u[n - N_1] - u[n - (N_2 + 1)]$$

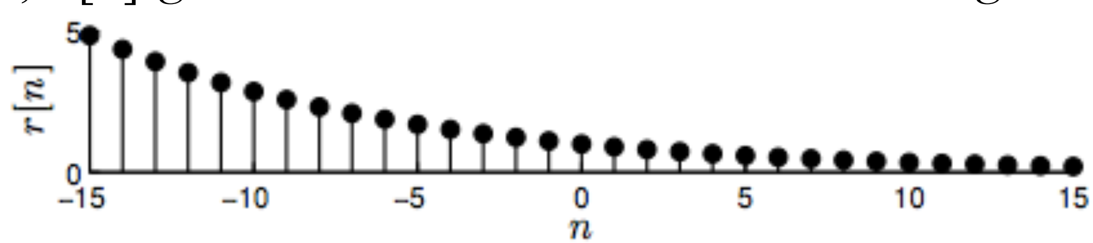
Real Exponential

DEFINITION The **real exponential** $r[n] = a^n$, $a \in \mathbb{R}$, $a \geq 0$

- For $a > 1$, $r[n]$ shrinks to the left and grows to the right; here $a = 1.1$

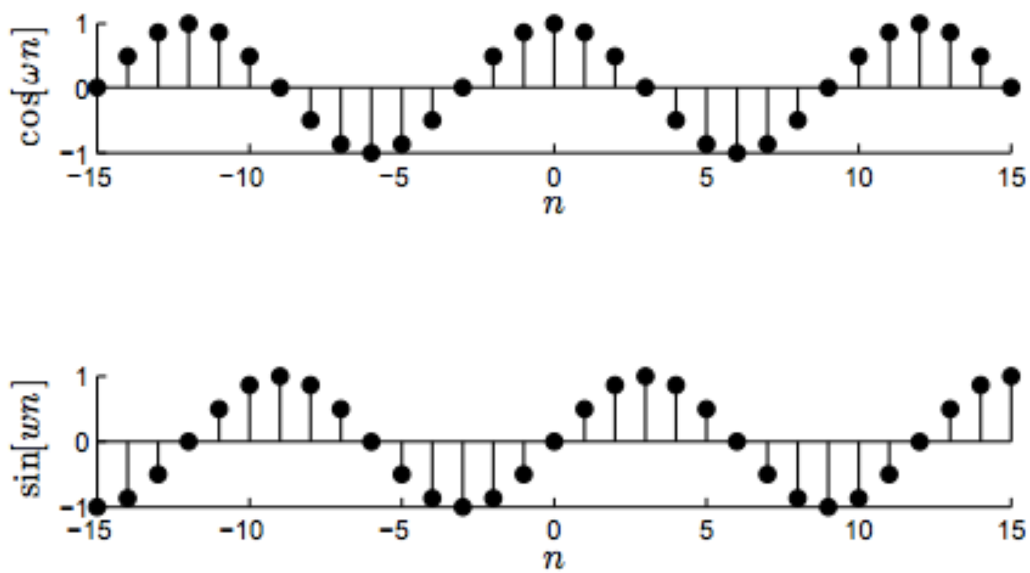


- For $0 < a < 1$, $r[n]$ grows to the left and shrinks to the right; here $a = 0.9$



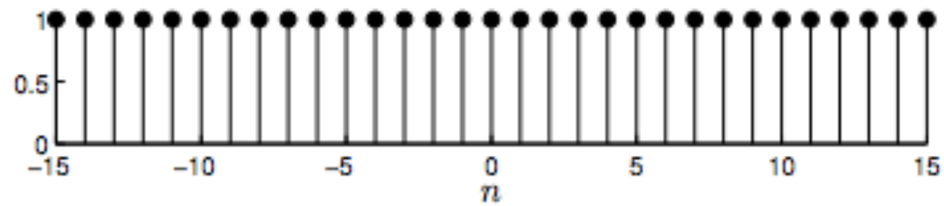
Sinusoids

- There are two natural real-value sinusoids: $\cos(\omega n + \phi)$ and $\sin(\omega n + \phi)$
- **Frequency:** ω (units: radians/sample)
- **Phase:** ϕ (units: radians)

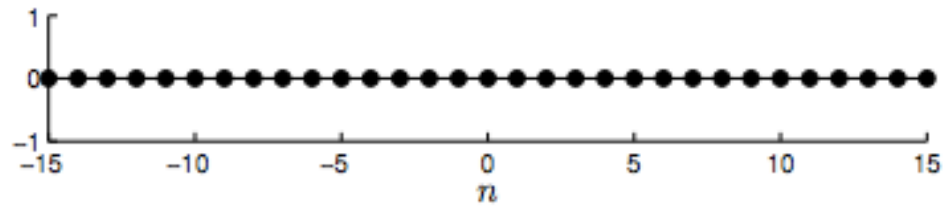


Sinusoid Examples

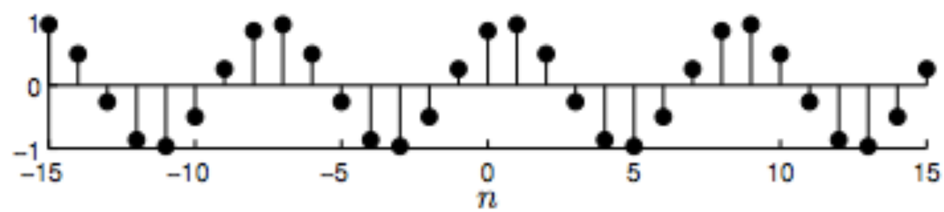
- $\cos(0n)$



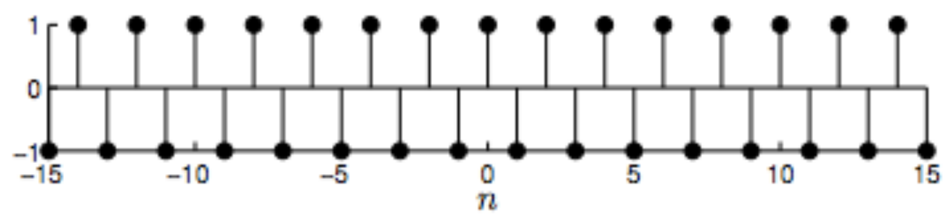
- $\sin(0n)$



- $\sin\left(\frac{\pi}{4}n + \frac{2\pi}{6}\right)$



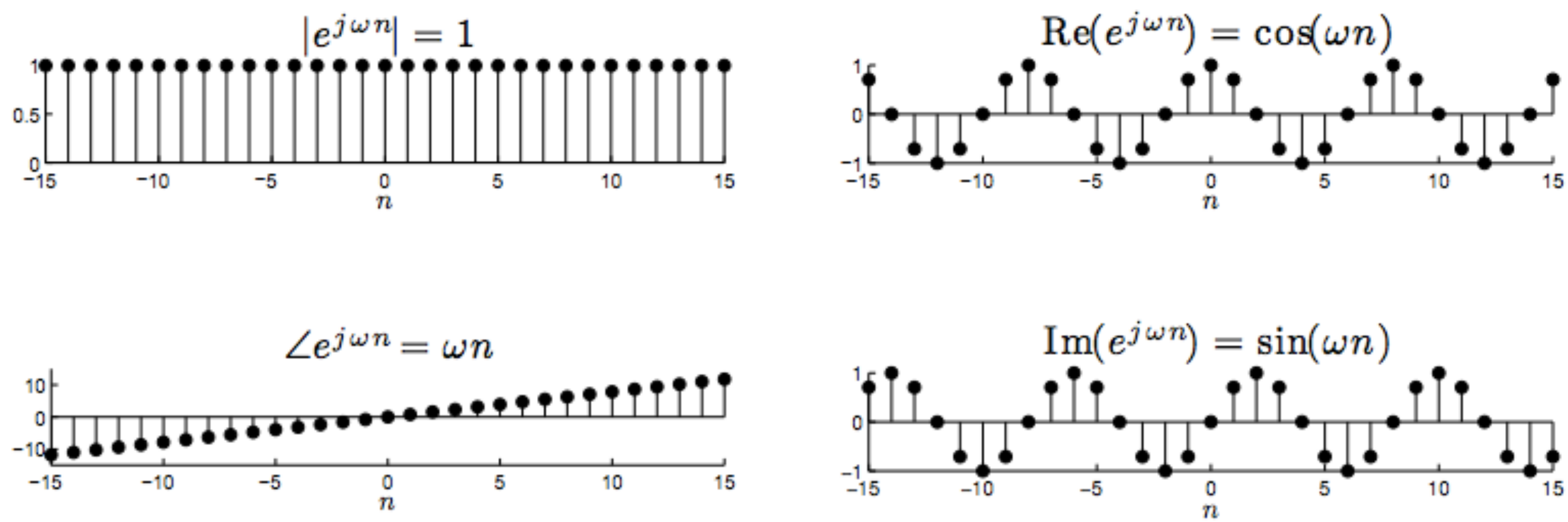
- $\cos(\pi n)$



Complex Sinusoid

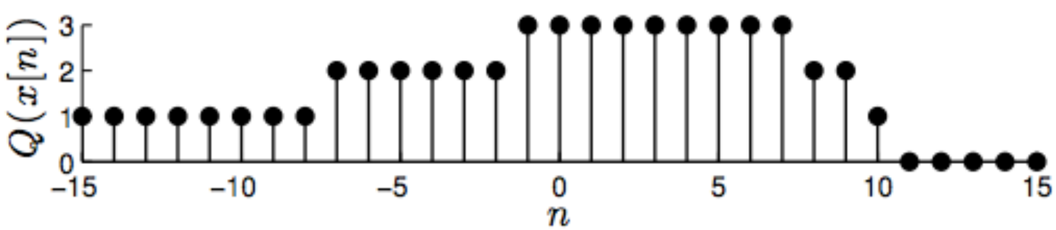
- The complex-value sinusoid combines both the cos and sin terms using Euler's identity:

$$e^{j(\omega n+\phi)} = \cos(\omega n+\phi) + j\sin(\omega n+\phi)$$



Digital Signals

- **Digital signals** are a special subclass of discrete-time signals
  - Independent variable is still an integer: $n \in \mathbb{Z}$
  - Dependent variable is from a finite set of integers: $x[n] \in \{0, 1, \ldots, D-1\}$
  - Typically, choose $D=2^q$ and represent each possible level of $x[n]$ as a digital code with q bits
  - Ex. Digital signal with $q=2$ bits --> D=4 levels



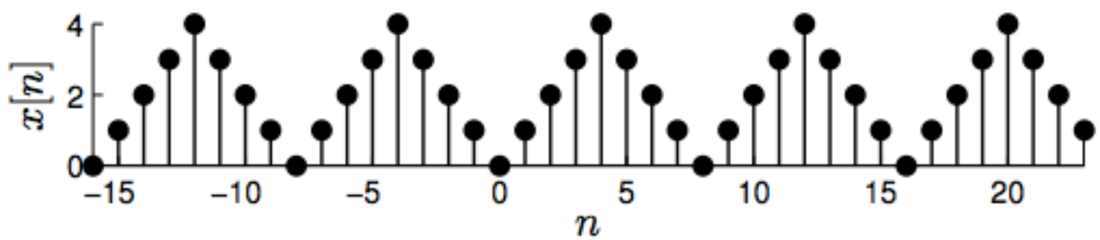
Signal Properties

Periodic Signals

DEFINITION

A discrete-time signal is **periodic** if it repeats with period $N \in \mathbb{Z}$:

$$x[n + mN] = x[n] \quad \forall m \in \mathbb{Z}$$



Notes:

- The period N must be an integer
- A periodic signal is infinite in length

DEFINITION

A discrete-time signal is **aperiodic** if it is not periodic

Discrete-Time Sinusoids

- Discrete-time sinusoids $e^{j(\omega n+\phi)}$ have two counterintuitive properties
- Both involve the frequency ω
- **Property #1**: Aliasing
- **Property #2**: Aperiodicity

Property #1: Aliasing of Sinusoids

- Consider two sinusoids with two different frequencies

$$\omega \quad \Rightarrow \quad x_1[n] = e^{j(\omega n + \phi)}$$
$$\omega + 2\pi \quad \Rightarrow \quad x_2[n] = e^{j((\omega + 2\pi)n + \phi)}$$

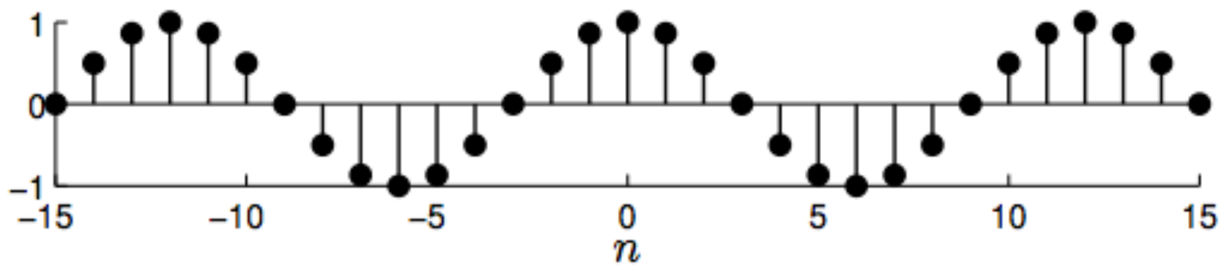
- But note that

$$x_2[n] = e^{j((\omega+2\pi)n+\phi)} = e^{j(\omega n+\phi)+j2\pi n} = e^{j(\omega n+\phi)}\, e^{j2\pi n} = e^{j(\omega n+\phi)} = x_1[n]$$

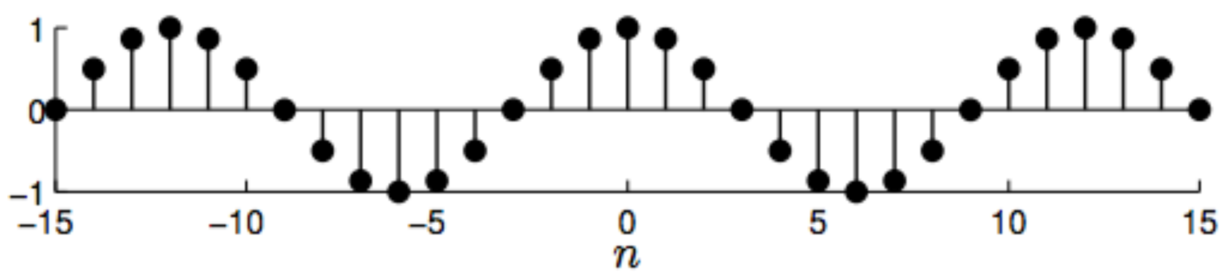
- The signals x_1 and x_2 have different frequencies but are **identical!**
- We say that x_1 and x_2 are aliases; this phenomenon is called aliasing
- Note: Any integer multiple of 2π will do; try with $x_3[n] = e^{j((\omega+2\pi m)n+\phi)},\ m \in \mathbb{Z}$

Aliasing Example

$x_1[n] = \cos\left(\frac{\pi}{6}n\right)$

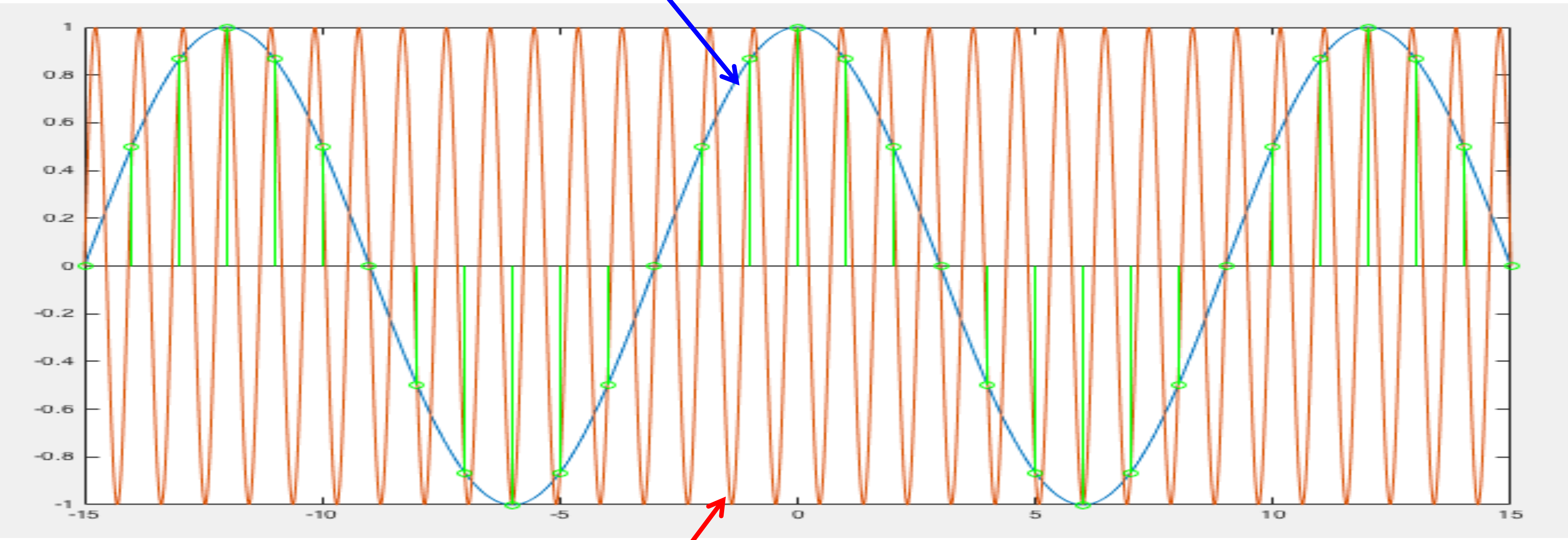


$x_2[n] = \cos\left(\frac{13\pi}{6}n\right) = \cos\left(\left(\frac{\pi}{6} + 2\pi\right)n\right)$



Aliasing Example

$x_1[n] = \cos\left(\frac{\pi}{6}n\right)$

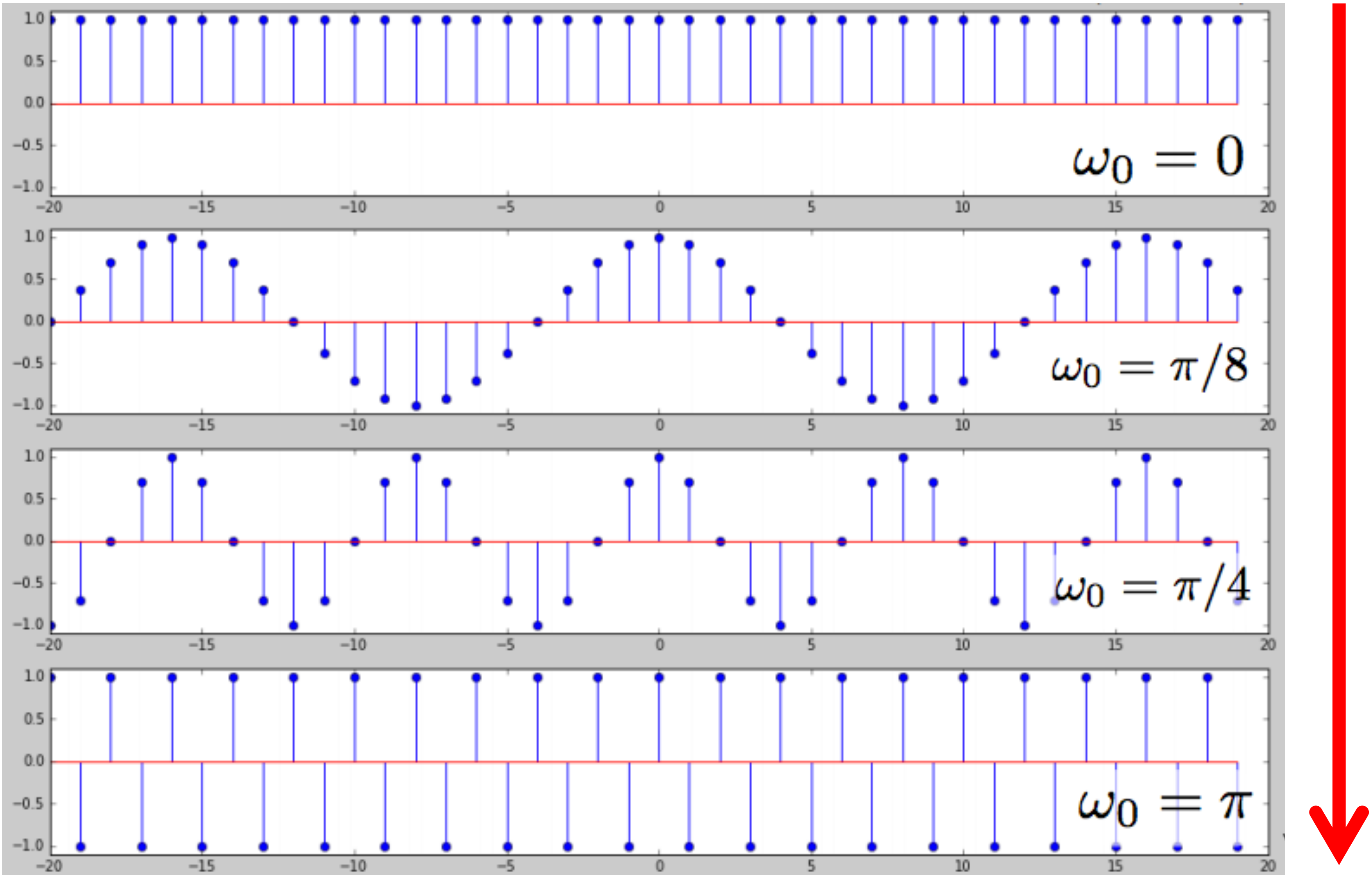


$x_2[n] = \cos\left(\frac{13\pi}{6}n\right) = \cos\left(\left(\frac{\pi}{6} + 2\pi\right)n\right)$

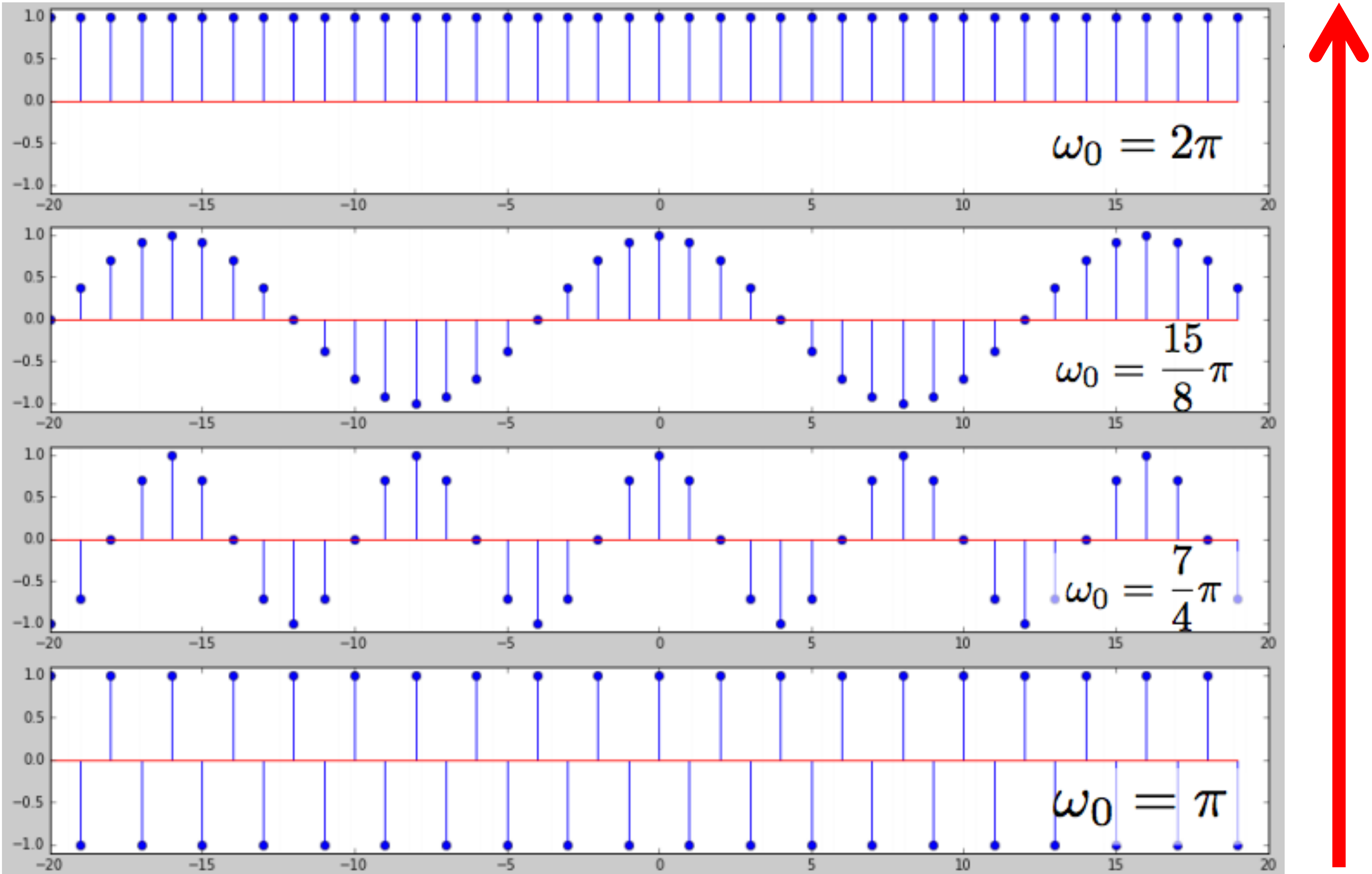
Which is higher in frequency?

- $\cos(\pi n)$ or $\cos(3\pi/2 n)$?

Increasing Frequency



Decreasing Frequency



Property #2: Periodicity of Sinusoids

- Consider $x_1[n] = e^{j(\omega n+\phi)}$ with frequency $\omega = \frac{2\pi k}{N}$, $\quad k, N \in \mathbb{Z}$ (harmonic frequency)

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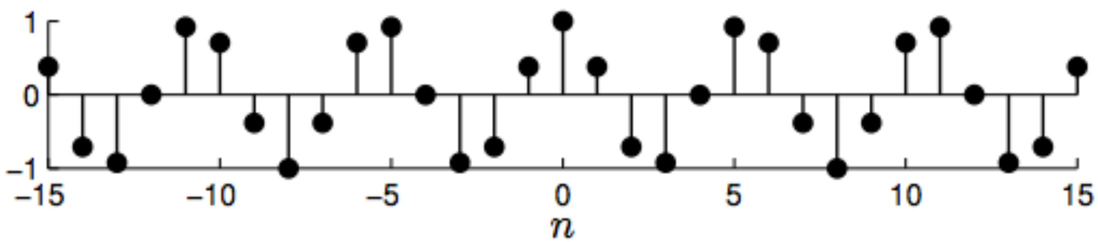
- Consider $x_1[n] = e^{j(\omega n+\phi)}$ with frequency $\omega = \frac{2\pi k}{N}$, $\quad k, N \in \mathbb{Z}$ (harmonic frequency)

- It is easy to show that x_1 is periodic with period N, since

$$x_1[n+N] = e^{j(\omega(n+N)+\phi)} = e^{j(\omega n+\omega N+\phi)} = e^{j(\omega n+\phi)}\, e^{j(\omega N)} = e^{j(\omega n+\phi)}\, e^{j(\frac{2\pi k}{N} N)} = x_1[n] \checkmark$$

- Ex: $x_1[n] = \cos(\frac{2\pi 3}{16} n)$, $\quad N = 16$



- Note: x_1 is periodic with the (smaller) period of $\frac{N}{k}$ when $\frac{N}{k}$ is an integer

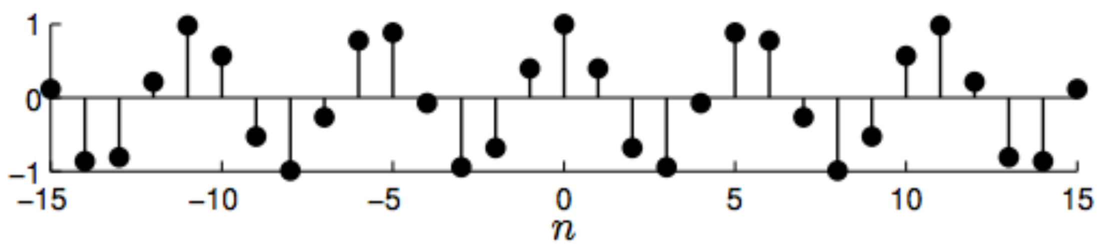
Aperiodicity of Sinusoids

- Consider $x_2[n] = e^{j(\omega n+\phi)}$ with frequency $\omega \neq \frac{2\pi k}{N}$, $\quad k, N \in \mathbb{Z}$ (not harmonic frequency)

- Is x_2 periodic?

$$x_2[n+N] = e^{j(\omega(n+N)+\phi)} = e^{j(\omega n+\omega N+\phi)} = e^{j(\omega n+\phi)}\, e^{j(\omega N)} \neq x_1[n] \quad \text{NO!}$$

- Ex: $x_2[n] = \cos(1.16\, n)$



- If its frequency ω is not harmonic, then a sinusoid oscillates but is not periodic!

Harmonic Sinusoids

$$e^{j(\omega n+\phi)}$$

- Semi-amazing fact: The **only** periodic discrete-time sinusoids are those with **harmonic frequencies**

$$\omega = \frac{2\pi k}{N}, \quad k, N \in \mathbb{Z}$$

- Which means that
  - **Most** discrete-time sinusoids are **not** periodic!
  - The harmonic sinusoids are somehow magical (they play a starring role later in the DFT)

Periodic or not?

- $\cos(5/7\pi n)$

- $\cos(\pi/5n)$

- What are N and k? (I.e How many samples is one period?

Periodic or not?

- $\cos(5/7\pi n)$
  - $N=14$, $k=5$
  - $\cos(5/14*2\pi n)$
  - Repeats every $N=14$ samples
- $\cos(\pi/5n)$
  - $N=10$, $k=1$
  - $\cos(1/10*2\pi n)$
  - Repeats every $N=10$ samples

Periodic or not?

- $\cos(5/7\pi n)$
  - $N=14$, $k=5$
  - $\cos(5/14*2\pi n)$
  - Repeats every $N=14$ samples
- $\cos(\pi/5n)$
  - $N=10$, $k=1$
  - $\cos(1/10*2\pi n)$
  - Repeats every $N=10$ samples

- $\cos(5/7\pi n)+\cos(\pi/5n)$?

Periodic or not?

- $\cos(5/7\pi n)+\cos(\pi/5n)$?
  - N=SCM{10,14}=70
  - $\cos(5/7*\pi n)+\cos(\pi/5n)$
    - $n=N=70 \rightarrow \cos(5/7*70\pi)+\cos(\pi/5*70)=\cos(25*2\pi)+\cos(7*2\pi)$

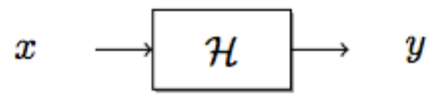
Discrete-Time Systems

Discrete Time Systems

DEFINITION

A discrete-time **system** $\mathcal{H}$ is a transformation (a rule or formula) that maps a discrete-time input signal x into a discrete-time output signal y

$$y = \mathcal{H}\{x\}$$



- Systems manipulate the information in signals
- Examples:
  - A speech recognition system converts acoustic waves of speech into text
  - A radar system transforms the received radar pulse to estimate the position and velocity of targets
  - A fMRI system transforms measurements of electron spin into voxel-by-voxel estimates of brain activity
  - A 30 day moving average smooths out the day-to-day variability in a stock price

System Properties

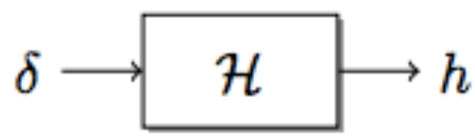
- Causality
  - y[n] only depends on x[m] for m<=n
- Linearity
  - Scaled sum of arbitrary inputs results in output that is a scaled sum of corresponding outputs
    - $Ax_1[n]+Bx_2[n] \rightarrow Ay_1[n]+By_2[n]$
- Memoryless
  - y[n] depends only on x[n]
- Time Invariance
  - Shifted input results in shifted output
    - $x[n-q] \rightarrow y[n-q]$
- BIBO Stability
  - A bounded input results in a bounded output (ie. max signal value exists for output if max)

LTI Systems

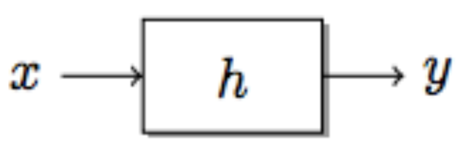
DEFINITION

A system $\mathcal{H}$ is **linear time-invariant** (LTI) if it is both linear and time-invariant

- LTI system can be completely characterized by its impulse response



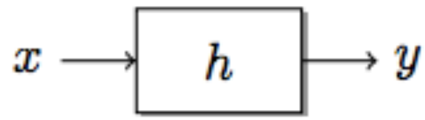
- Then the output for an arbitrary input is a sum of weighted, delay impulse responses



$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

$$y[n] = x[n] * h[n]$$

Convolution



- Convolution formula:

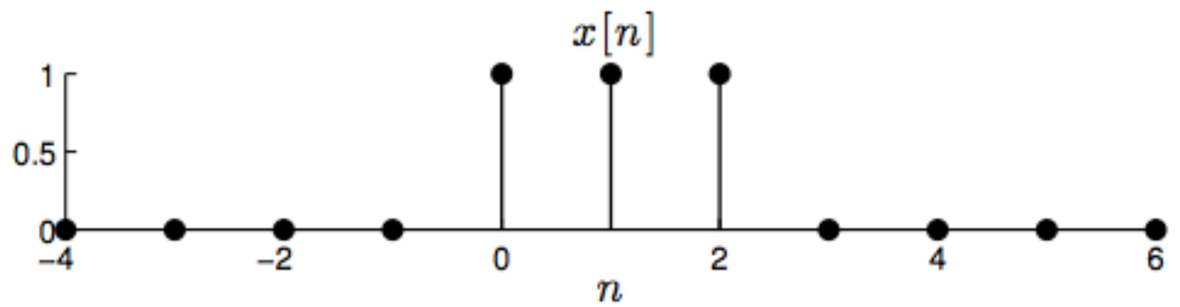
$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- Convolution method:
  - 1) Time reverse the impulse response and shift it *n* time steps to the right
  - 2) Compute the inner product between the shifted impulse response and the input vector
  - Repeat for evey *n*

Convolution Example

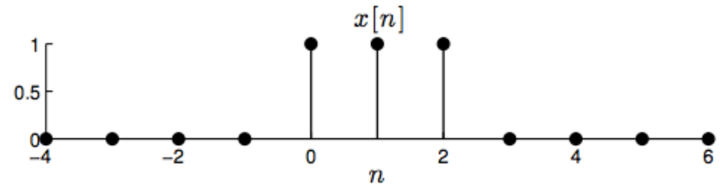
$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- Convolve a unit pulse with itself



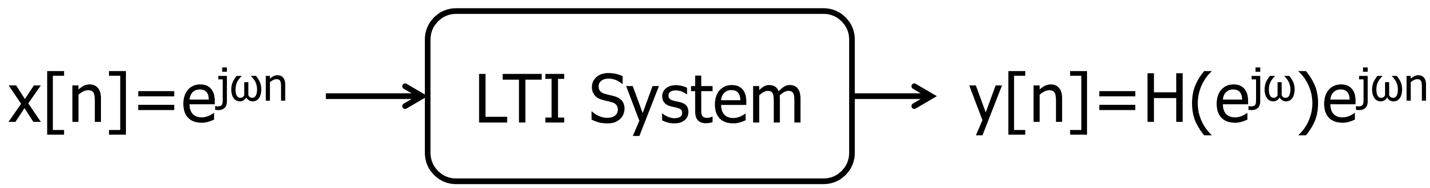
Convolution Example

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$



LTI System Frequency Response

- (DT)Fourier Transform of impulse response



$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

Eigenvalue (frequency response)

- $x[n]=e^{j\omega n}$

$$y[n]=\sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

$$=\sum_{k=-\infty}^{\infty} e^{j\omega(n-k)}h[k]$$

$$=e^{j\omega n}\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

$$=H(e^{j\omega})e^{j\omega n}$$

$$H(e^{j\omega})=\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

- Describes the change in amplitude and phase of signal at frequency ω
- Frequency response
- Complex value
  - Re and Im
  - Mag and Phase

DT Frequency Response

- $H(e^{j(\omega+2\pi)})$?

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

DT Frequency Response

- $H(e^{j(\omega+2\pi)})$?

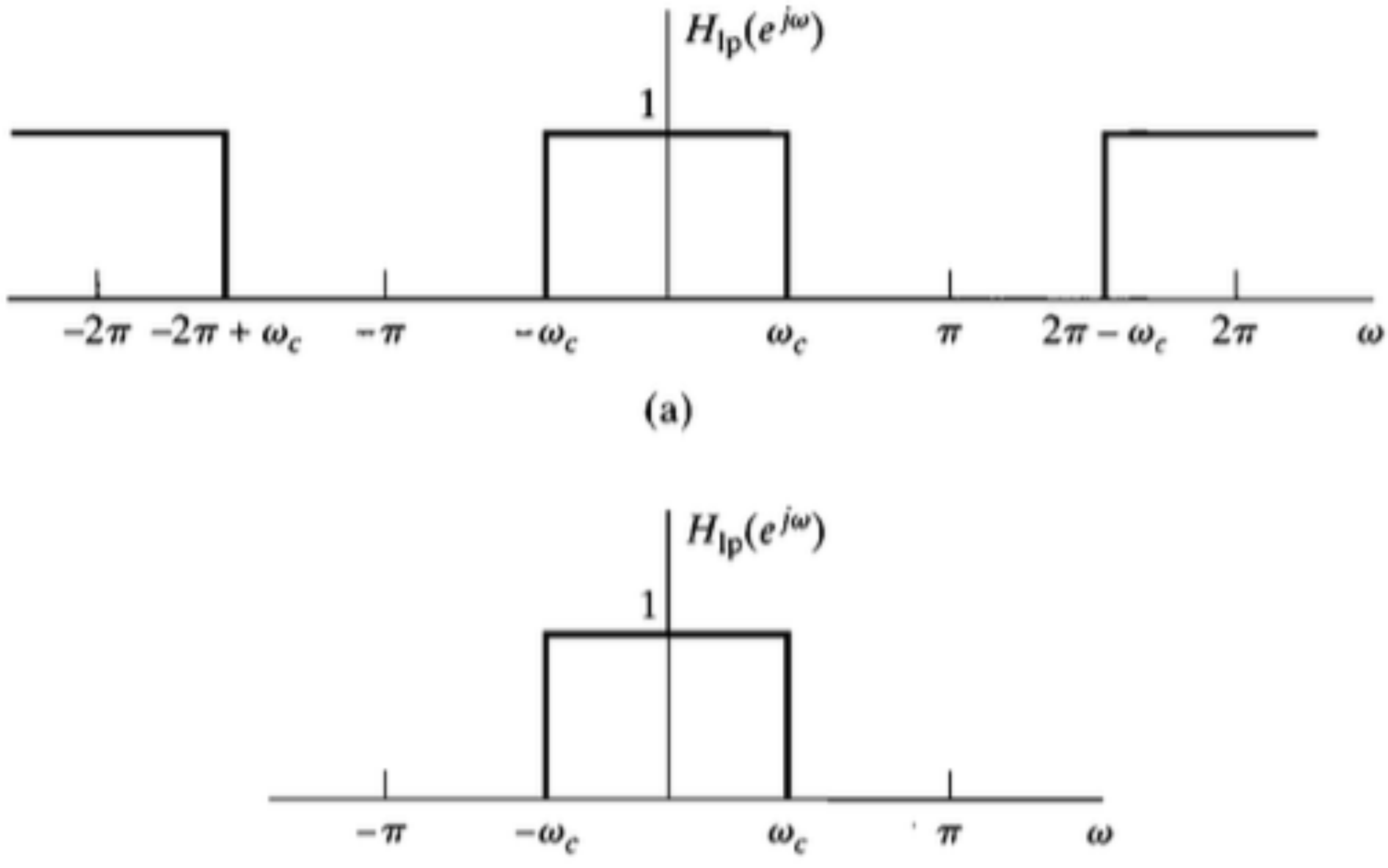
$$H(e^{j(\omega+2\pi)n}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j(\omega+2\pi)k}$$

$$= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}e^{-j2\pi k}$$

$$= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

$$= H(e^{j\omega n})$$

Periodicity of Low Pass Freq Response

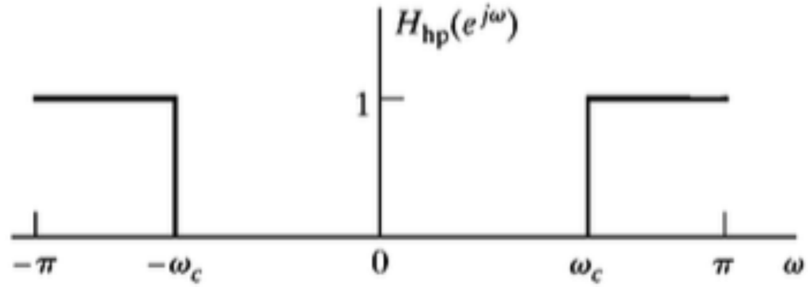


(a)

(b)

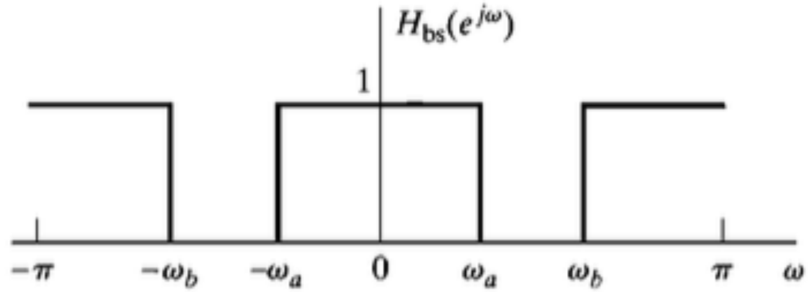
Other Filters

High-pass



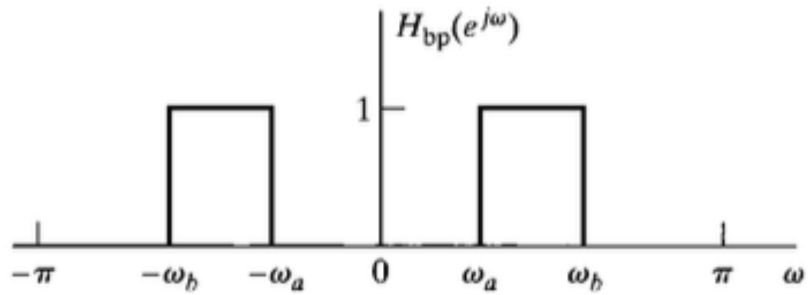
(a)

Band-stop



(b)

Band-pass



(c)

Big Ideas

- Discrete Time Signals
  - Sample at Nyquist to avoid aliasing
- Discrete Time Systems
  - LTI Systems are predictable and practical to implement
  - Convolution to evaluate output in time domain
  - Frequency Response to evaluate output in frequency domain
    - Put a frequency in, get the same frequency out

Admin

- Lab tomorrow and next week
  - Populate PCBs