ESE 3400: Medical Devices Lab

Lec 14: November 2, 2022

Digital Filters Pt 1

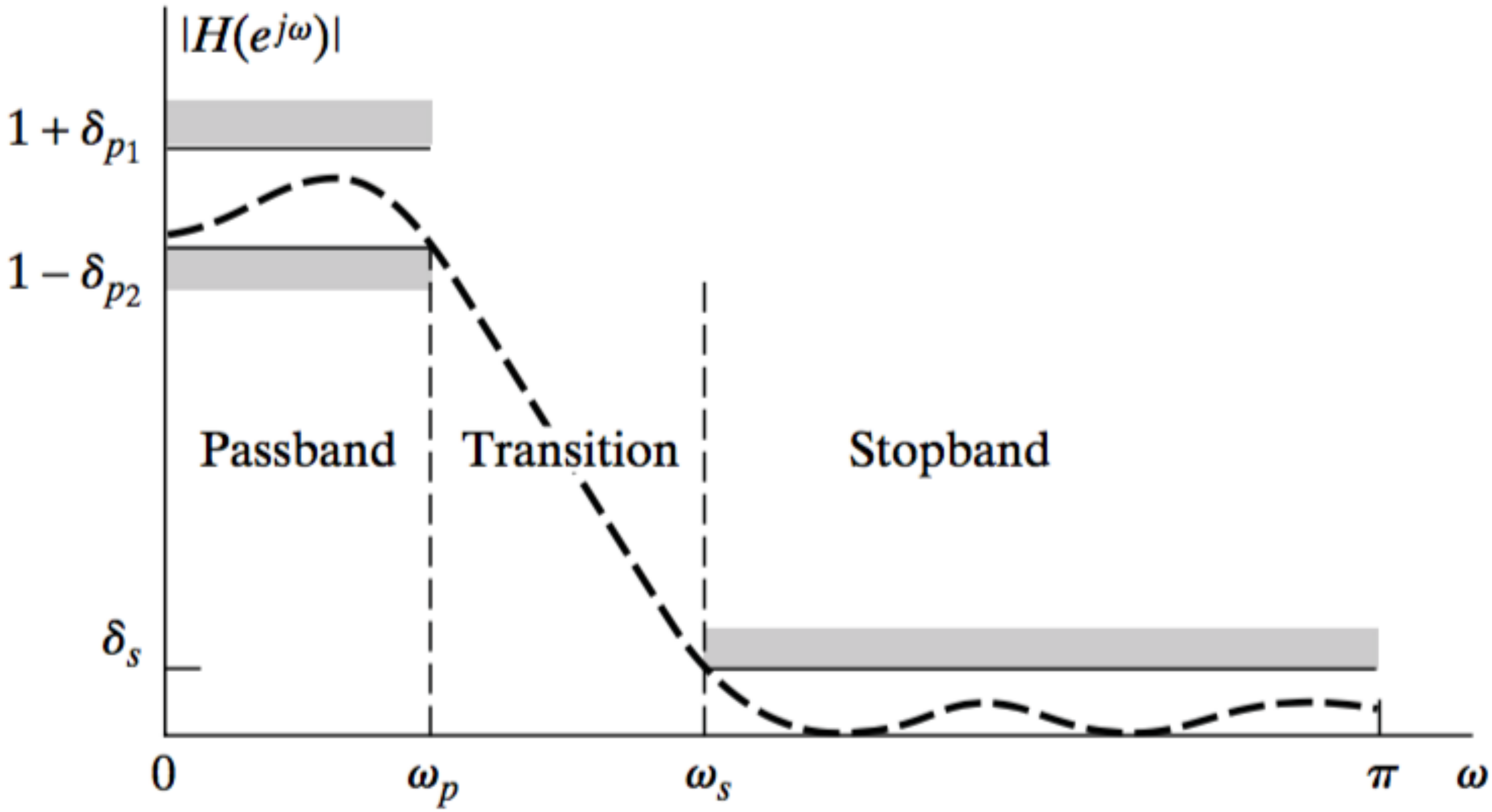
Linear Filter Design

- Used to be an art
  - Now, lots of tools to design optimal filters
- For DSP there are two common classes
  - Infinite impulse response IIR
  - Finite impulse response FIR
- We will focus on FIR designs

What is a Linear Filter?

- Attenuates certain frequencies
- Passes certain frequencies
- Affects both phase and magnitude

Filter Specifications



CT Filters

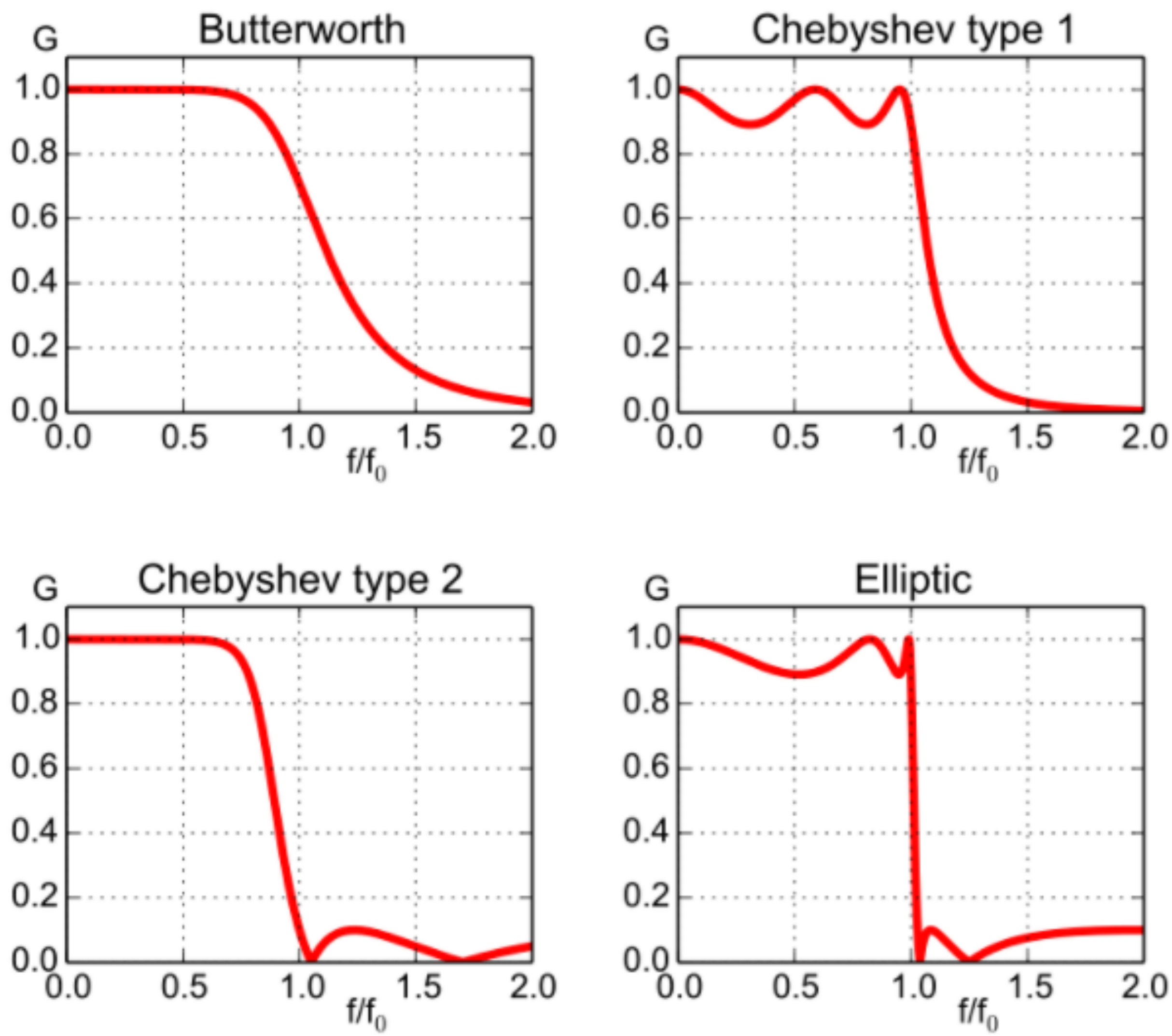
- Butterworth
  - Monotonic in pass and stop bands
- Chebyshev, Type I
  - Equiripple in pass band and monotonic in stop band
- Chebyshev, Type II
  - Monotonic in pass band and equiripple in stop band
- Elliptic
  - Equiripple in pass and stop bands

Design Comparison

- Design specifications
  - passband edge frequency $\omega_p = 0.5\pi$
  - stopband edge frequency $\omega_s = 0.6\pi$
  - maximum passband gain = 0 dB
  - minimum passband gain = -0.3dB
  - maximum stopband gain =-30dB

- Use bilinear transformation to design DT low pass filter for each type

Comparisons



DTFT Definition

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

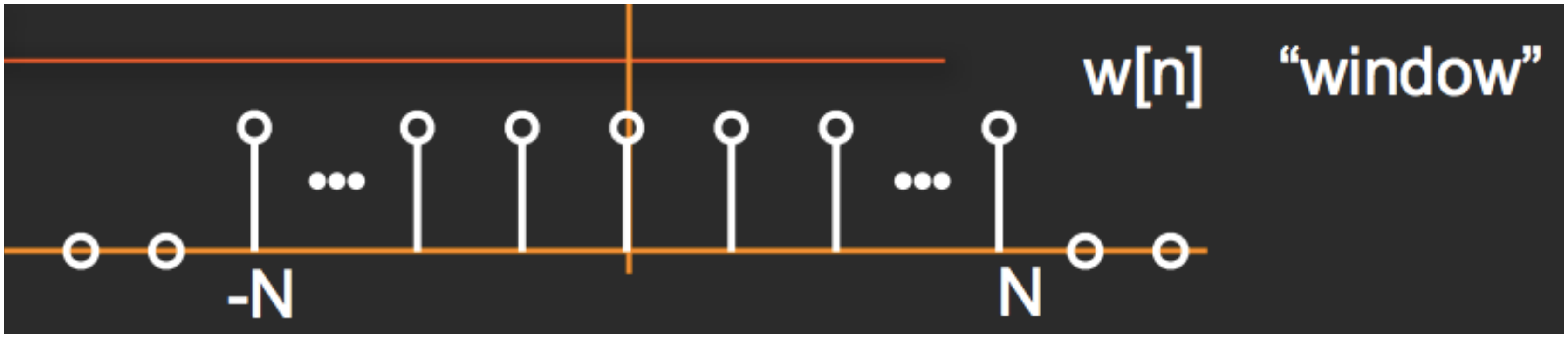
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Alternate

$$X(f) = \sum_{k=-\infty}^{\infty} x[k]e^{-j2\pi fk}$$

$$x[n] = \int_{-0.5f_s}^{0.5f_s} X(f)e^{j2\pi fn}df$$

Example: Window DTFT



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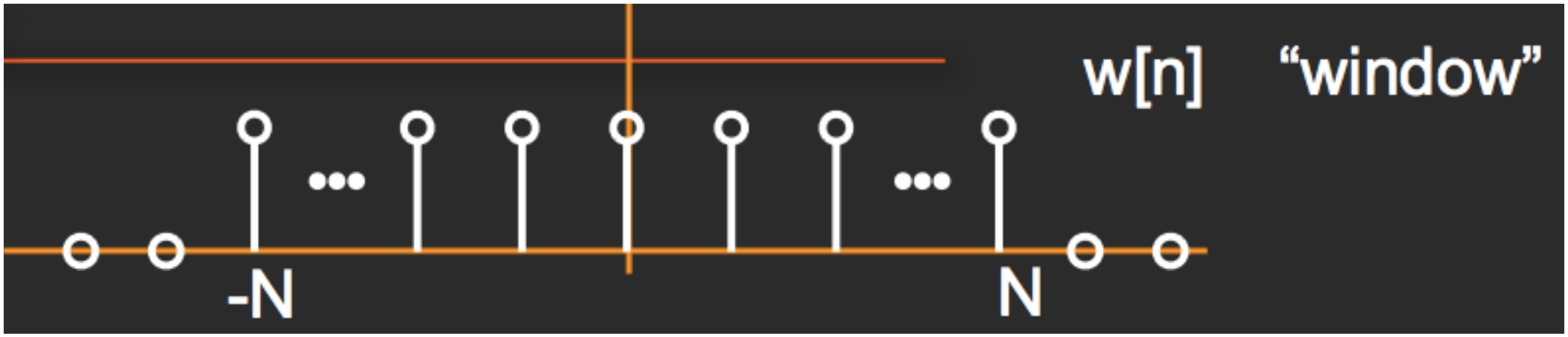

$$W(e^{j\omega}) = \sum_{k=-\infty}^{\infty} w[k]e^{-j\omega k}$$

$$= \sum_{k=-N}^{N} e^{-j\omega k}$$

Example: Window DTFT

$$W(e^{j\omega}) = \sum_{k=-N}^{N} e^{-j\omega k}$$

Useful sum: $1 + p + p^2 + \ldots + p^M = \dfrac{1 - p^{M+1}}{1 - p}$

Example: Window DTFT

$$W(e^{j\omega}) = \sum_{k=-N}^{N} e^{-j\omega k}$$

$$= e^{j\omega N} + e^{j\omega(N-1)} + \ldots + e^{j\omega 0} + \ldots e^{-j\omega(N-1)} + e^{-j\omega N}$$

$$= e^{-j\omega N}(1 + e^{j\omega} + \ldots + e^{j\omega N} + \ldots + e^{j\omega(2N-1)} + e^{j\omega 2N})$$

Useful sum: $1 + p + p^2 + \ldots + p^M = \dfrac{1 - p^{M+1}}{1 - p}$

Example: Window DTFT

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Useful sum: $1 + p + p^2 + \ldots + p^M = \dfrac{1-p^{M+1}}{1-p}$

$$p = e^{j\omega} \qquad M = 2N$$

$$W(e^{j\omega}) = e^{-j\omega N}\frac{1-e^{j\omega(2N+1)}}{1-e^{j\omega}}$$

Example: Window DTFT

$$W(e^{j\omega}) = e^{-j\omega N}\frac{1-e^{j\omega(2N+1)}}{1-e^{j\omega}}$$

Example: Window DTFT

$$W(e^{j\omega}) = e^{-j\omega N}\frac{1-e^{j\omega(2N+1)}}{1-e^{j\omega}}$$

$$= \frac{e^{-j\omega N} - e^{j\omega(N+1)}}{1-e^{j\omega}}$$

$$= \frac{e^{-j\omega N} - e^{j\omega(N+1)}}{1-e^{j\omega}} \times \frac{e^{-j\omega/2}}{e^{-j\omega/2}}$$

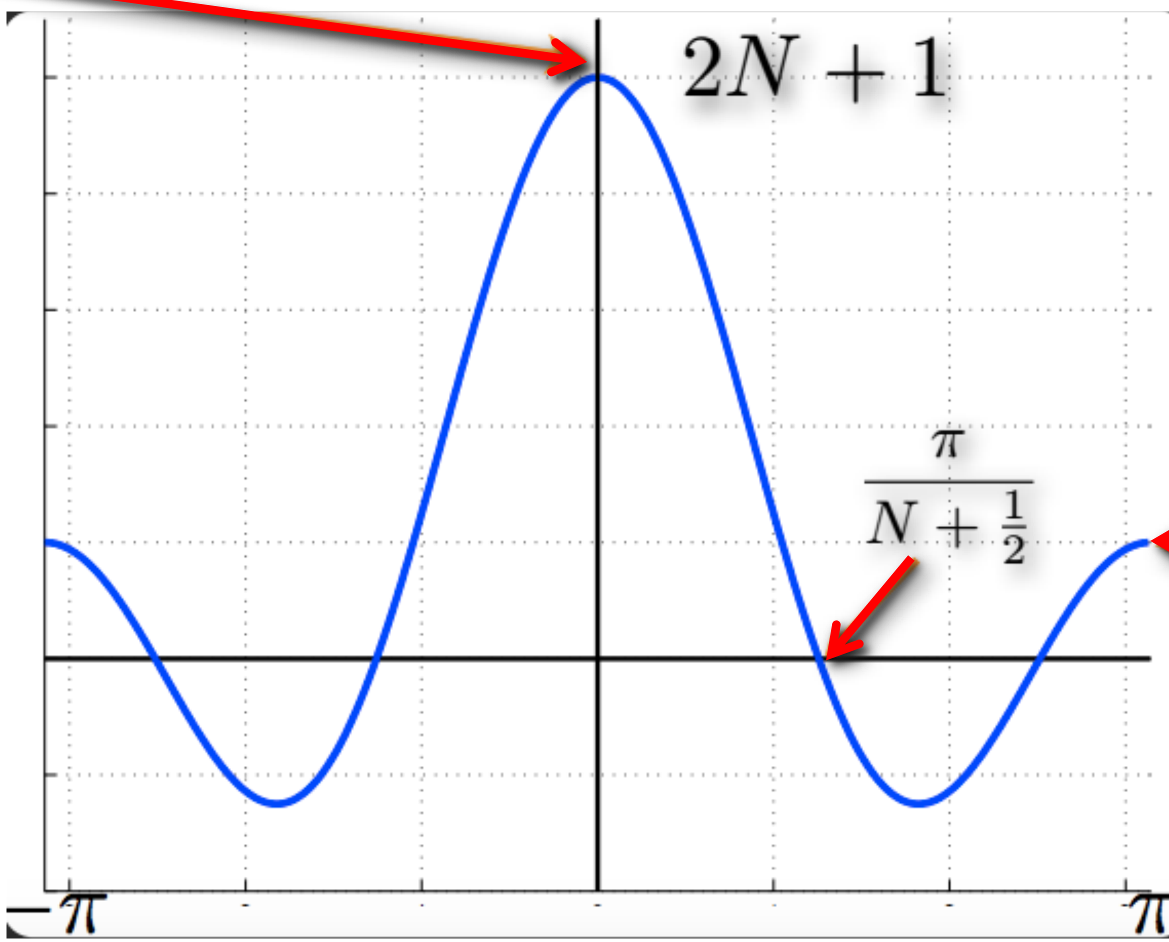
$$= \frac{e^{-j\omega(N+1/2)} - e^{j\omega(N+1/2)}}{e^{-j\omega/2} - e^{j\omega/2}} = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Periodic sinc

Example: Window DTFT

$$W(e^{j\omega}) = \frac{\sin((N+1/2)\omega)}{\sin(\omega/2)}$$

Also, Σw[n]



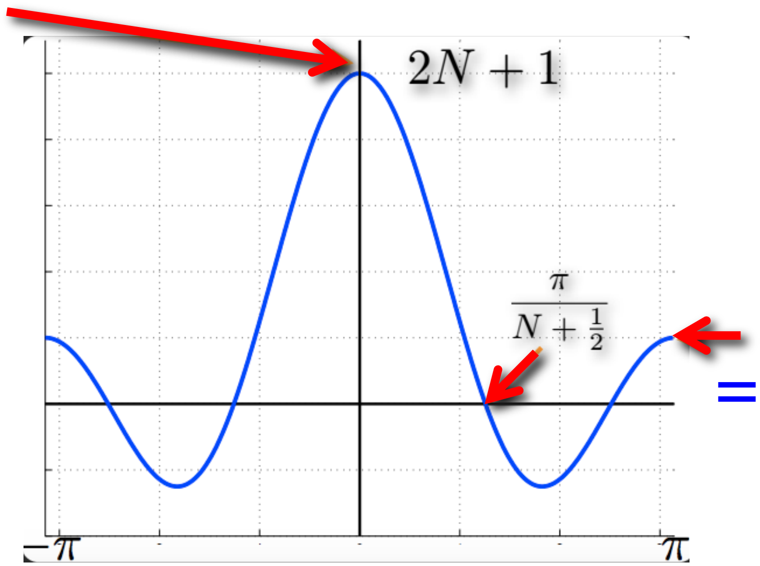
=1 why?

Plot for N=2

Example: Window DTFT

$$W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Also, Σw[n]



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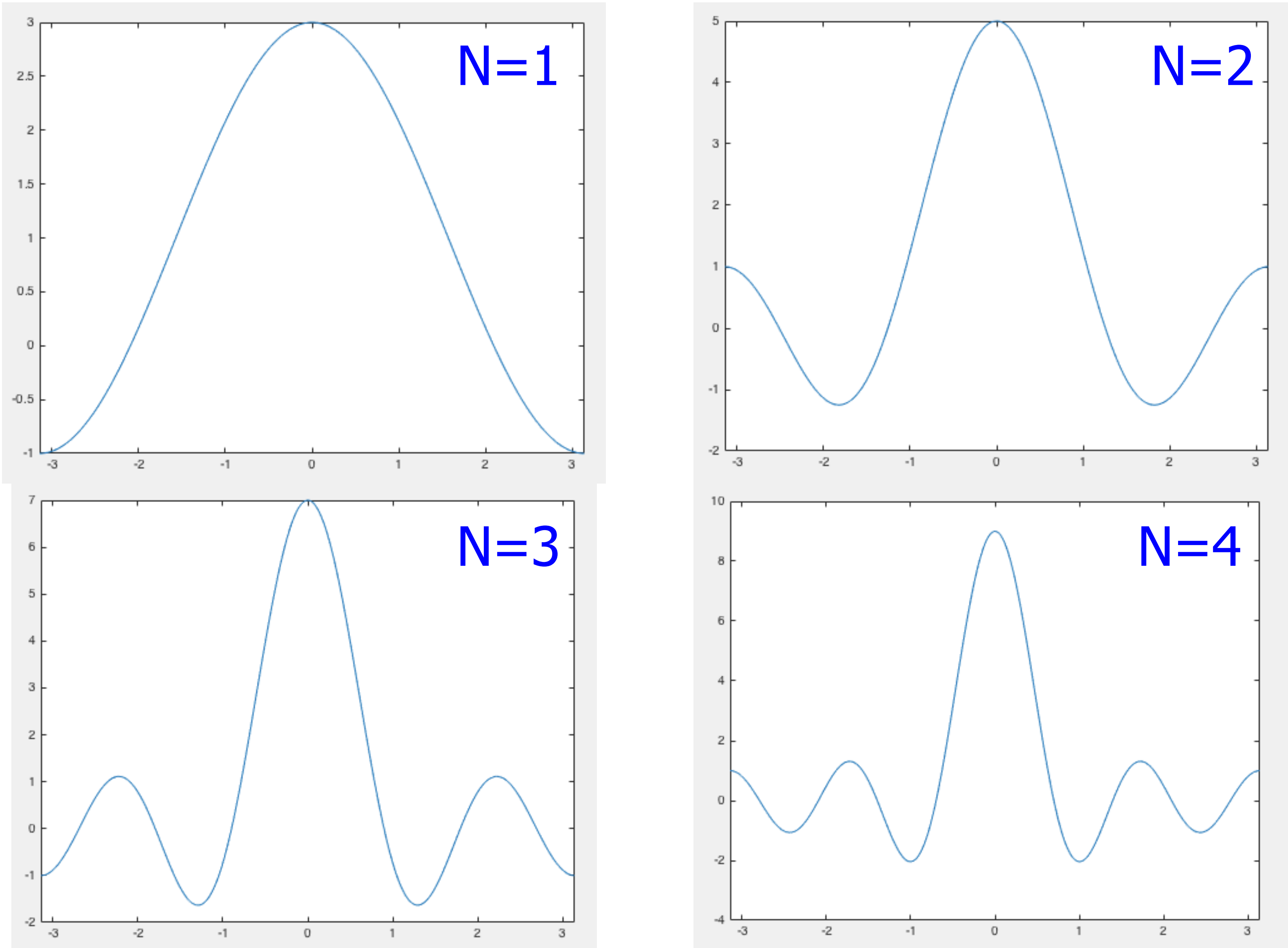
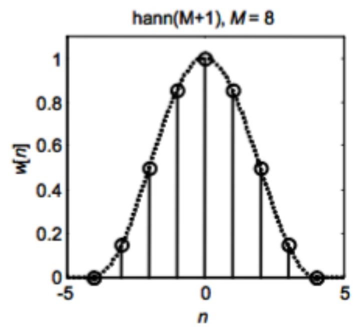
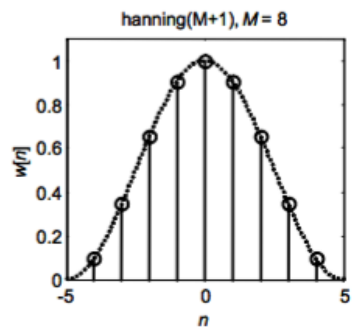
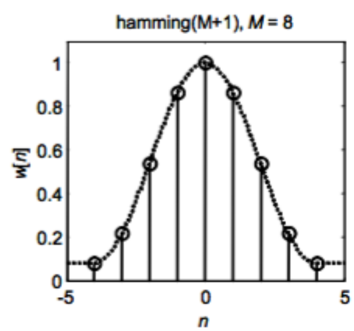
Plot for N=2

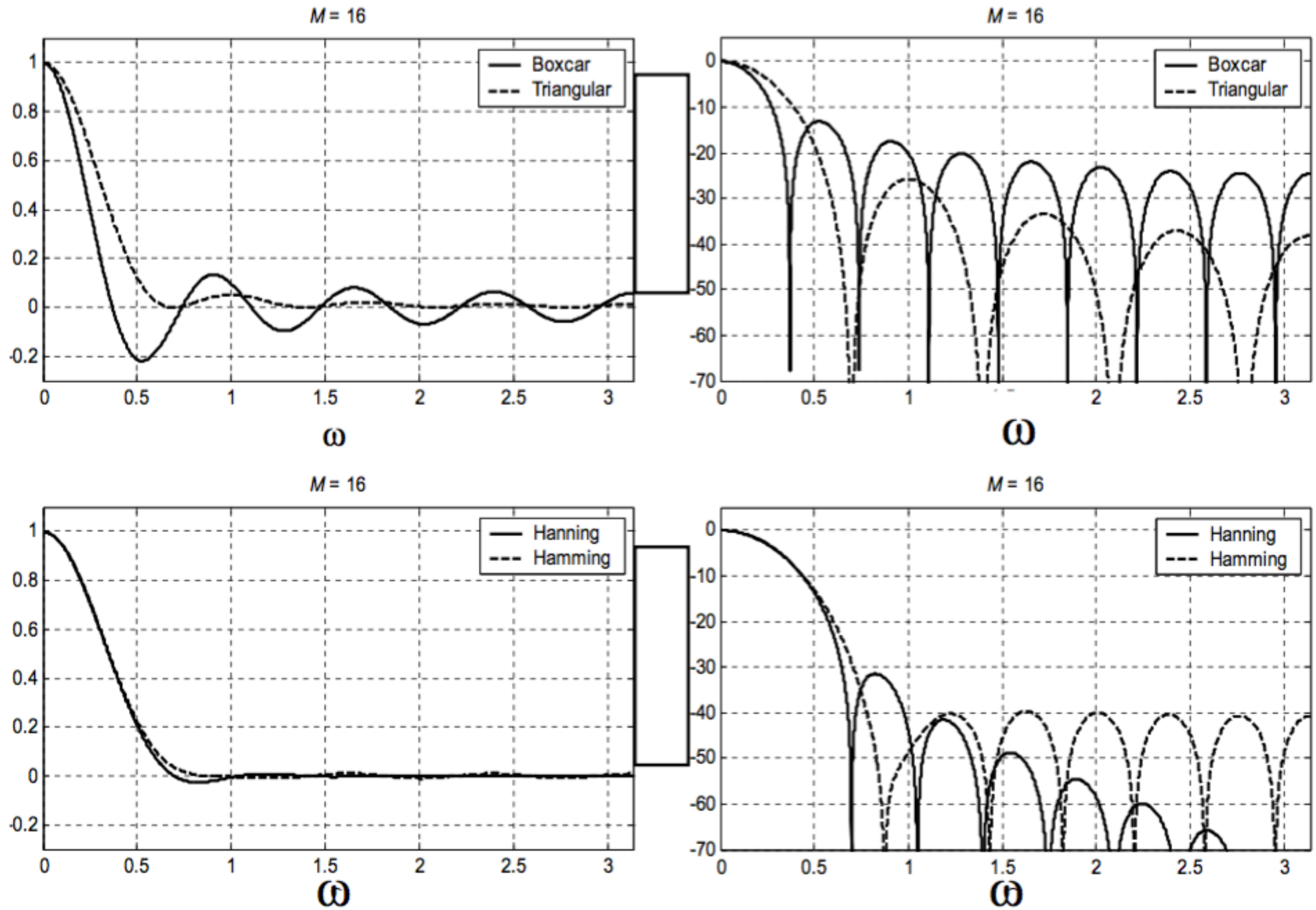
Periodic Sinc



Tapered Windows

Name(s)	Definition	MATLAB Command	Graph ($M = 8$)
Hann	$w[n] = \begin{cases} \frac{1}{2}\left[1+\cos\left(\frac{\pi n}{M/2}\right)\right] & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hann(M+1)`	
Hanning	$w[n] = \begin{cases} \frac{1}{2}\left[1+\cos\left(\frac{\pi n}{M/2+1}\right)\right] & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hanning(M+1)`	
Hamming	$w[n] = \begin{cases} 0.54+0.46\cos\left(\frac{\pi n}{M/2}\right) & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hamming(M+1)`	

Tradeoff – Ripple vs. Transition Width



Commonly Used Windows

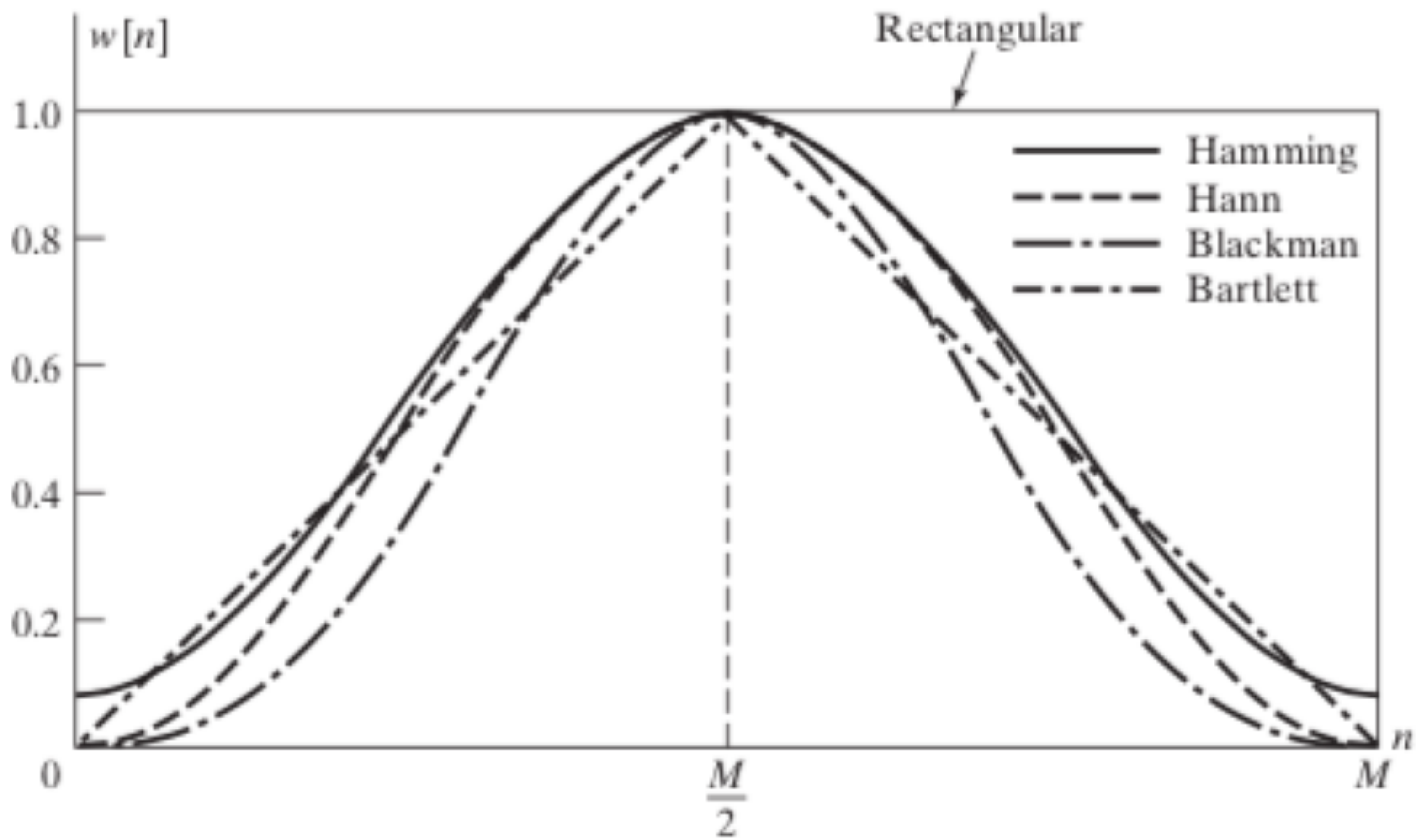
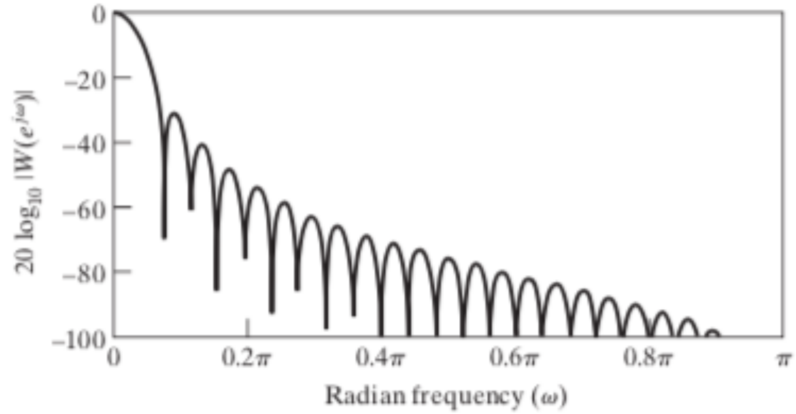


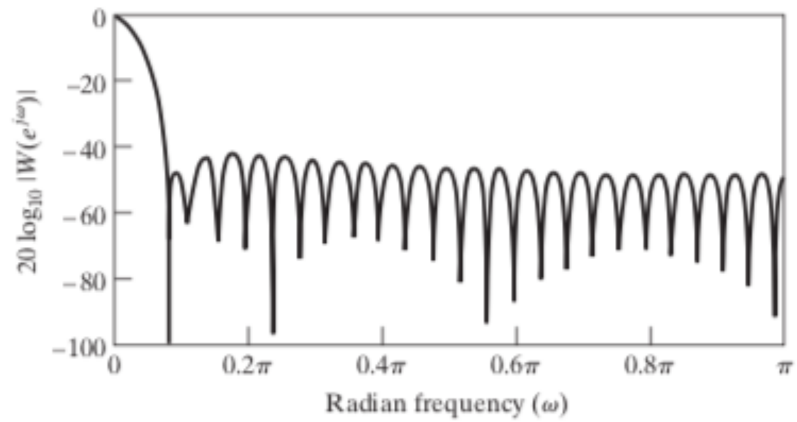
Figure 7.29 Commonly used windows.

Hann



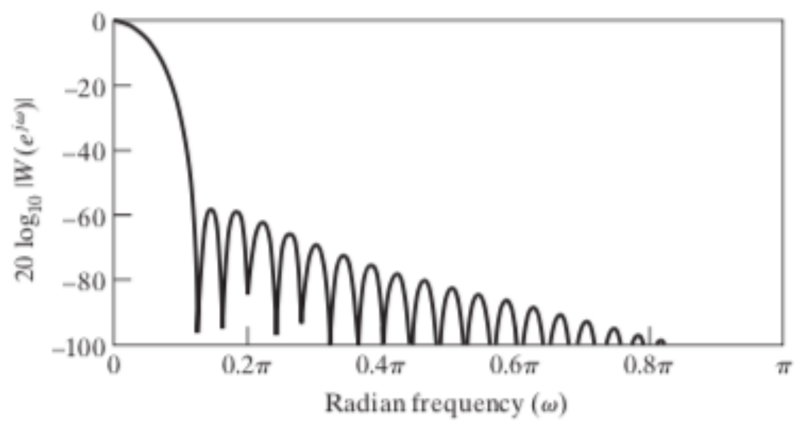
(c)

Hamming



(d)

Blackman



(e)

Kaiser Window

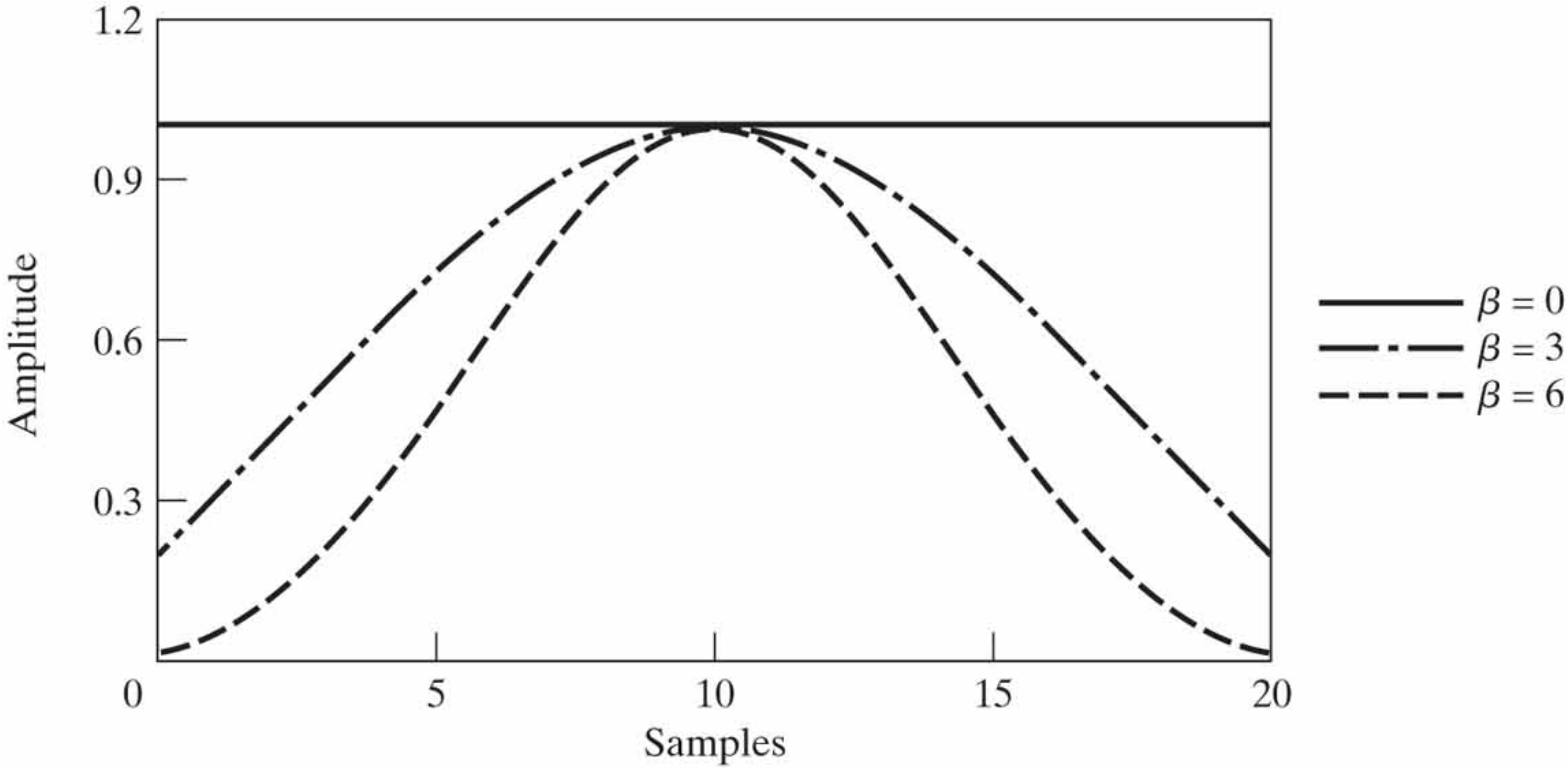
- Near optimal window quantified as the window maximally concentrated around $\omega=0$

$$w[n] = \begin{cases} \dfrac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \le n \le M, \\ 0, & \text{otherwise}, \end{cases}$$

- Two parameters – M and β
- $\alpha = M/2$
- $I_0(x)$ – zeroth order Bessel function of the first kind

Kaiser Window

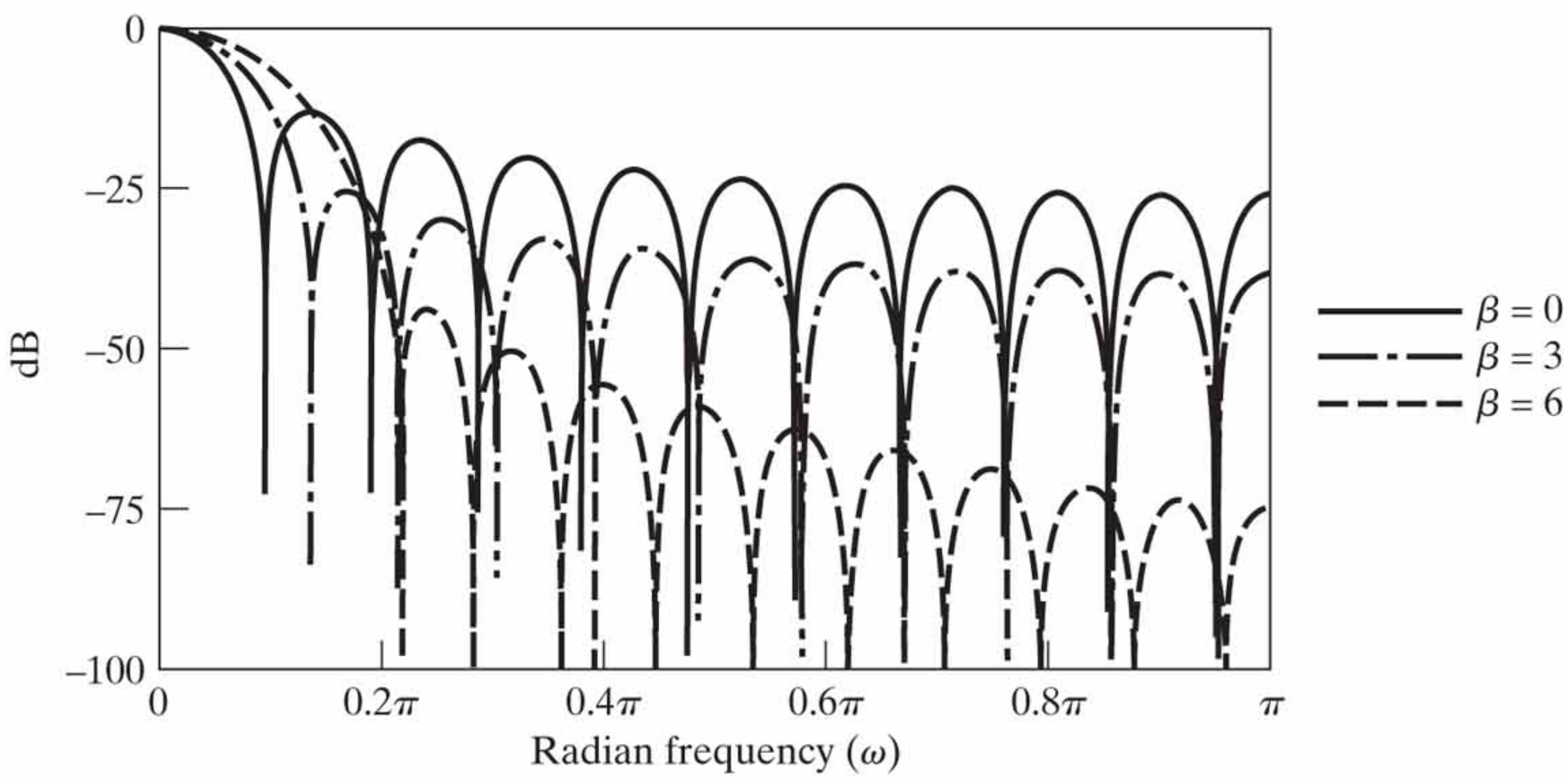
- M=20



(a)

Kaiser Window

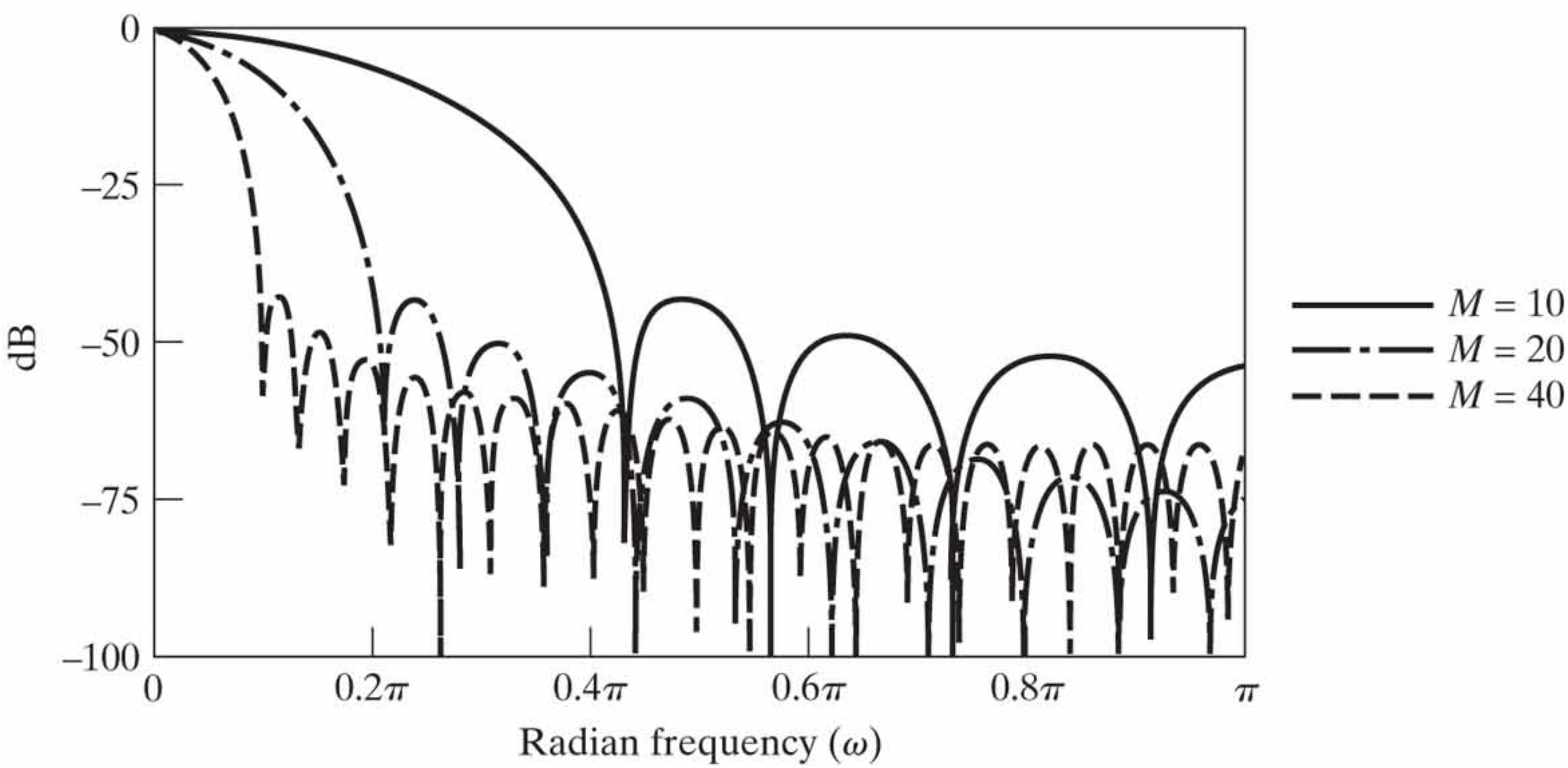
- M=20



(b)

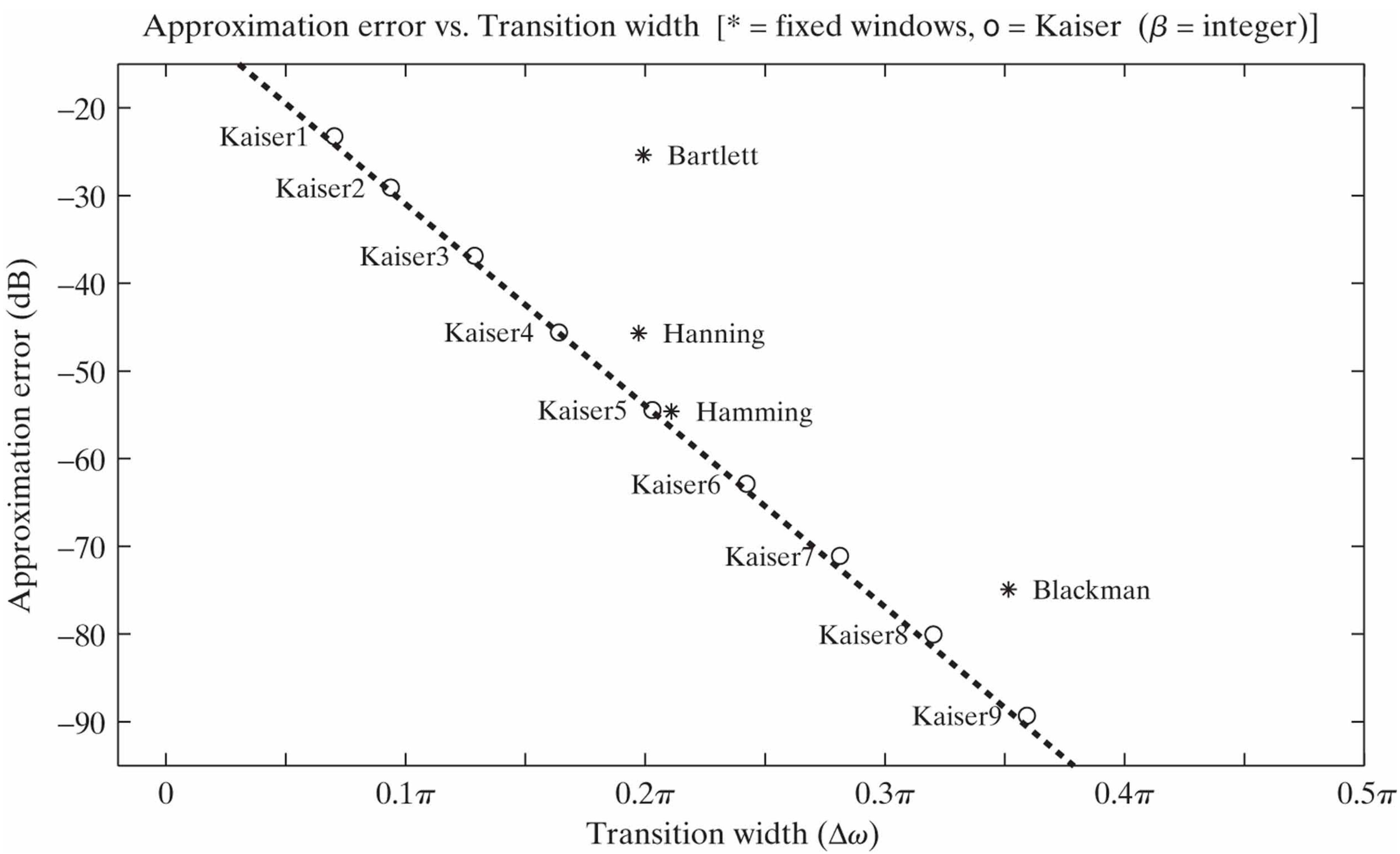
Kaiser Window

- β=6



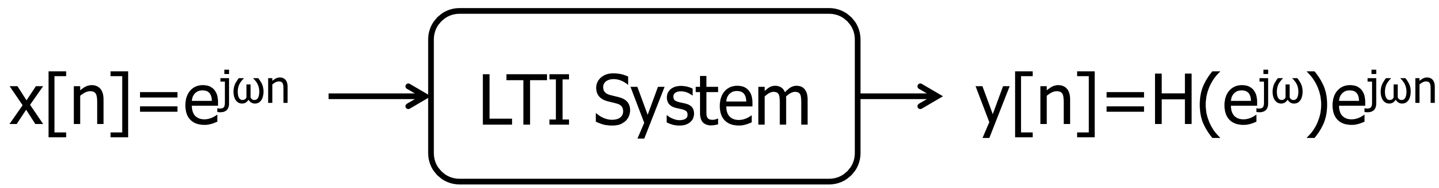
(c)

Approximation Error



LTI System Frequency Response

- (DT)Fourier Transform of impulse response



$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

Properties of the DTFT

- Linearity:

$$a x_1[n] + b x_2[n] \leftrightarrow a X_1(e^{j\omega}) + b X_2(e^{j\omega})$$

- Time/Freq Shifting:

$$x[n] \leftrightarrow X(e^{j\omega})$$

$$x[n-n_d] \leftrightarrow e^{-j\omega n_d} X(e^{j\omega})$$

$$e^{j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$$

Example: Moving Average

- Moving Average Filter
  - Causal: $M_1=0$, $M_2=M$

$$y[n]=\frac{x[n-M]+\ldots+x[n]}{M+1}$$

Example: Moving Average

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$$y[n] = \frac{x[n-M] + \ldots + x[n]}{M+1}$$

Impulse response

Example: Moving Average

- Moving Average Filter
  - Causal: $M_1=0$, $M_2=M$

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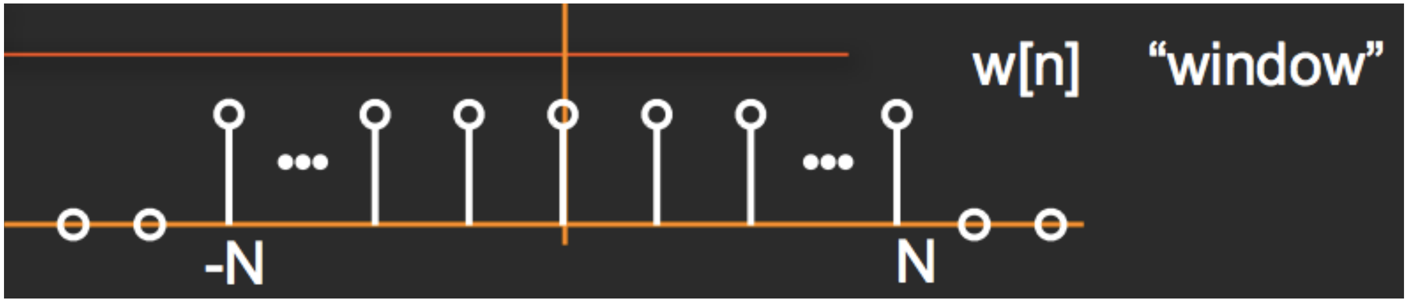
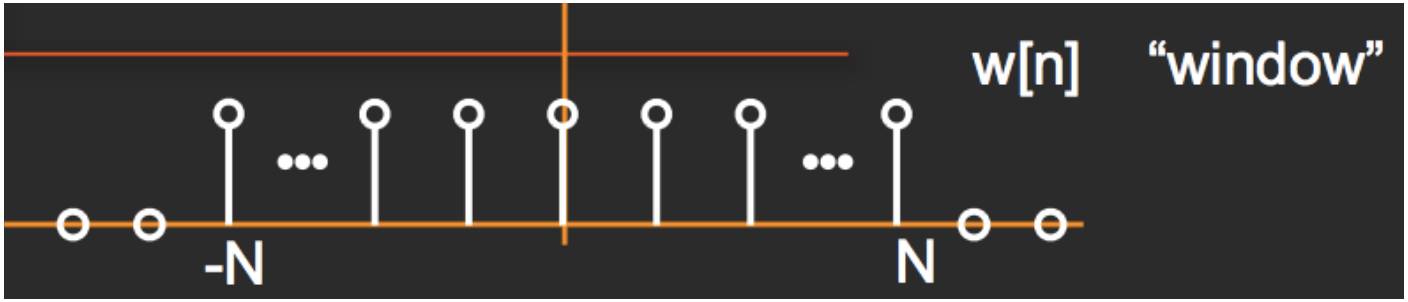
Impulse response

Scaled &Time Shifted window

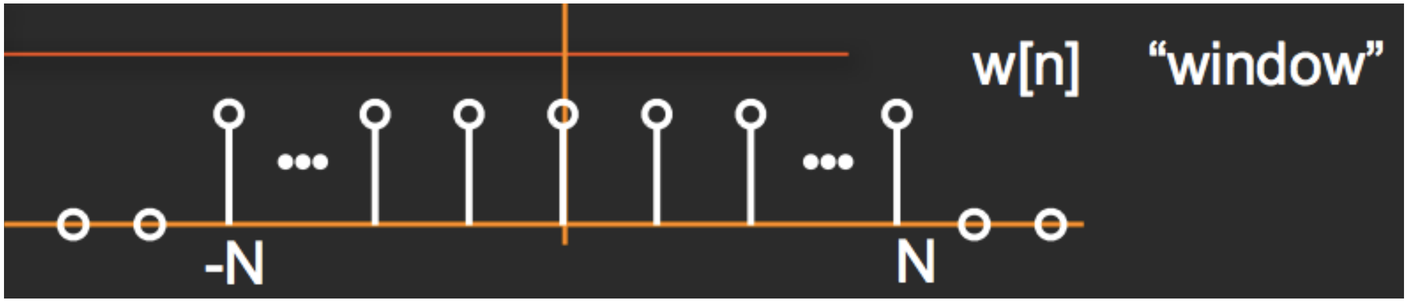


Example: Moving Average



$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

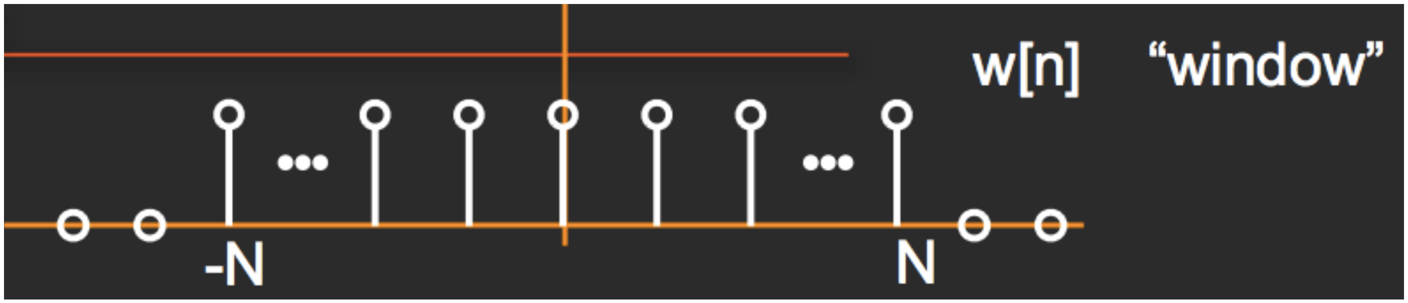
Example: Moving Average



$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

$$h[n] =$$

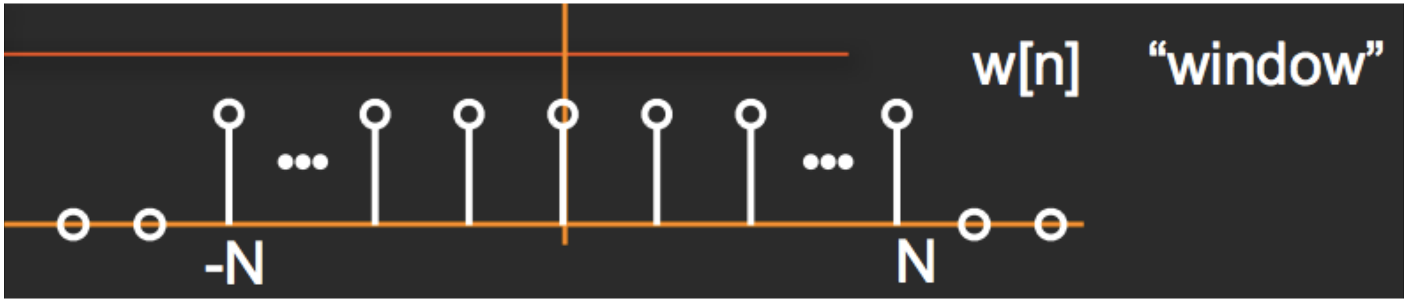
Example: Moving Average



$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

$$h[n] = \frac{1}{M+1} w[n - M/2] \leftrightarrow H(e^{j\omega}) =$$

Example: Moving Average



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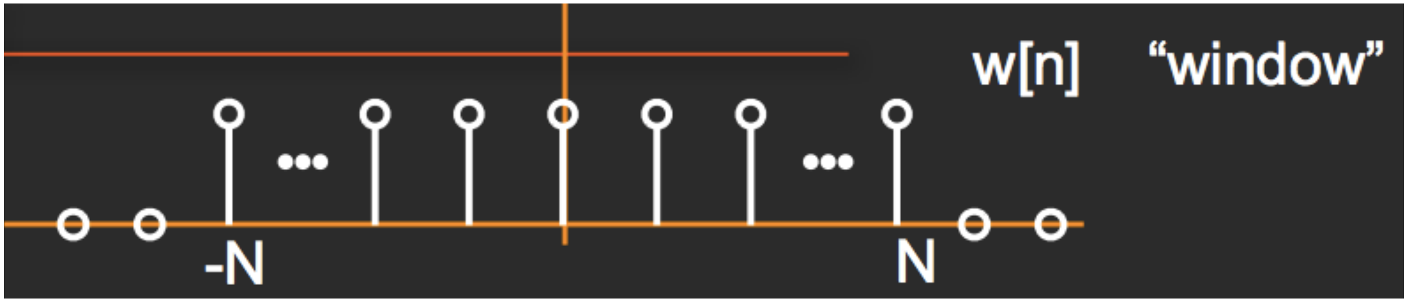
$$x[n-n_d] \leftrightarrow e^{-j\omega n_d} X(e^{j\omega})$$

Example: Moving Average



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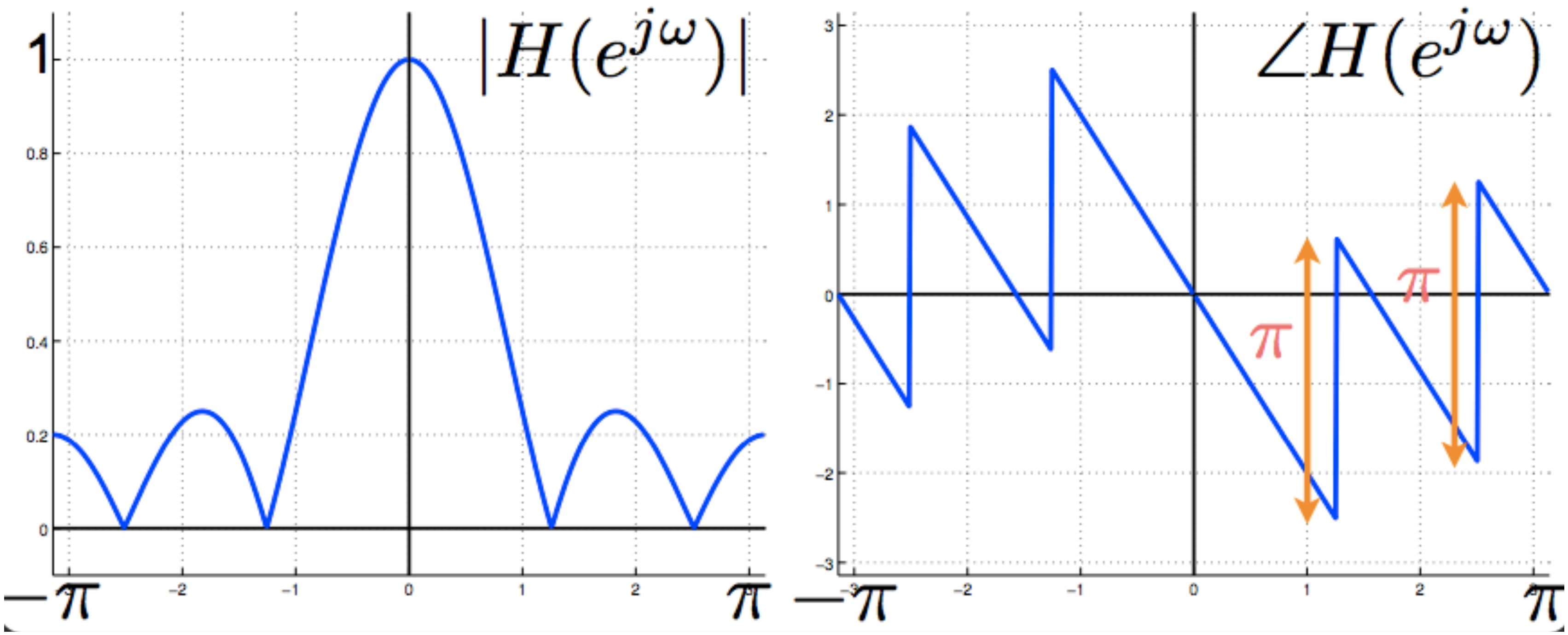
$$h[n] = \frac{1}{M+1} w[n - M/2] \leftrightarrow H(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin\left((M/2+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Example: Moving Average



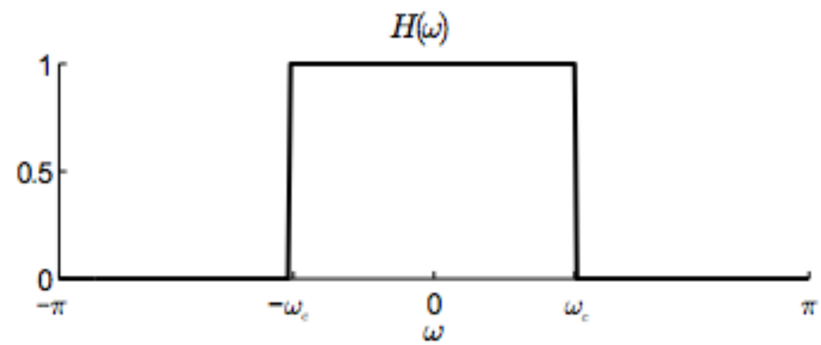
M=4
(N=2)

$$H(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin\left((M/2+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Example: Ideal Low-Pass Filter

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm\pi$)

$$H(\omega) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$



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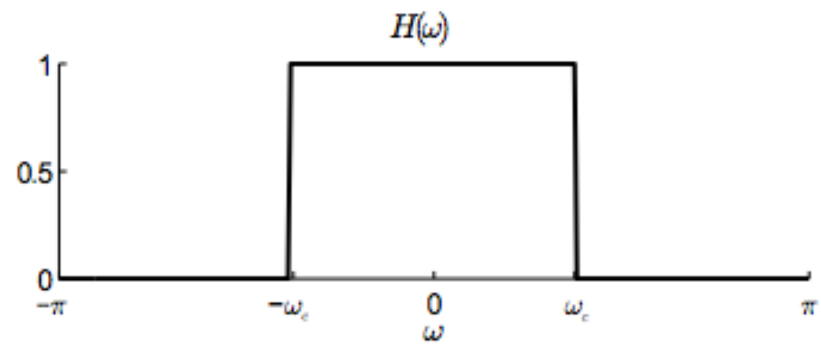

- Compute the impulse response $h[n]$ given this $H(\omega)$
- Apply the inverse DTFT

$$h[n] = \int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n} \frac{d\omega}{2\pi} = \int_{-\omega_c}^{\omega_c} e^{j\omega n} \frac{d\omega}{2\pi} = \left. \frac{e^{j\omega n}}{2\pi j n} \right|_{-\omega_c}^{\omega_c} = \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2\pi j n} = \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n}$$

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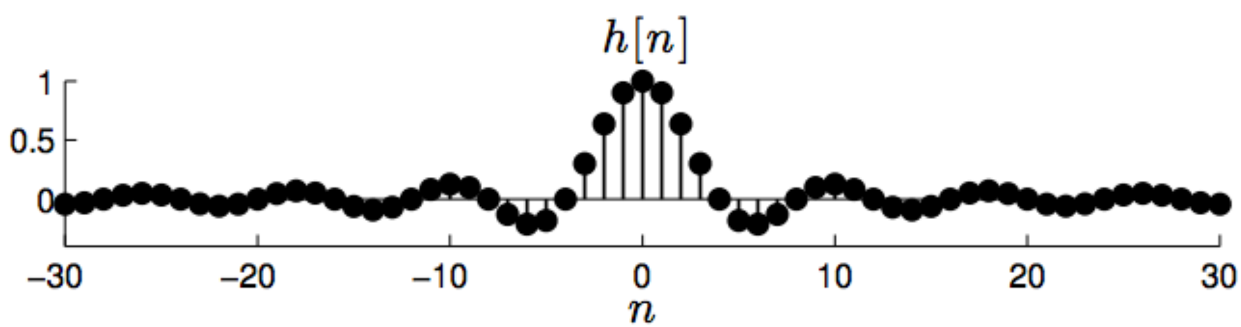

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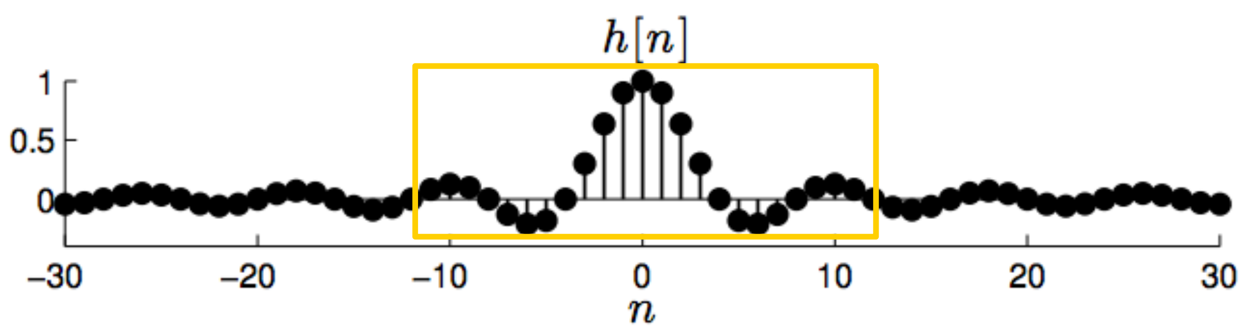
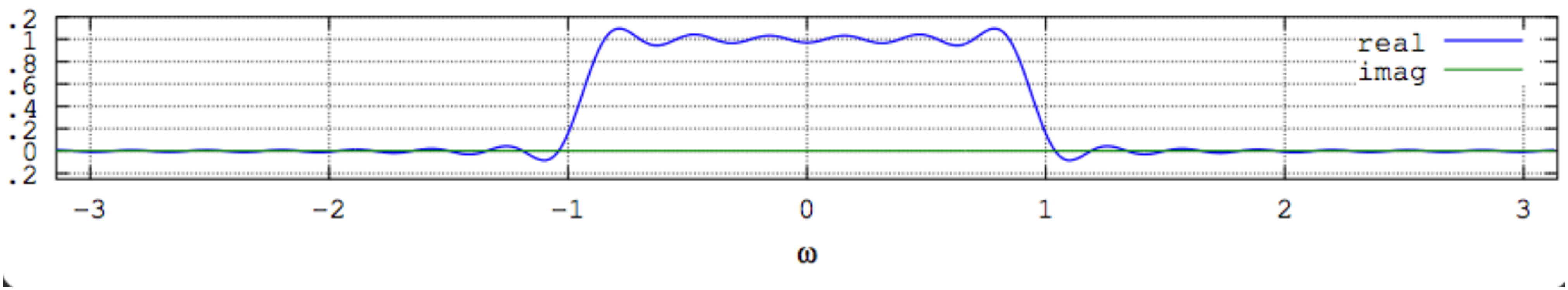
$$h[n] = 2\omega_c \frac{\sin(\omega_c n)}{\omega_c n}$$



Truncate and shift

$$h_{LP}[n] = w_N[n-N] \cdot h[n-N]$$

Example: Practical LP Filter



- Pass band smeared and rippled
  - Smearing determined by width of main lobe
  - Rippling determined by size of side lobes

Admin

- Finish Lab 8 by next lab
  - Submit Jupyter Notebook in Canvas