ESE 3400: Medical Devices Lab

Lec 9: October 10, 2022

DTFT, Sampling and Reconstruction

Lecture Outline

- DTFT
- Sampling/Reconstruction
- Data Converters
  - Anti-Aliasing Filtering
  - Sampling Rate
  - Transfer Characteristics

Discrete-Time Fourier Transform (DTFT)

DTFT Definition

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

DTFT Definition

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Alternate

$$X(f) = \sum_{k=-\infty}^{\infty} x[k]e^{-j2\pi fk}$$

$$x[n] = \int_{-0.5f_s}^{0.5f_s} X(f)e^{j2\pi fn}df$$

Fourier Transform Pairs

TABLE 2.3 FOURIER TRANSFORM PAIRS

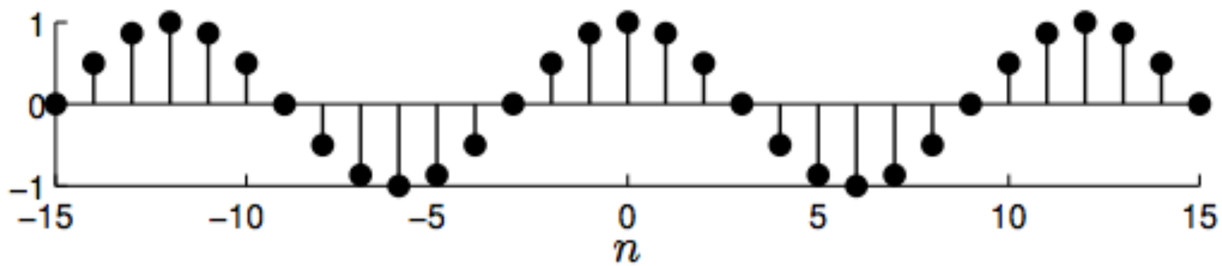
Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\|r\| < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \|\omega\| < \omega_c, \\ 0, & \omega_c < \|\omega\| \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

Video Example

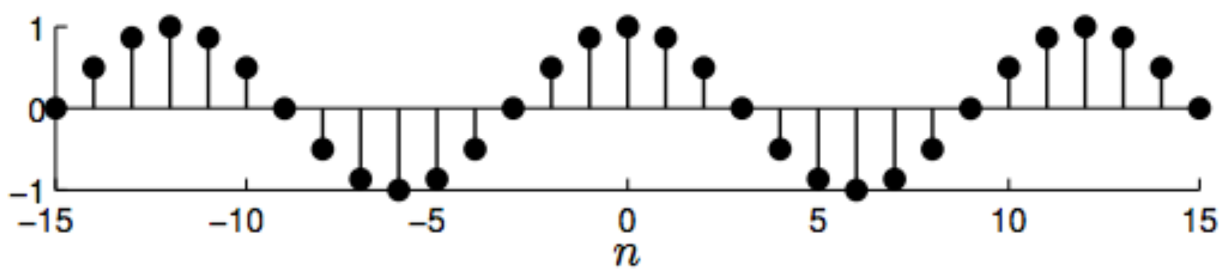
- https://www.youtube.com/watch?v=ByTsISFXUoY

Aliasing Example

$x_1[n] = \cos\left(\frac{\pi}{6}n\right)$



$x_2[n] = \cos\left(\frac{13\pi}{6}n\right) = \cos\left(\left(\frac{\pi}{6} + 2\pi\right)n\right)$



Aliasing Example

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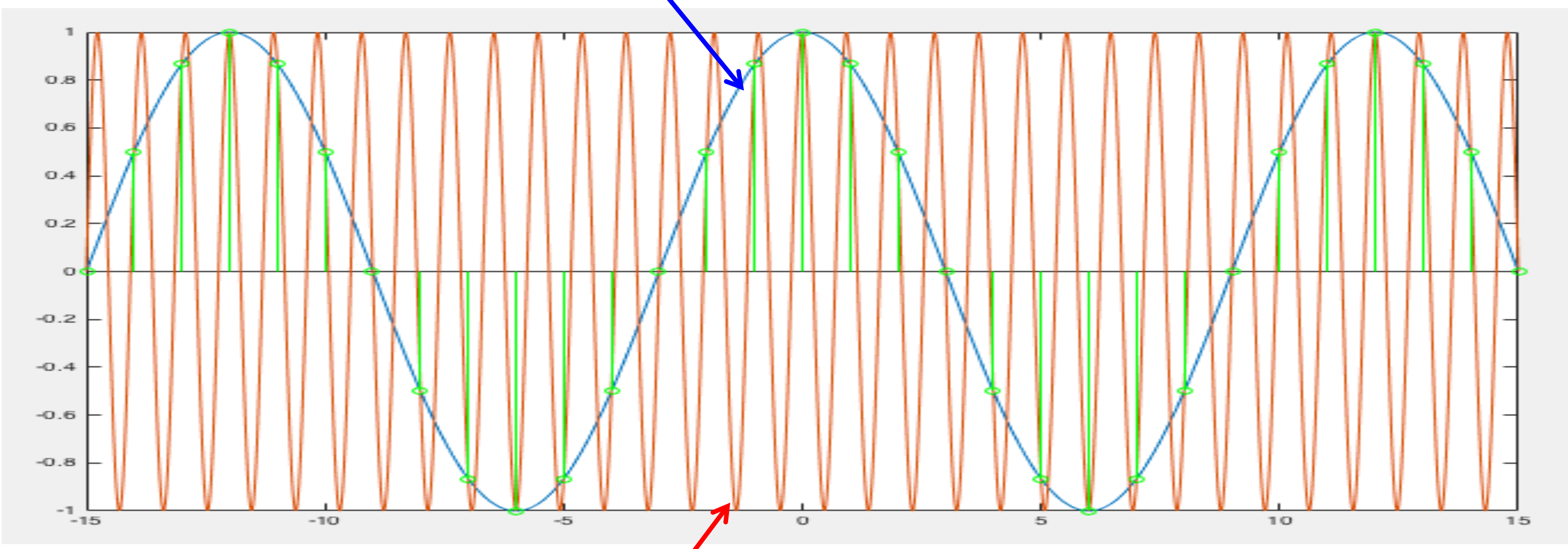

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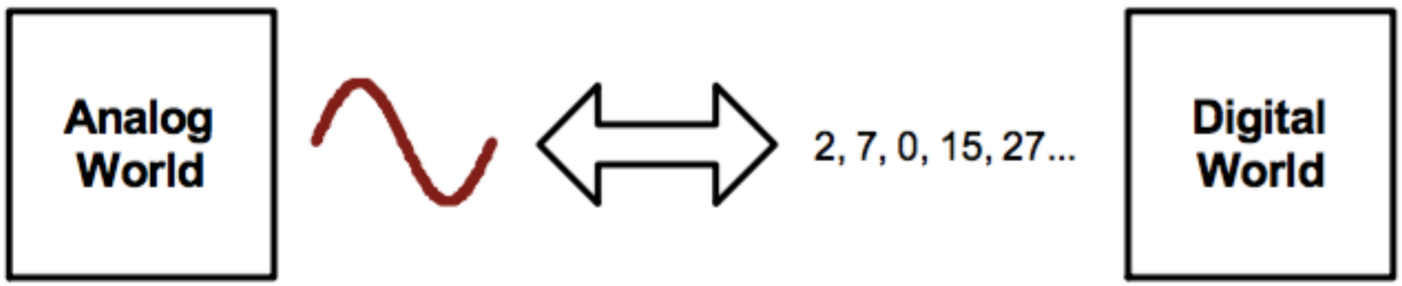
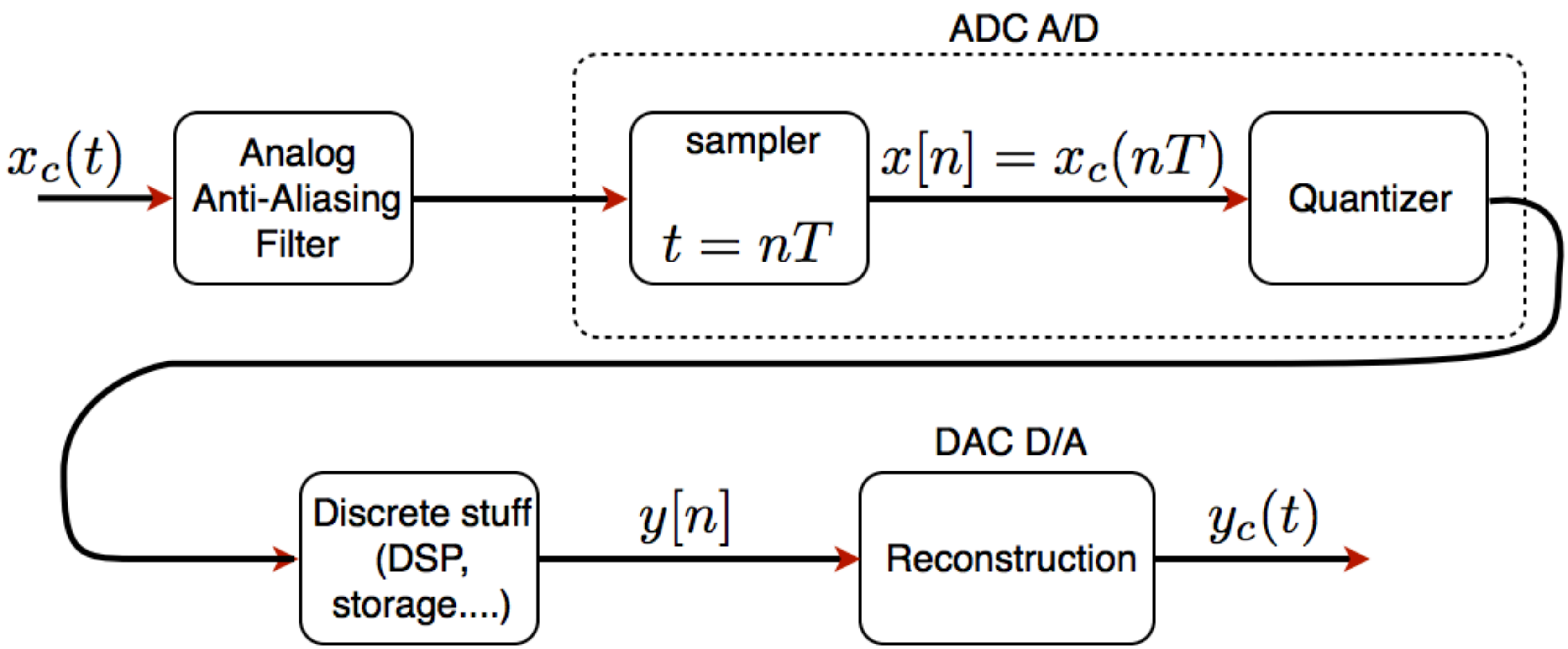
The Data Conversion Problem



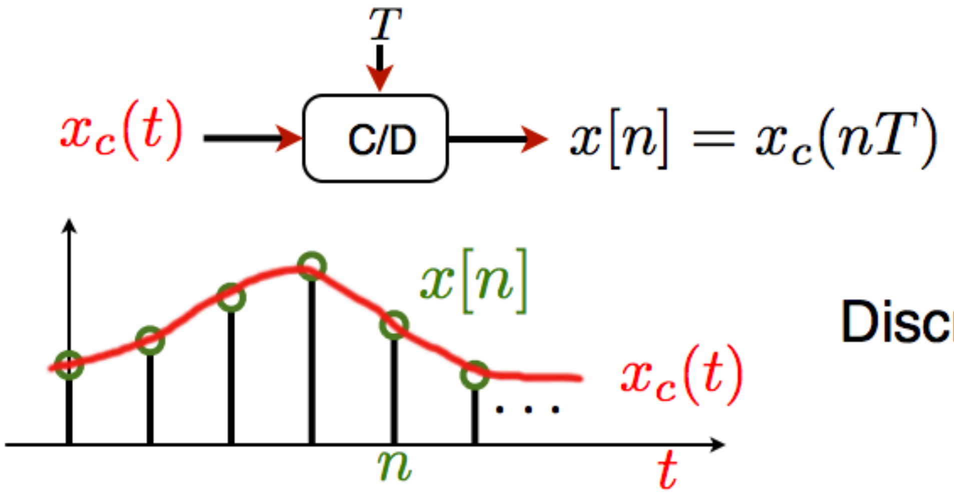
- Real world signals
  - Continuous time, continuous amplitude
- Digital abstraction
  - Discrete time, discrete amplitude
- Two problems
  - How to go discretize in time and amplitude
    - A/D conversion
  - How to "undescretize" in time and amplitude
    - D/A conversion

DSP System



ADC A/D
$x_c(t)$
Analog Anti-Aliasing Filter
sampler
$t = nT$
$x[n] = x_c(nT)$
Quantizer
DAC D/A
Discrete stuff (DSP, storage....)
$y[n]$
Reconstruction
$y_c(t)$

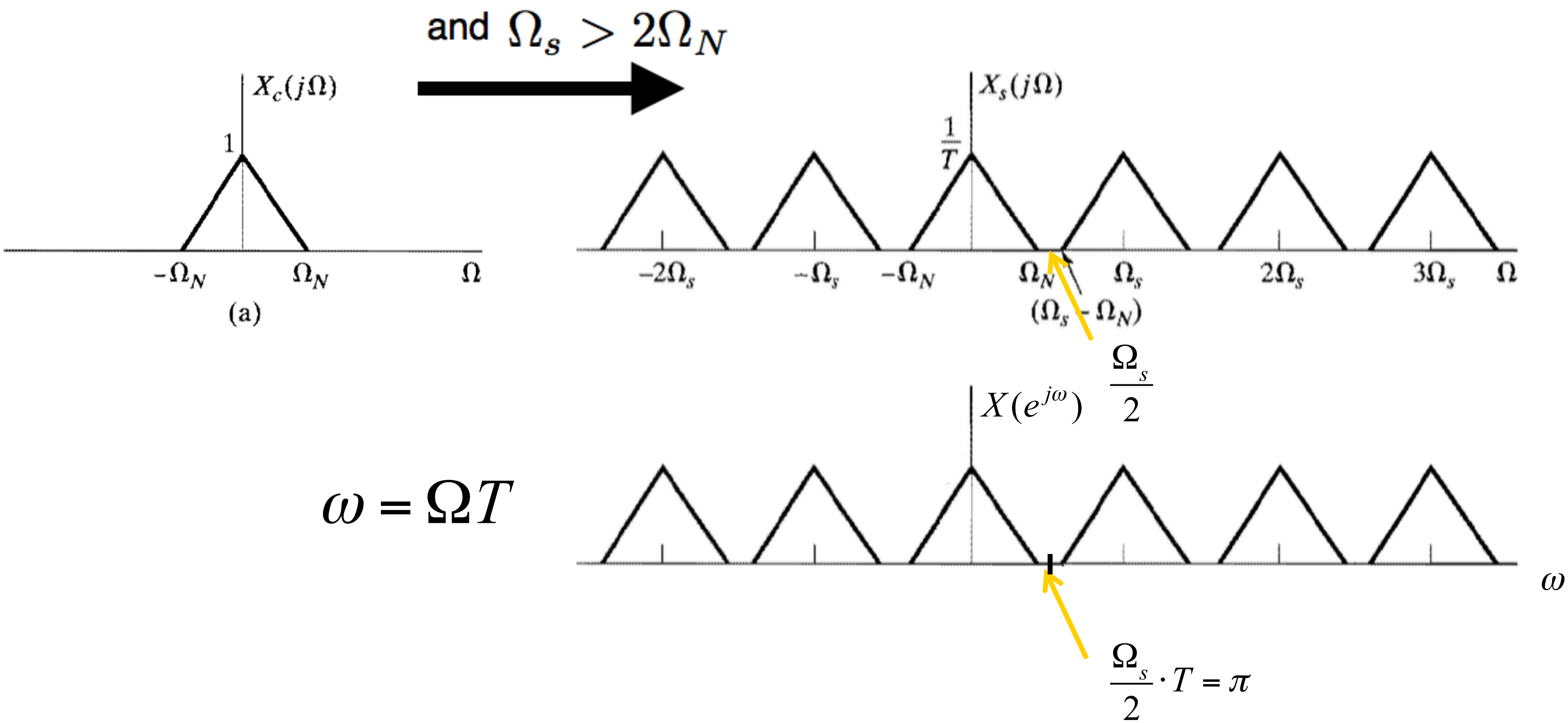
Ideal Sampling Model



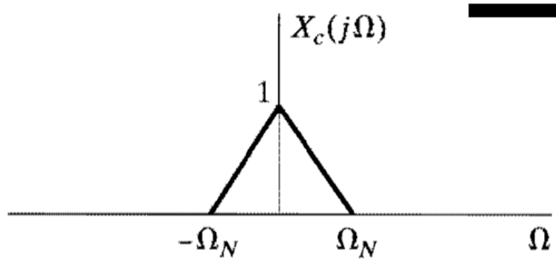
Discrete and Continuous

- Ideal continuous-to-discrete time (C/D) converter
  - T is the sampling period
  - f_s=1/T is the sampling frequency
  - Ω_s=2π/T

Frequency Domain Analysis

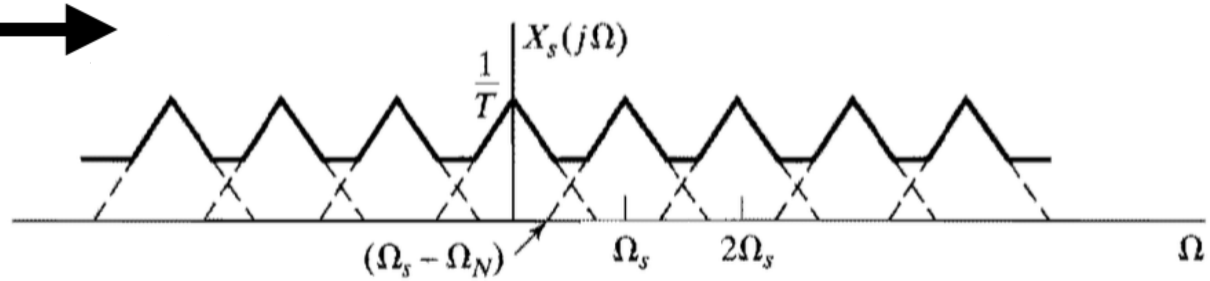


Frequency Domain Analysis w/Aliasing

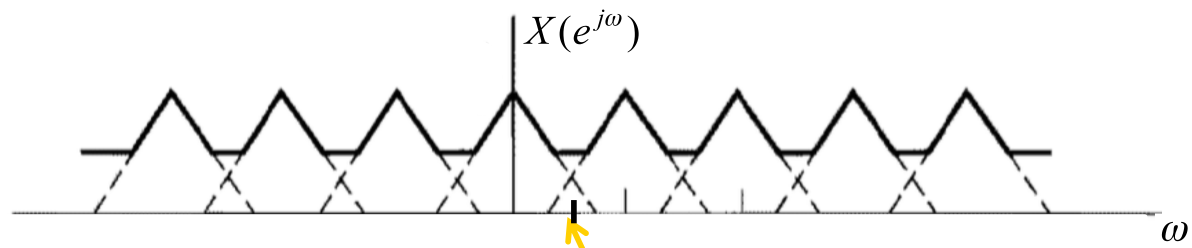


(a)

and $\Omega_s < 2\Omega_N$

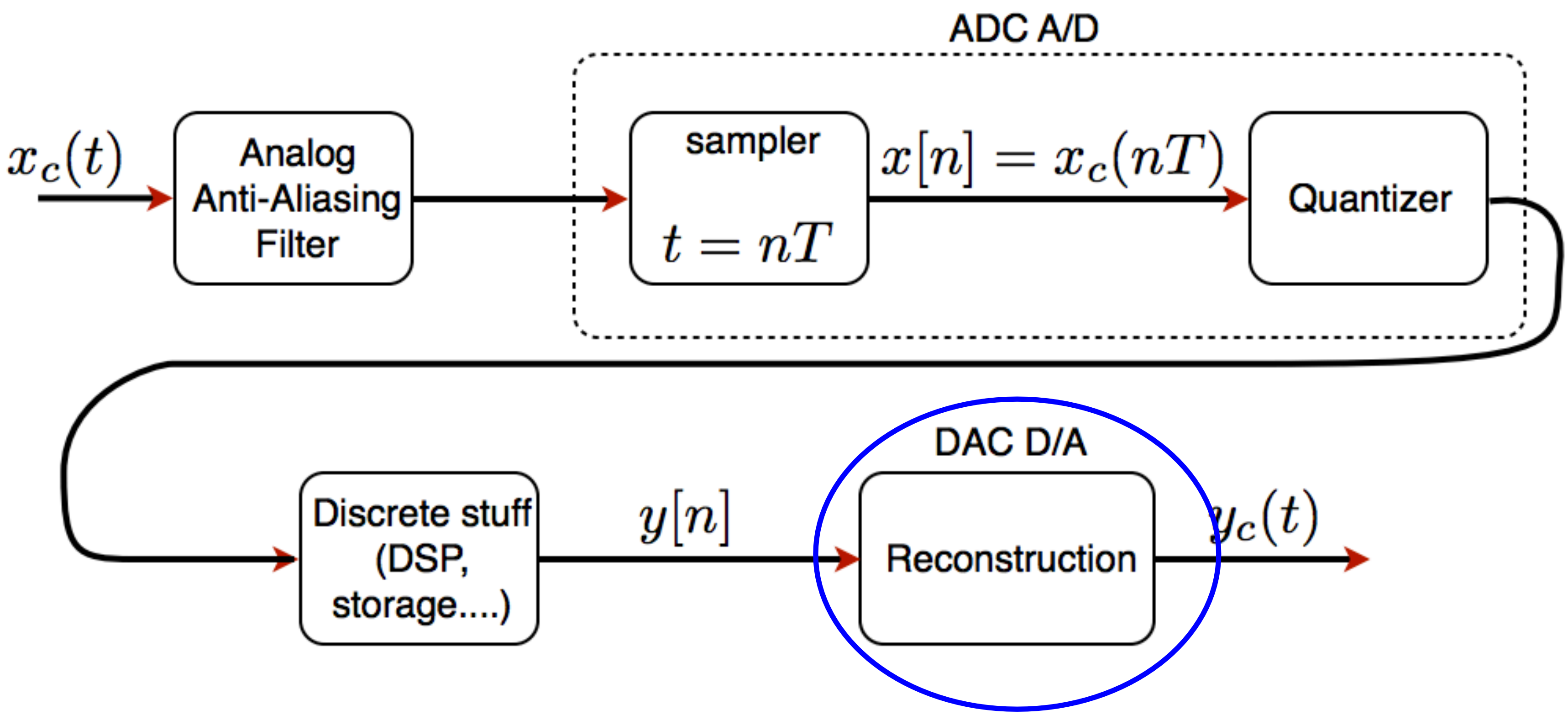


$$\omega = \Omega T$$



$$\frac{\Omega_s}{2} \cdot T = \pi$$

DSP System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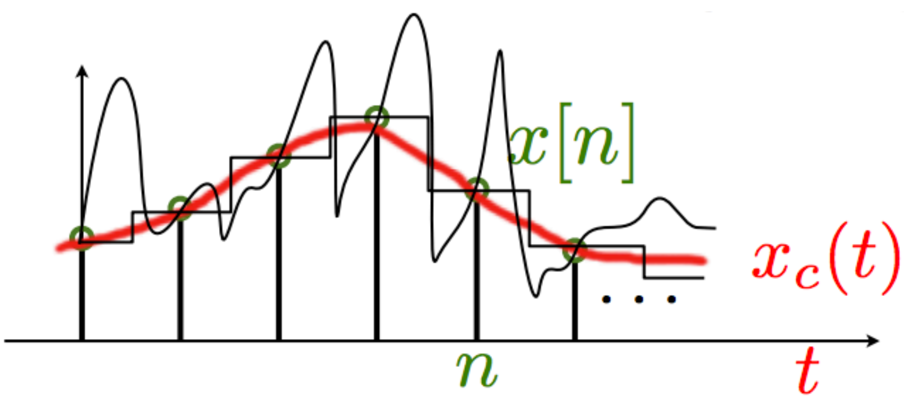
$y[n]$
Reconstruction
$y_c(t)$

Reconstruction of Bandlimited Signals

- Nyquist Sampling Theorem: Suppose $x_c(t)$ is bandlimited. I.e.

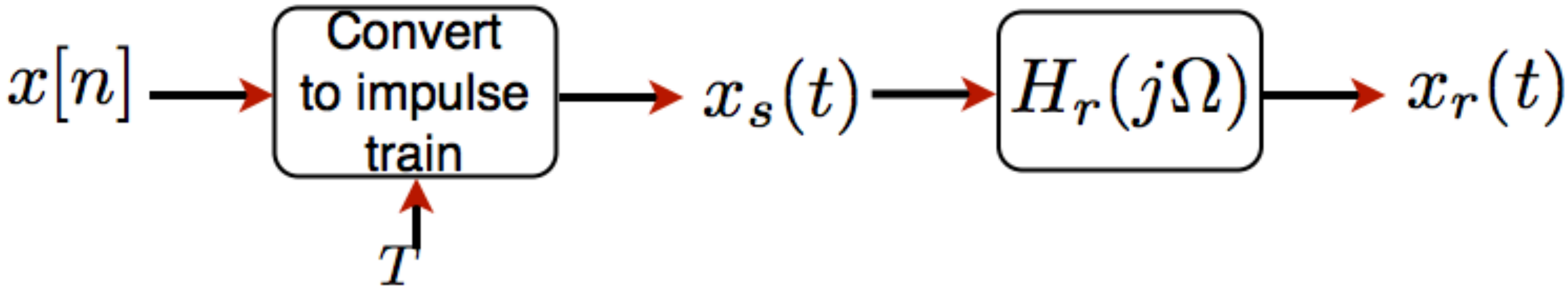
$$X_c(j\Omega) = 0 \quad \forall \quad |\Omega| \geq \Omega_N$$

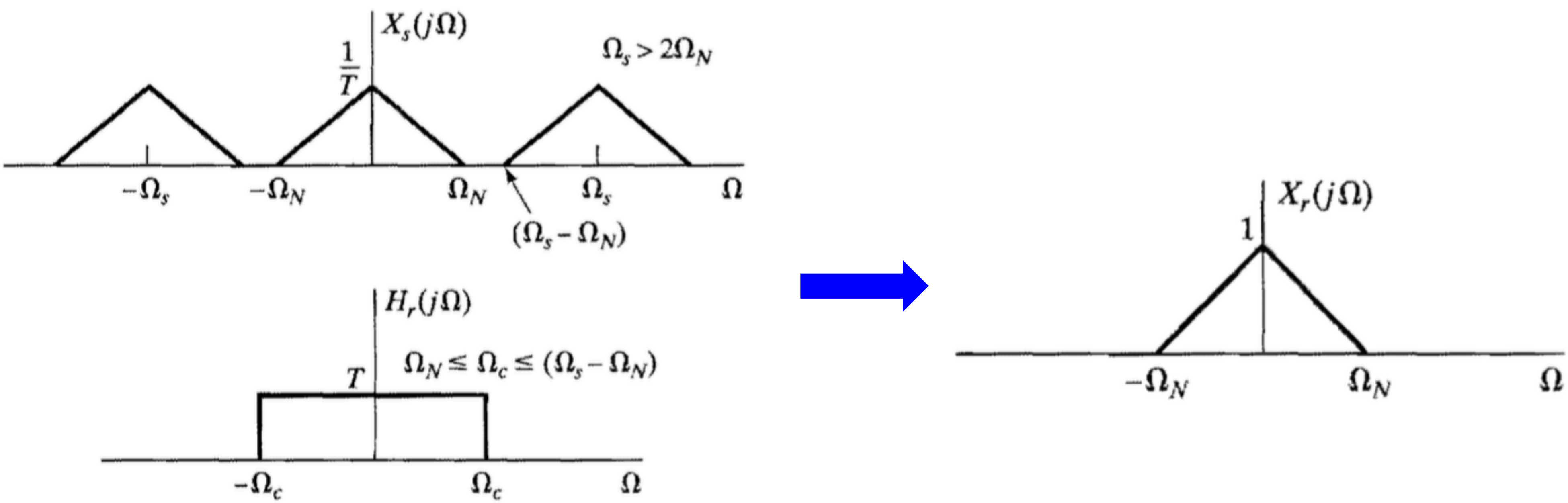
- If $\Omega_s \geq 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n]=x_c(nT)$
- Bandlimitedness is the key to uniqueness



Mulitiple signals go through the samples, but only one is bandlimited within our sampling band

Reconstruction in Frequency Domain





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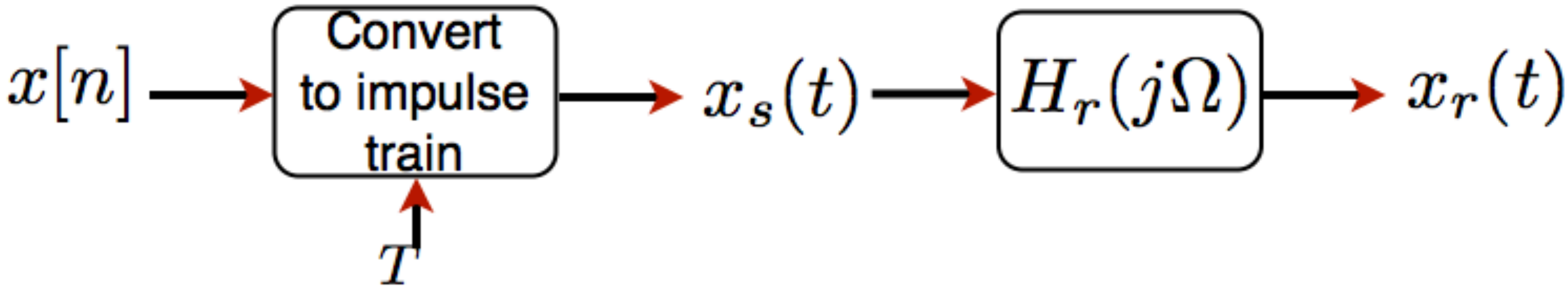
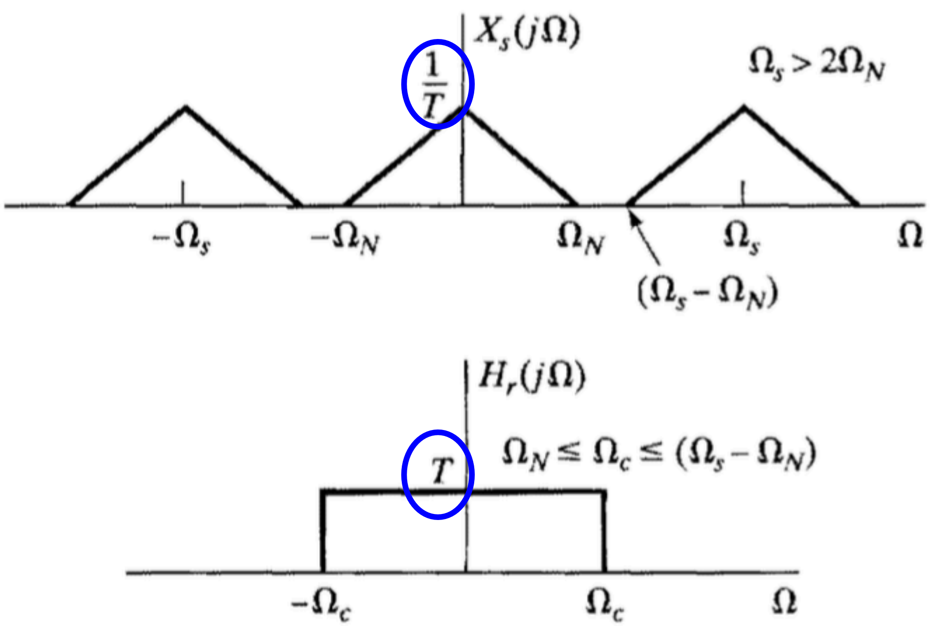
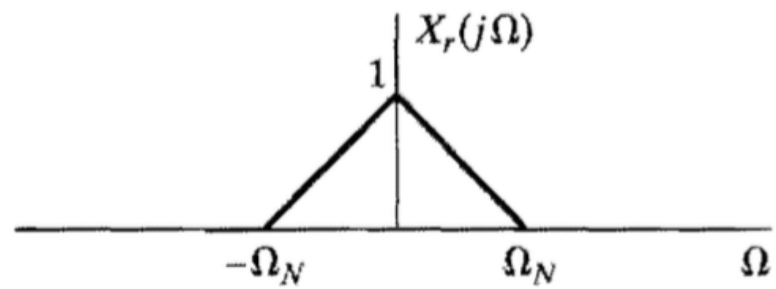






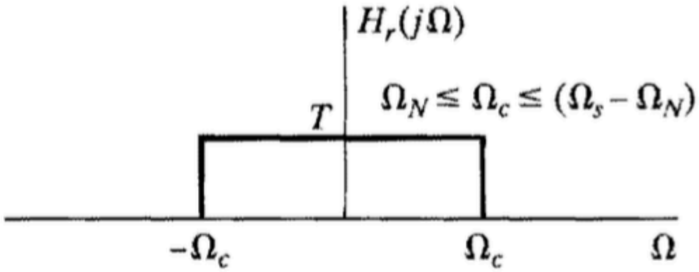
$2\Omega_N = \Omega_s$

$\Omega_N = \Omega_C = \Omega_s/2$

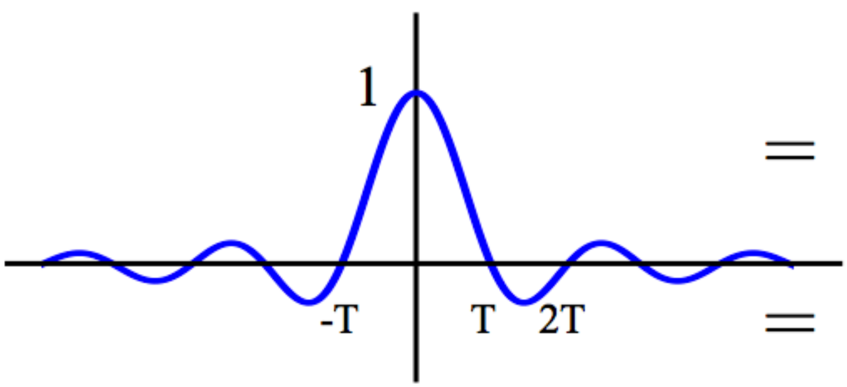
Sample at Nyquist and filter at signal badnwidth

Reconstruction in Time Domain

$\Omega_N = \Omega_C = \Omega_s/2$



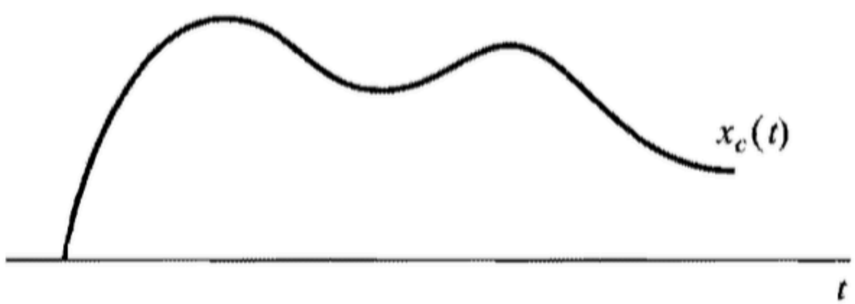
$$
\begin{aligned}
h_r(t) &= \frac{1}{2\pi}\int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t}\, d\Omega \\
&= \frac{T}{2\pi}\frac{1}{jt} e^{j\Omega t}\Big|_{-\Omega_s/2}^{\Omega_s/2} \\
&= \frac{T}{\pi t}\,\frac{e^{j\frac{\Omega_s}{2}t} - e^{-j\frac{\Omega_s}{2}t}}{2j} \\
&= \frac{T}{\pi t}\sin\left(\frac{\Omega_s}{2}t\right) = \frac{T}{\pi t}\sin\left(\frac{\pi}{T}t\right) \\
&= \mathrm{sinc}\left(\frac{t}{T}\right)
\end{aligned}
$$

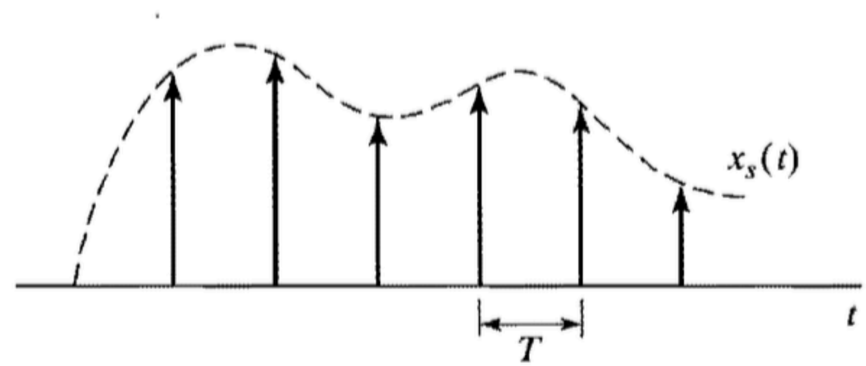


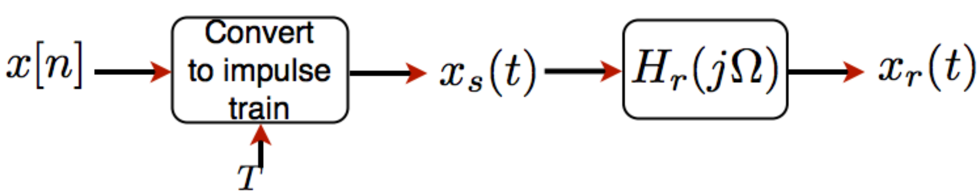
Reconstruction in Time Domain

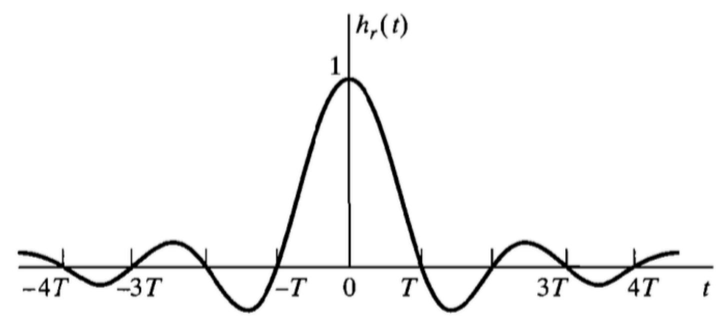
$$x_r(t) = x_s(t) * h_r(t) = \left(\sum_n x[n]\delta(t - nT) \right) * h_r(t)$$

$$= \sum_n x[n] h_r(t - nT)$$









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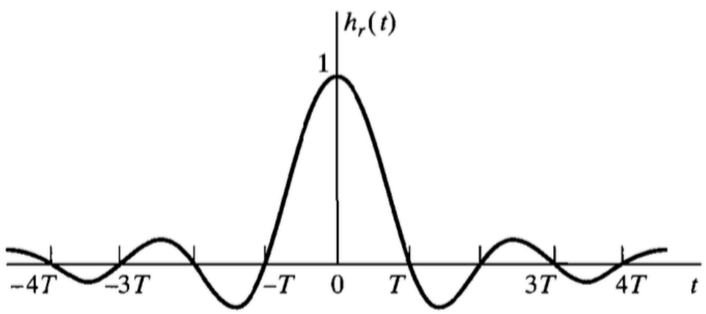

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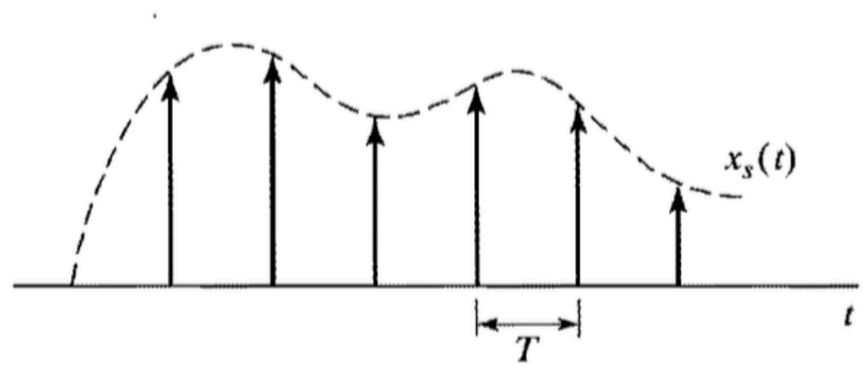

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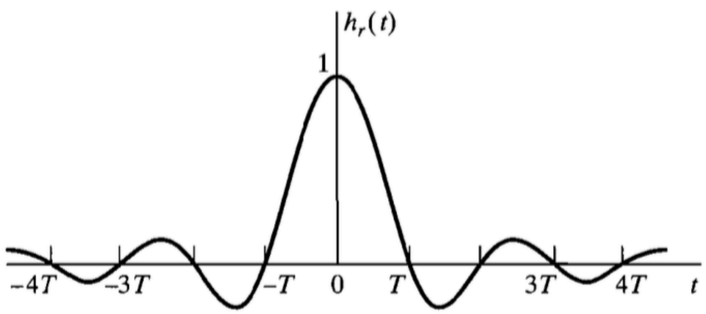

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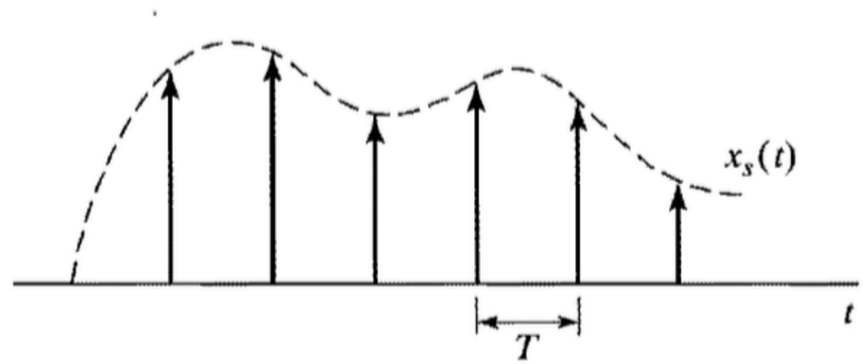

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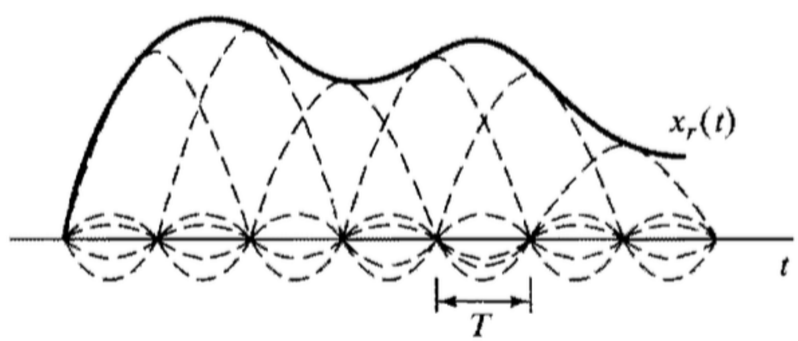
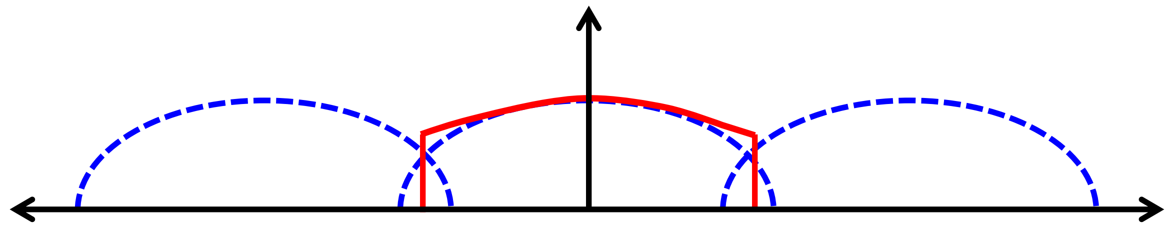


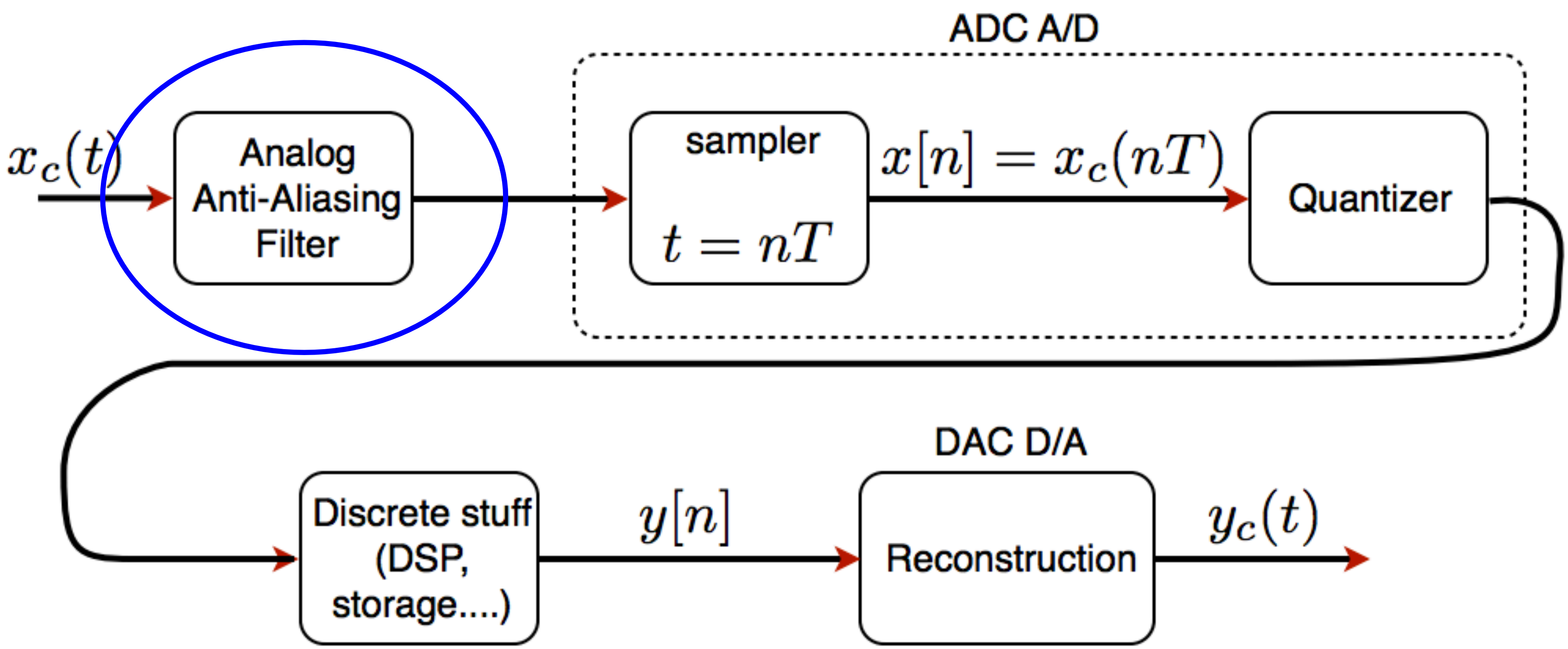
The sum of "sincs" gives $x_r(t)$ → unique signal that is bandlimited by sampling bandwidth

Aliasing

- If $\Omega_N > \Omega_s/2$, $x_r(t)$ is an aliased version of $x_c(t)$



DSP System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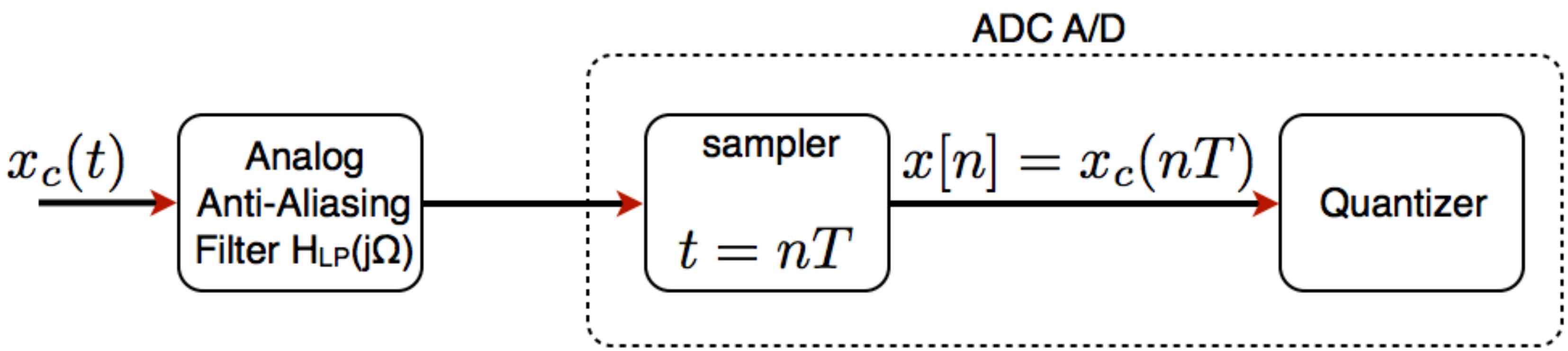
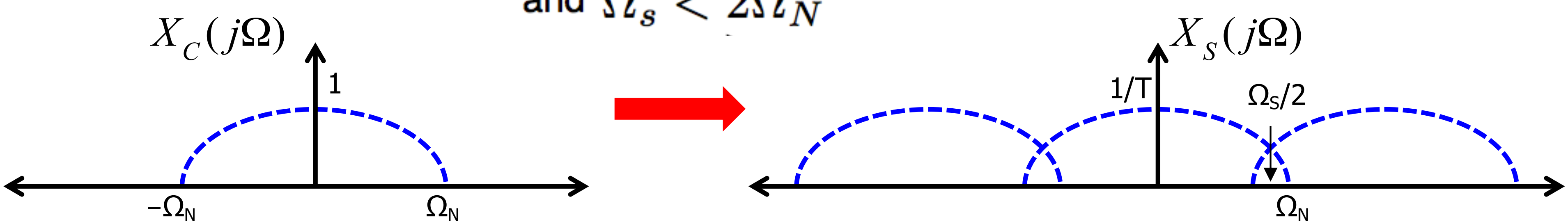
Anti-Aliasing Filter



and $\Omega_s < 2\Omega_N$



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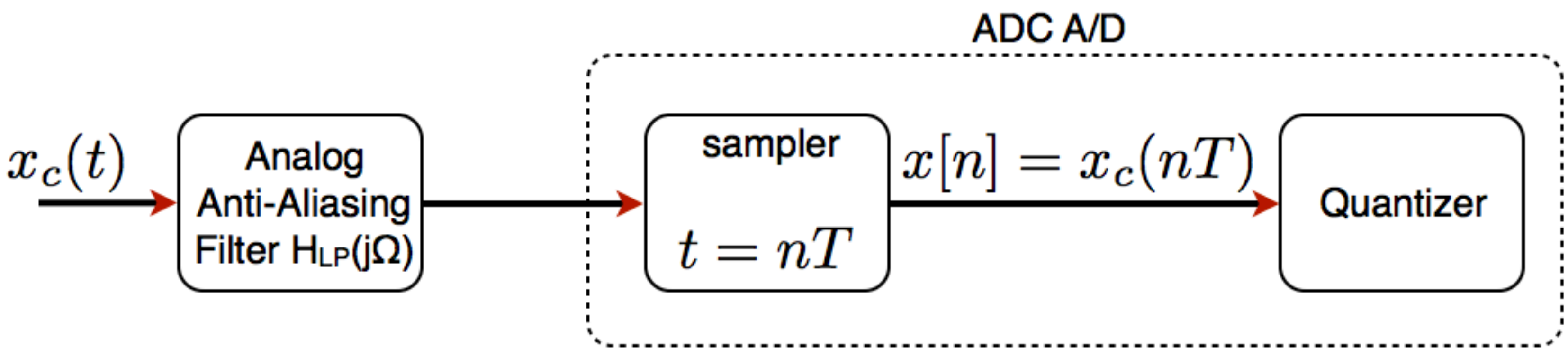

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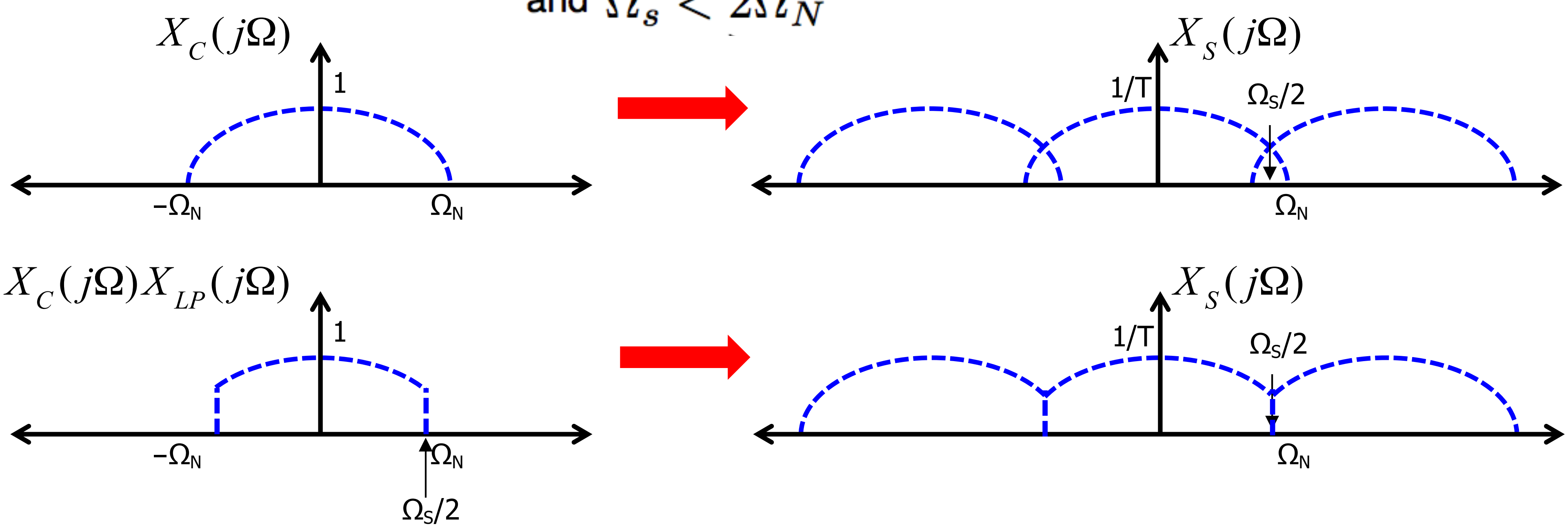
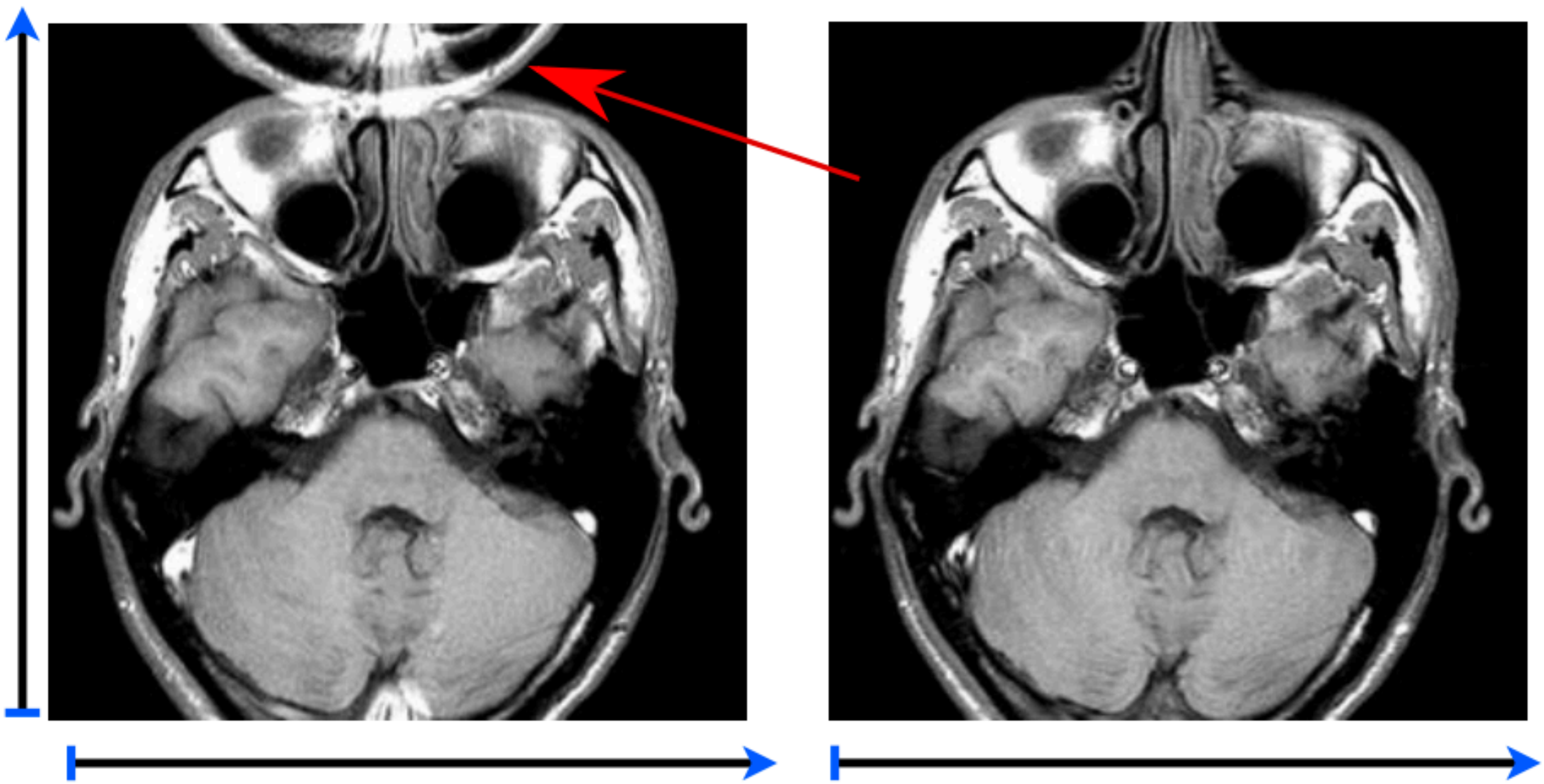


MRI aliasing example



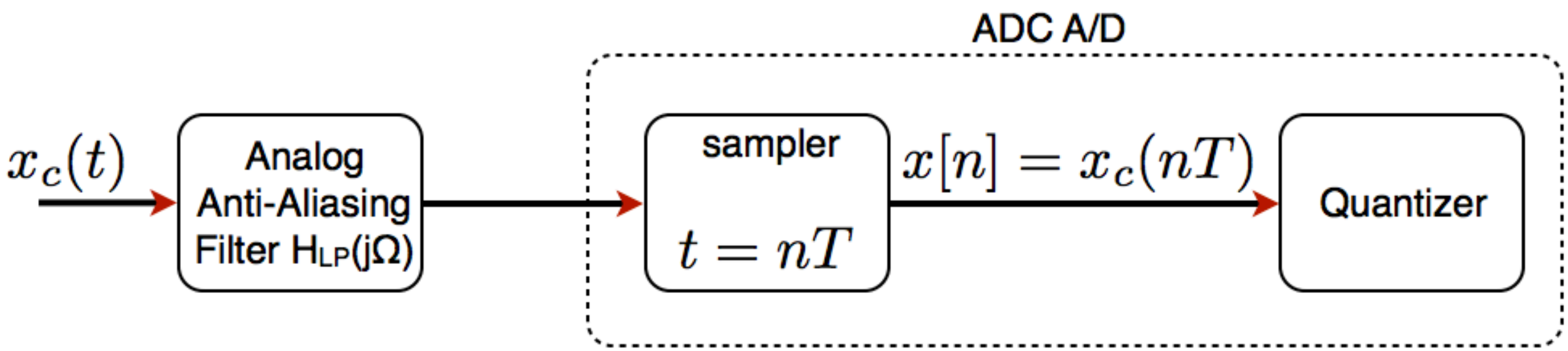
MRI anti-aliasing example

MRI anti-aliasing example

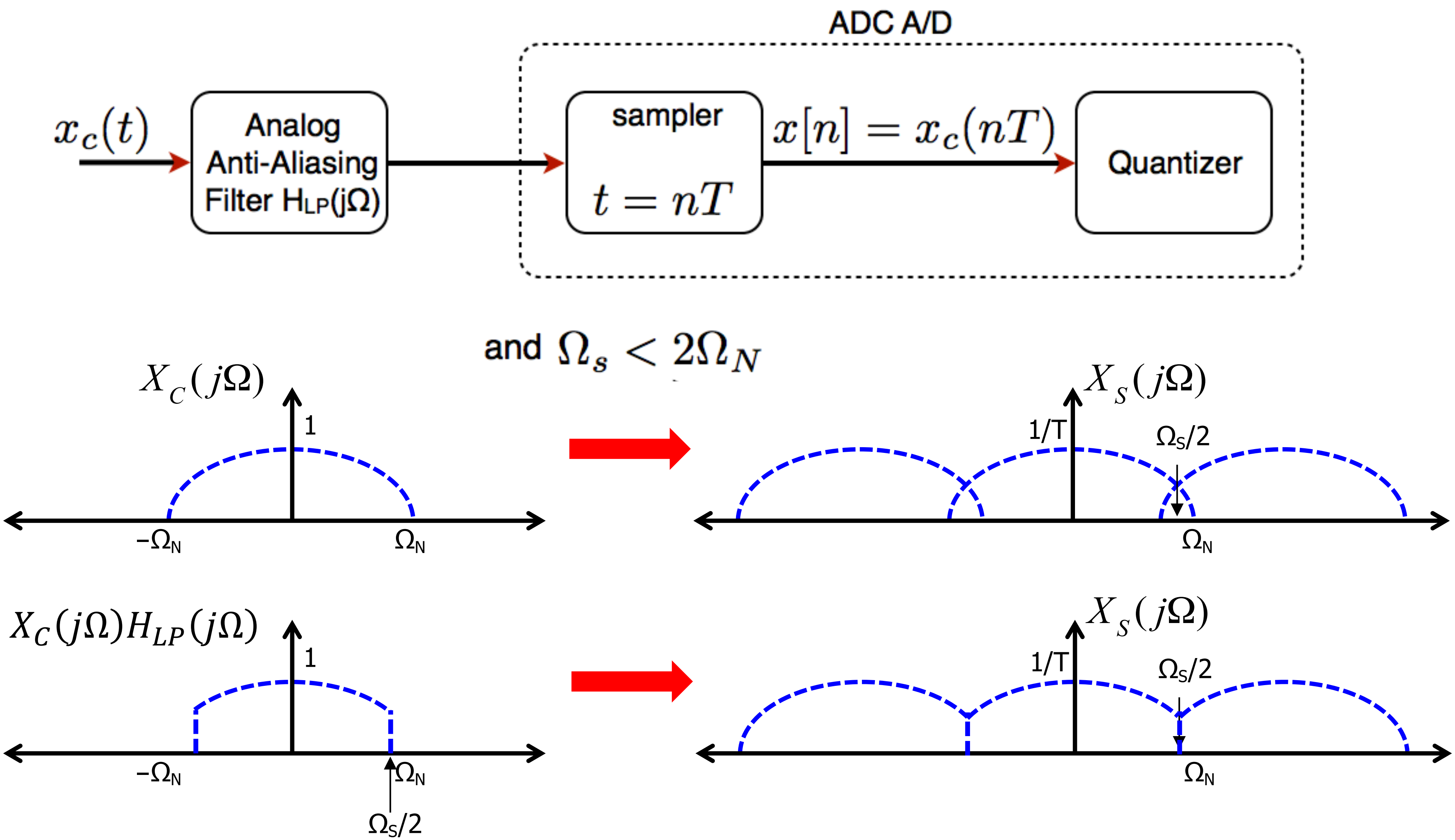
ADC

Analog to Digital Converter

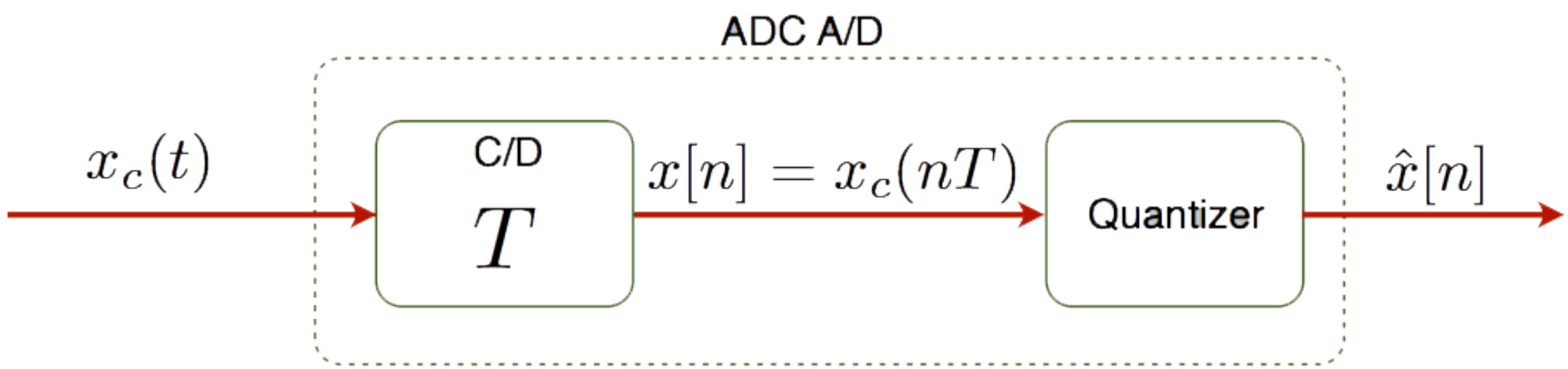
Anti-Aliasing Filter with ADC

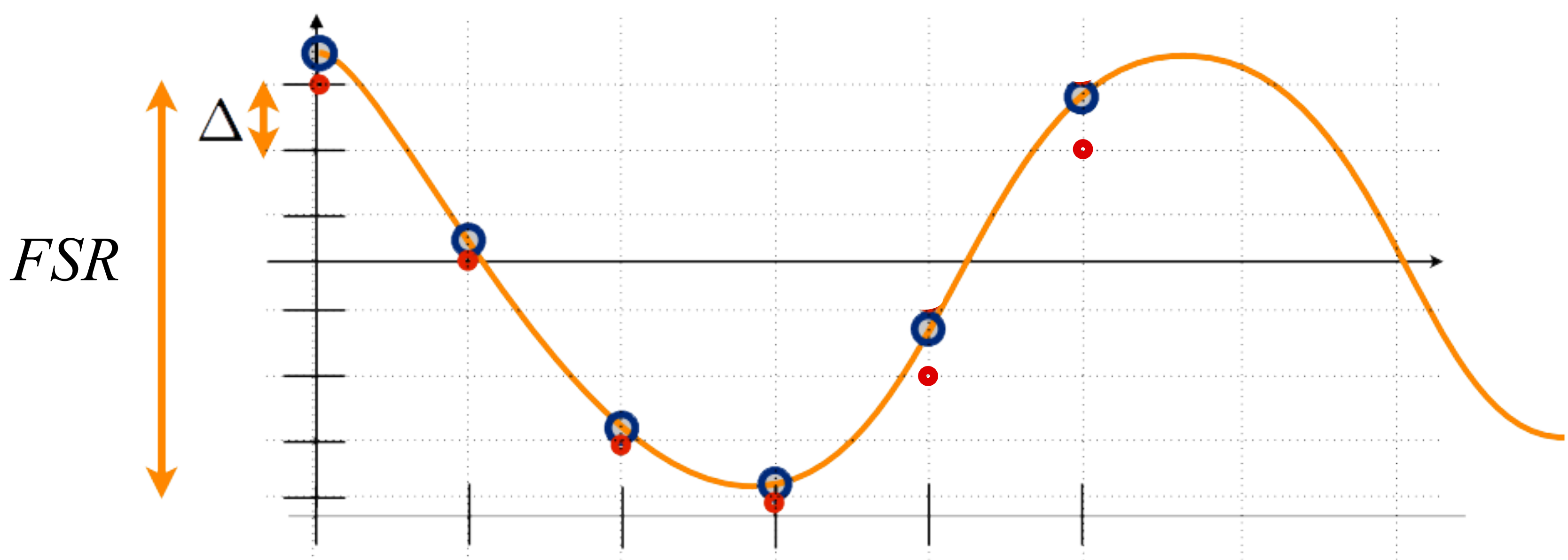


Anti-Aliasing Filter with ADC



Sampling and Quantization

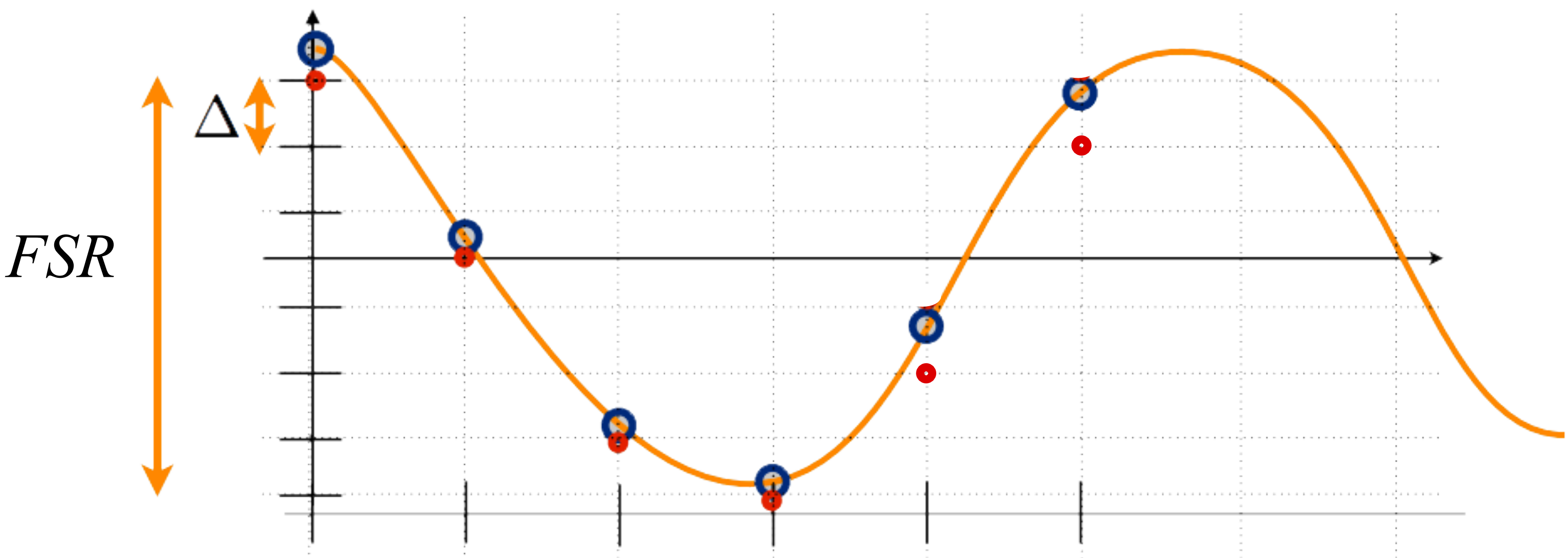




Sampling and Quantization

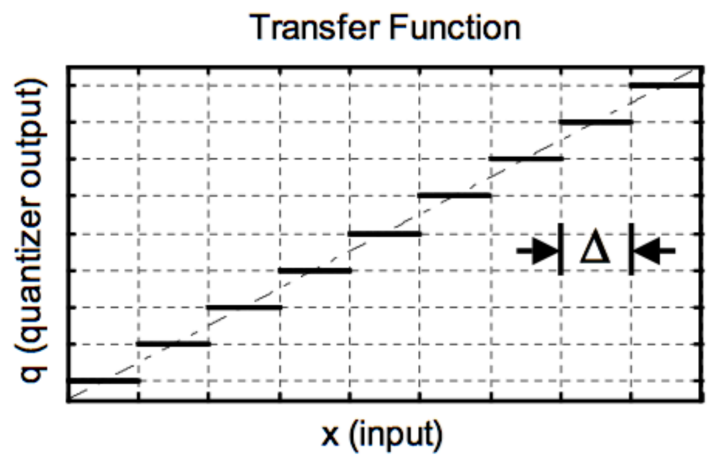
- For an input signal with V_{pp}=FSR with B bits

$$\Delta = \frac{FSR}{2^B}$$



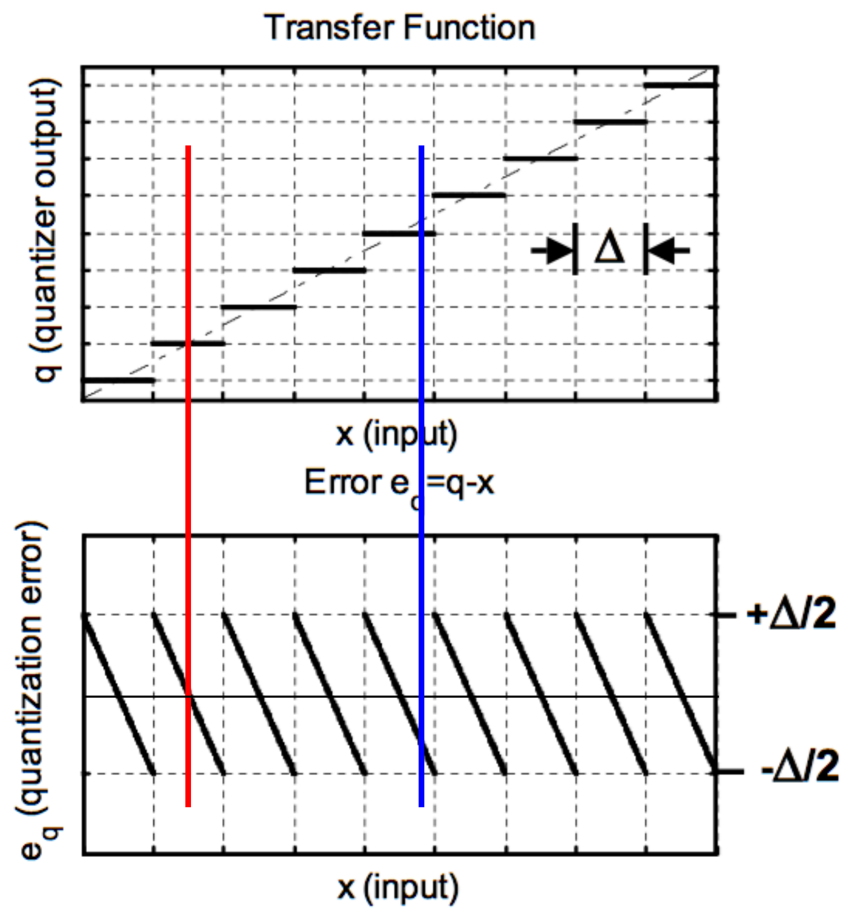
Ideal Quantizer

- Quantization step Δ



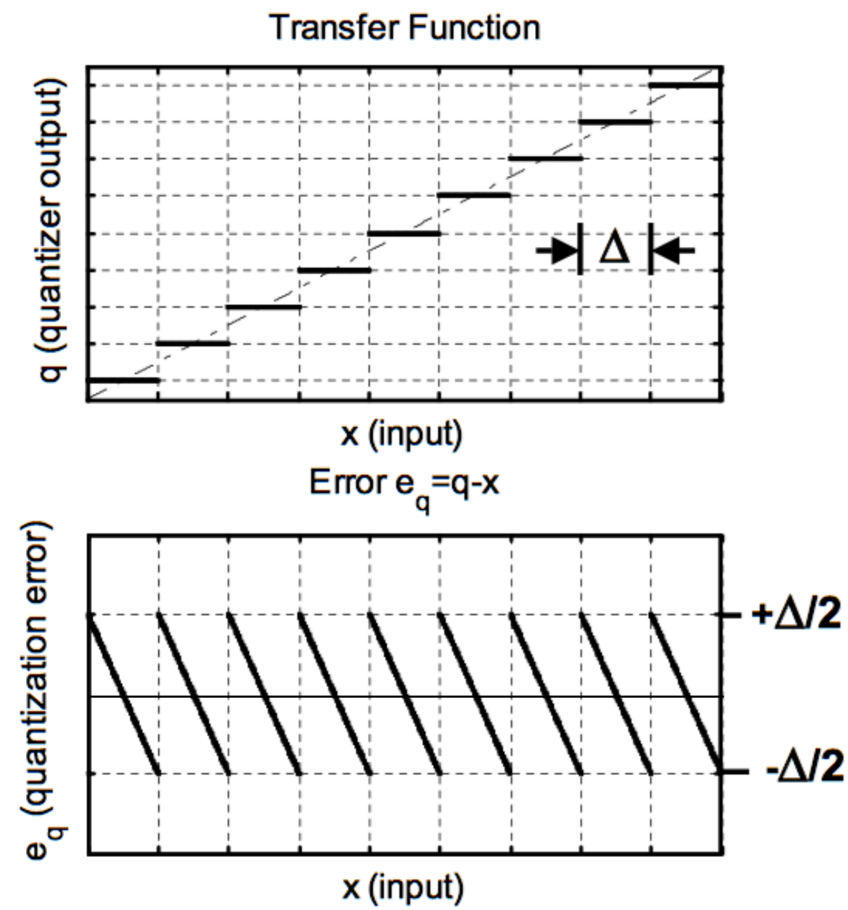
Ideal Quantizer

- Quantization step Δ
- Quantization error has sawtooth shape
  - Bounded by –Δ/2, +Δ/2



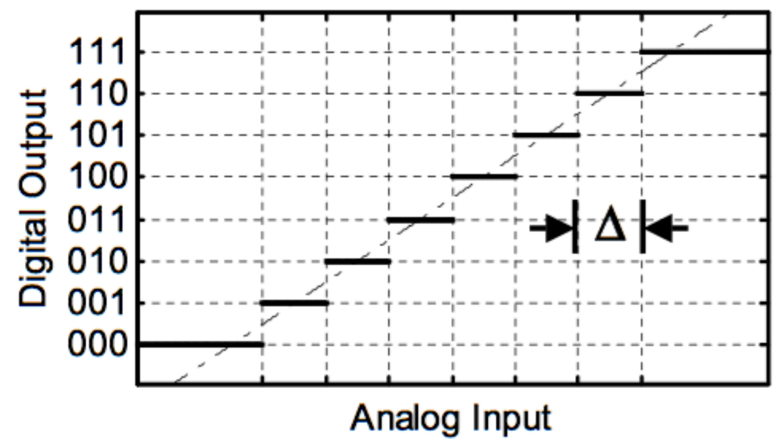
Ideal Quantizer

- Quantization step Δ
- Quantization error has sawtooth shape
  - Bounded by –Δ/2, +Δ/2
- Ideally infinite input range and infinite number of quantization levels



Ideal B-bit Quantizer

- Practical quantizers have a limited input range and a finite set of output codes
- E.g. a 3-bit quantizer can map onto $2^3=8$ distinct output codes



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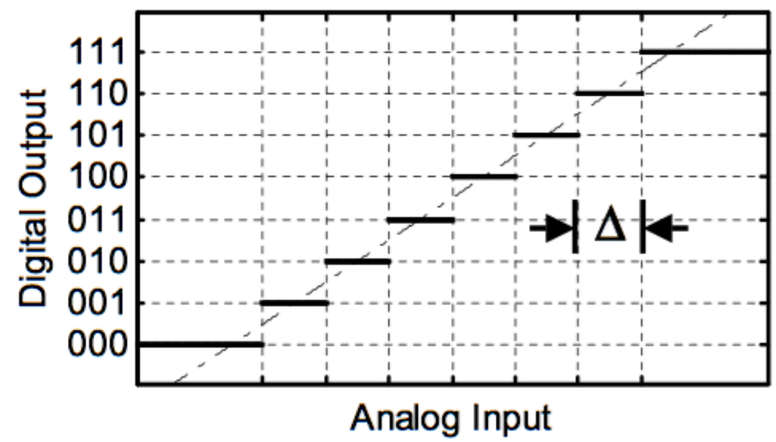
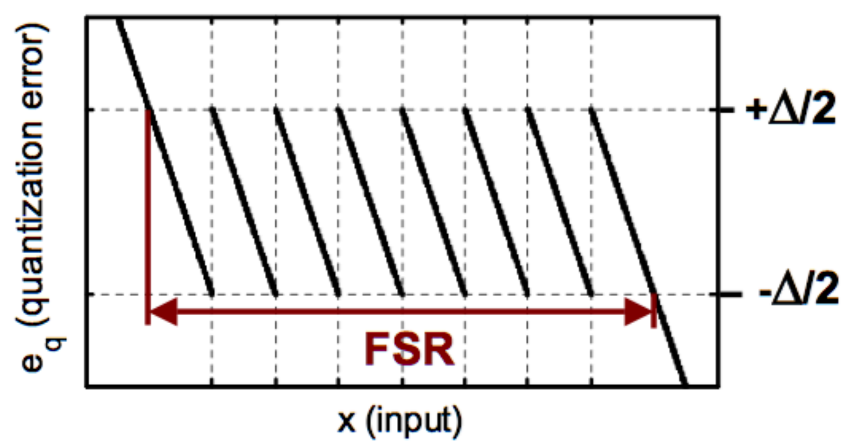


- Quantization error grows out of bounds beyond code boundaries
- We define the full scale range (FSR) as the maximum input range that satisfies $|e_q| \leq \Delta/2$
  - Implies that $FSR = 2^B \cdot \Delta$



Big Ideas

- DTFT
  - Represent signals in time and frequency
  - Find frequency content of signal
- Sampling and Reconstruction
  - Must sample at greater than the Nyquist rate
  - Actually oversample most of the time
    - More on that later…
- ADC transfer function
  - LSB based on FSR and bit resolution of ADC
  - Quantization error imposed

Admin

- Quiz 1 on Wednesday
  - Covers lecture 1-8, labs 1-4`
  - Closed book/note
  - Can bring a calculator
  - Starts at exactly 5:15pm-6:15pm