ESE 3400: Medical Devices Lab

Lec 5: October 4, 2023

Discrete Time Signals and Systems, DTFT, Sampling and Reconstruction

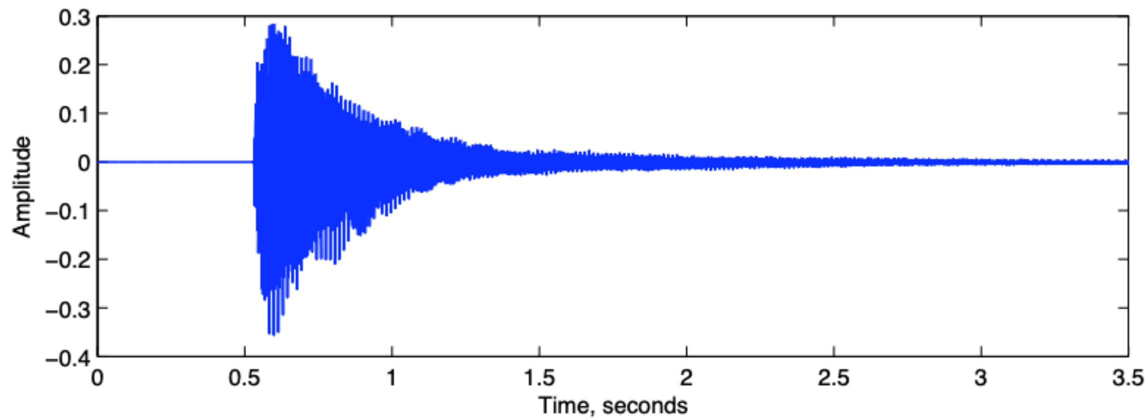
Lecture Outline

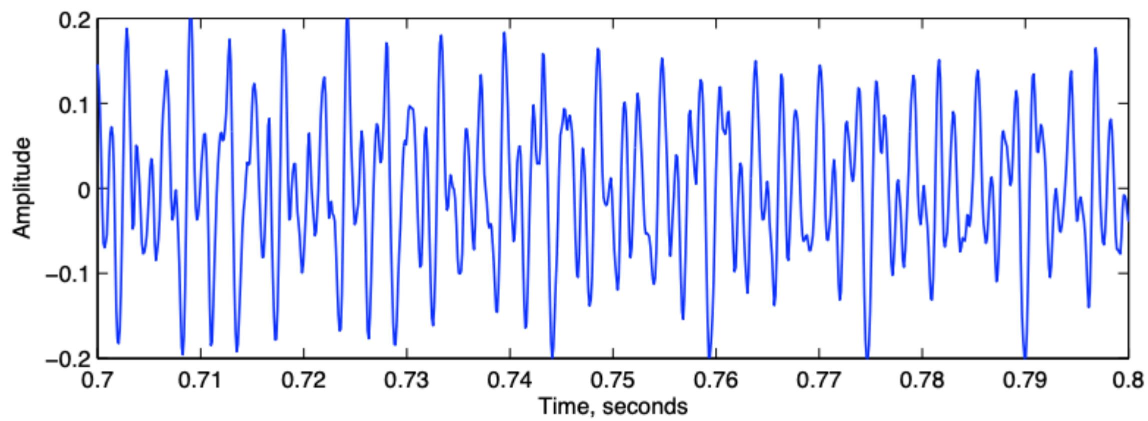
- Continuous Time Signals
  - Laplace Transform
- Discrete Time Signals
  - Aliasing and periodicity
- DTFT/Z-transform
- Sampling/Reconstruction
- Discrete Time Systems
  - LTI Systems
  - Convolution
  - Frequency Response

Continuous Time Signals

Time Domain

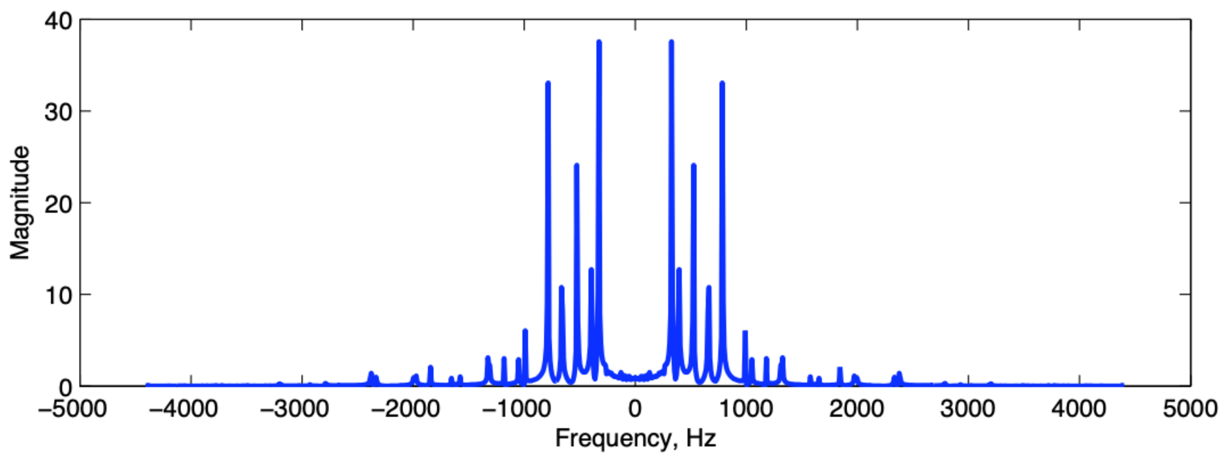
- If you listen to a piano chord, you hear several notes being struck, and fading away. This is waveform is plotted below:

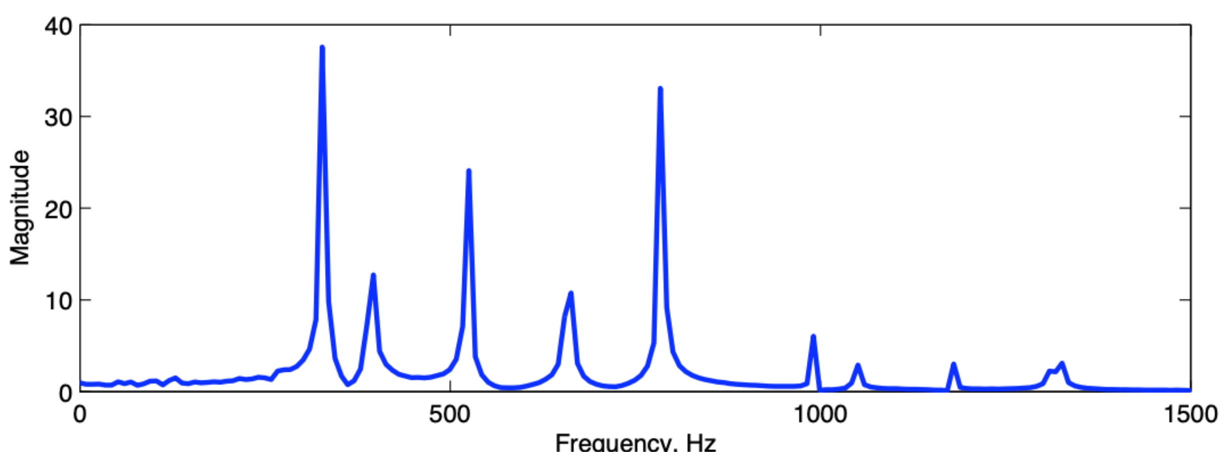




Frequency Domain

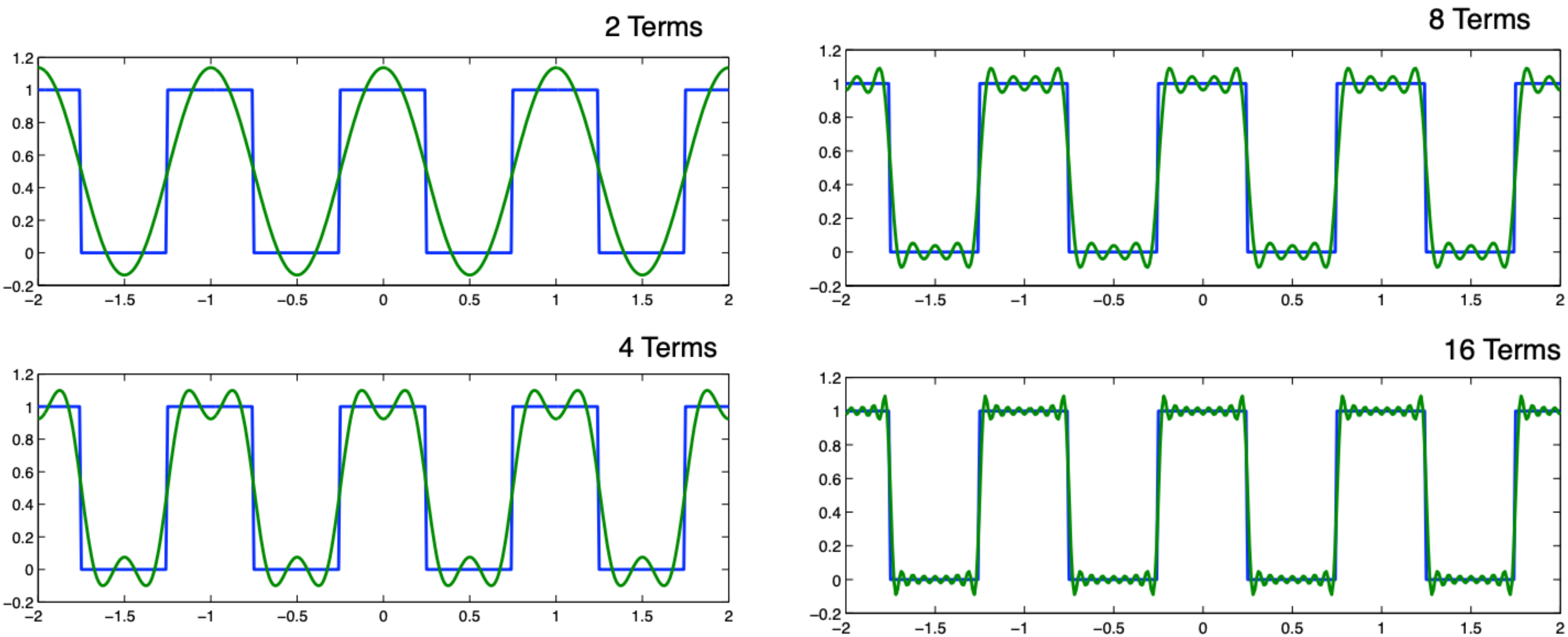
- If we represent the waveform as a sum of sinusoids at different frequencies, and plot the amplitude at each frequency, the plot is much simpler to understand.





Fourier Series

- Fourier Series approximation to a square wave



Fourier Transform

- Given a continuous time signal x(t), define its Fourier transform as the function of a real f :

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt,$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft}\, df.$$

Laplace Transform

- The Laplace transform of a signal f(t) is the function F = L(f) defined by

$$F(s) = \int_0^\infty f(t)e^{-st}\, dt$$

$$f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}\, ds$$

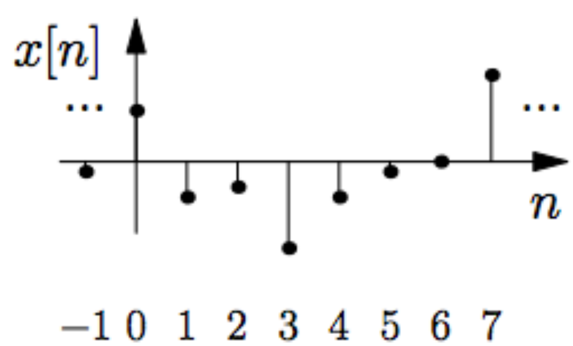
Discrete Time Signals

Signals are Functions

DEFINITION

A **signal** is a function that maps an independent variable to a dependent variable.

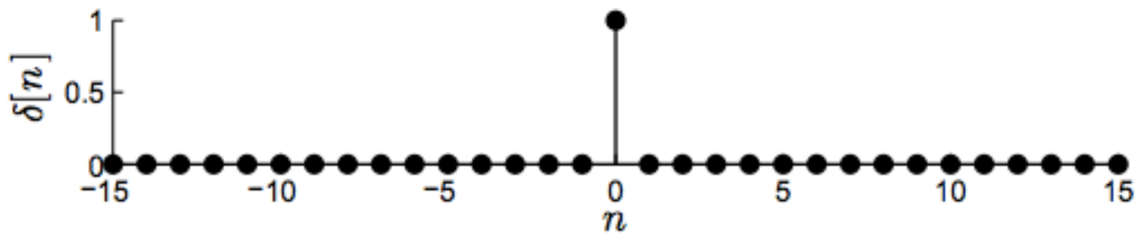
- Signal $x[n]$: each value of n produces the value $x[n]$
- In this course we will focus on **discrete-time** signals:
  - Independent variable is an **integer**: $n \in Z$ (will refer to n as time)
  - Dependent variable is a real or complex number: $x[n] \in R$



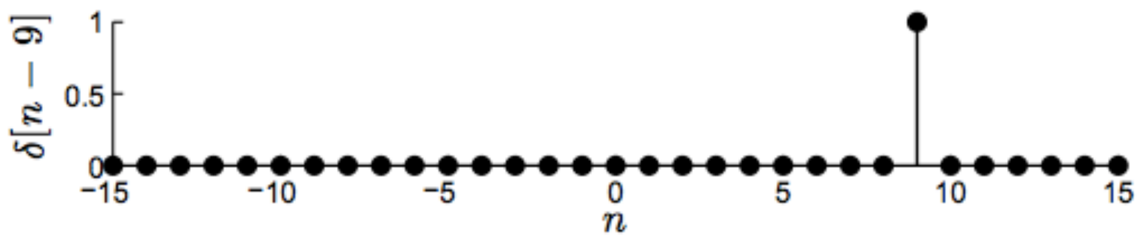
Unit Sample

DEFINITION

The **delta function** (aka unit impulse) $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$



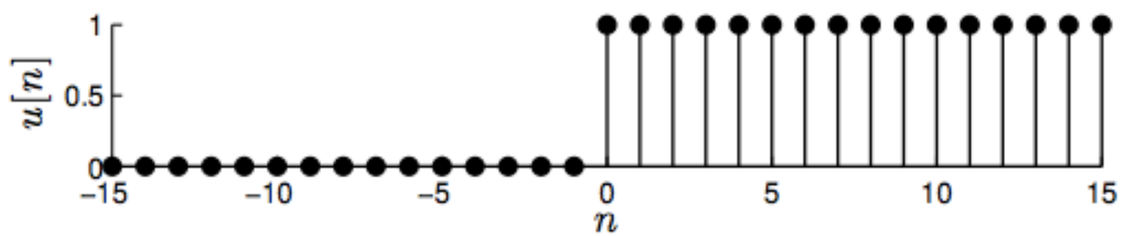
- The shifted delta function $\delta[n-m]$ peaks up at $n = m$; here $m = 9$



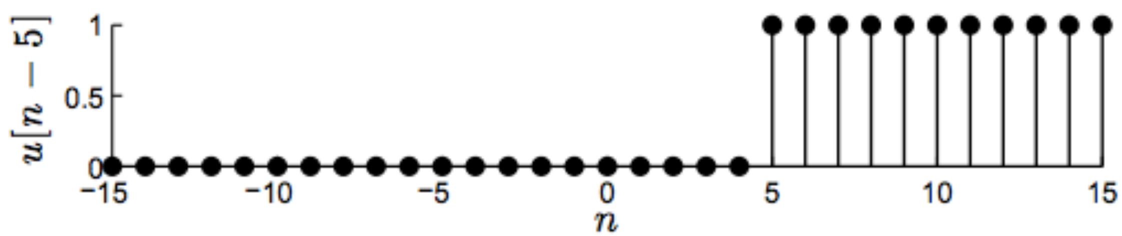
Unit Step

DEFINITION

The **unit step** $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$



- The shifted unit step $u[n-m]$ jumps from 0 to 1 at $n = m$; here, m=5

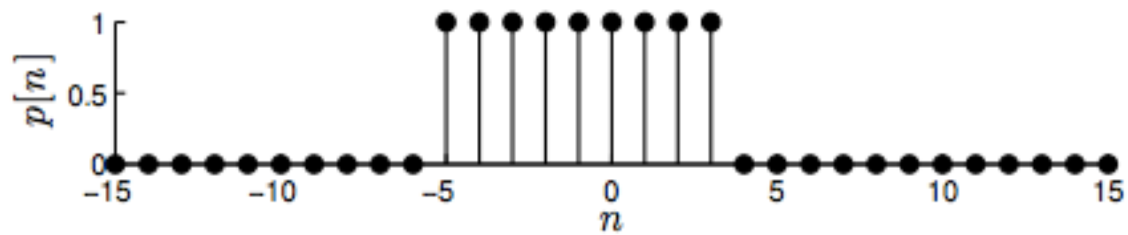


Unit Pulse

DEFINITION

The **unit pulse** (aka boxcar) $p[n] = \begin{cases} 0 & n < N_1 \\ 1 & N_1 \leq n \leq N_2 \\ 0 & n > N_2 \end{cases}$

- Ex: $p[n]$ for $N_1 = -5$ and $N_2 = 3$



- One of many different formulas for the unit pulse

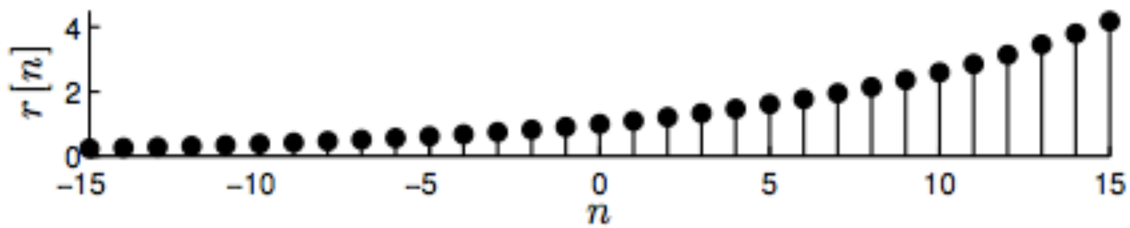
$$p[n] = u[n - N_1] - u[n - (N_2 + 1)]$$

Real Exponential

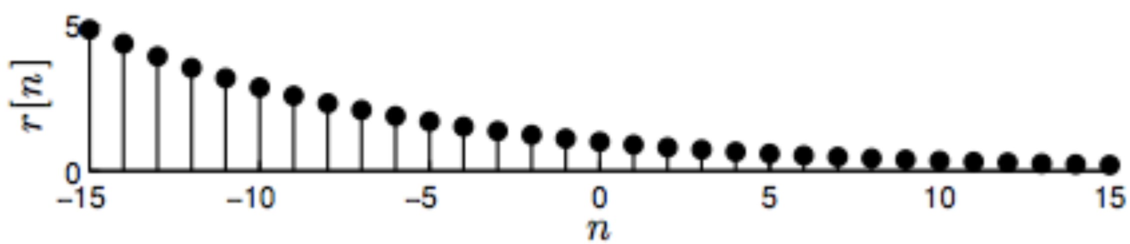
DEFINITION

The **real exponential** $r[n] = a^n$, $a \in \mathbb{R}$, $a \geq 0$

- For $a > 1$, $r[n]$ shrinks to the left and grows to the right; here $a = 1.1$

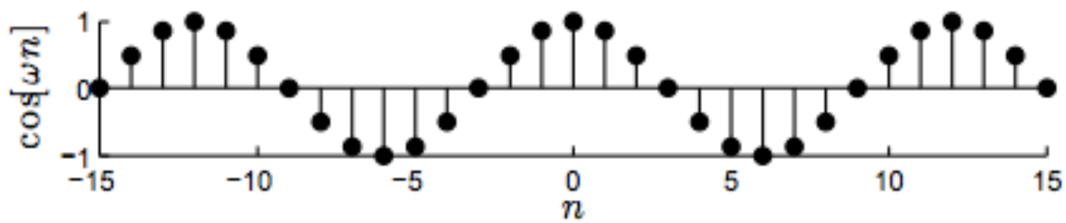


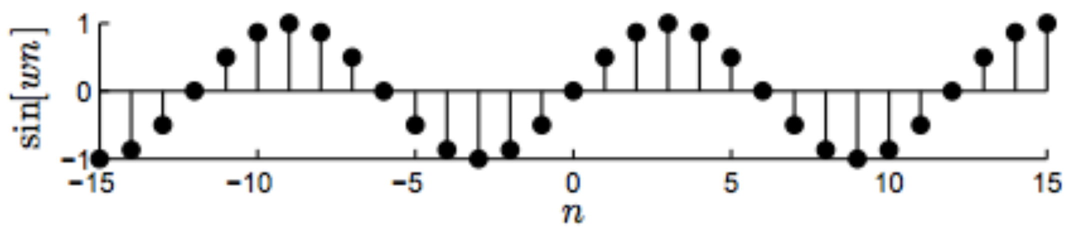
- For $0 < a < 1$, $r[n]$ grows to the left and shrinks to the right; here $a = 0.9$



Sinusoids

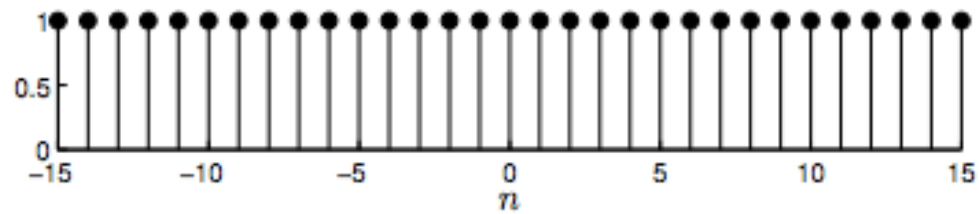
- There are two natural real-value sinusoids: $\cos(\omega n+\phi)$ and $\sin(\omega n+\phi)$
- **Frequency:** ω (units: radians/sample)
- **Phase:** ϕ (units: radians)



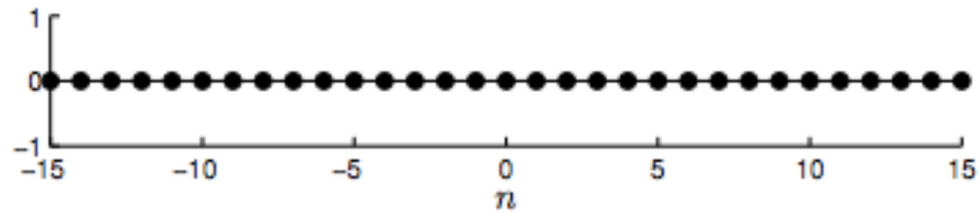


Sinusoid Examples

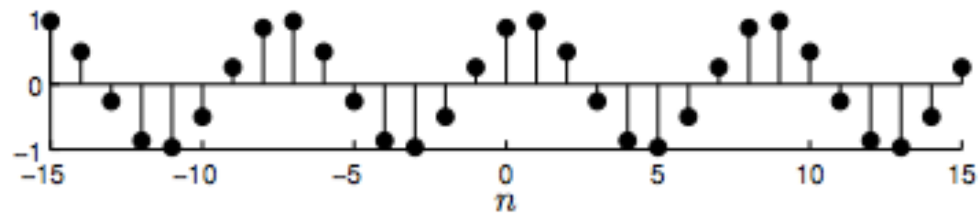
- $\cos(0n)$



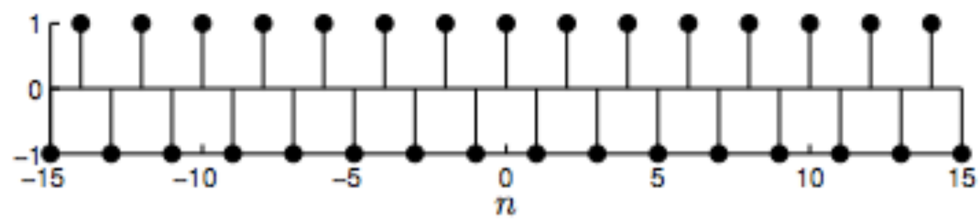
- $\sin(0n)$



- $\sin\left(\frac{\pi}{4}n + \frac{2\pi}{6}\right)$



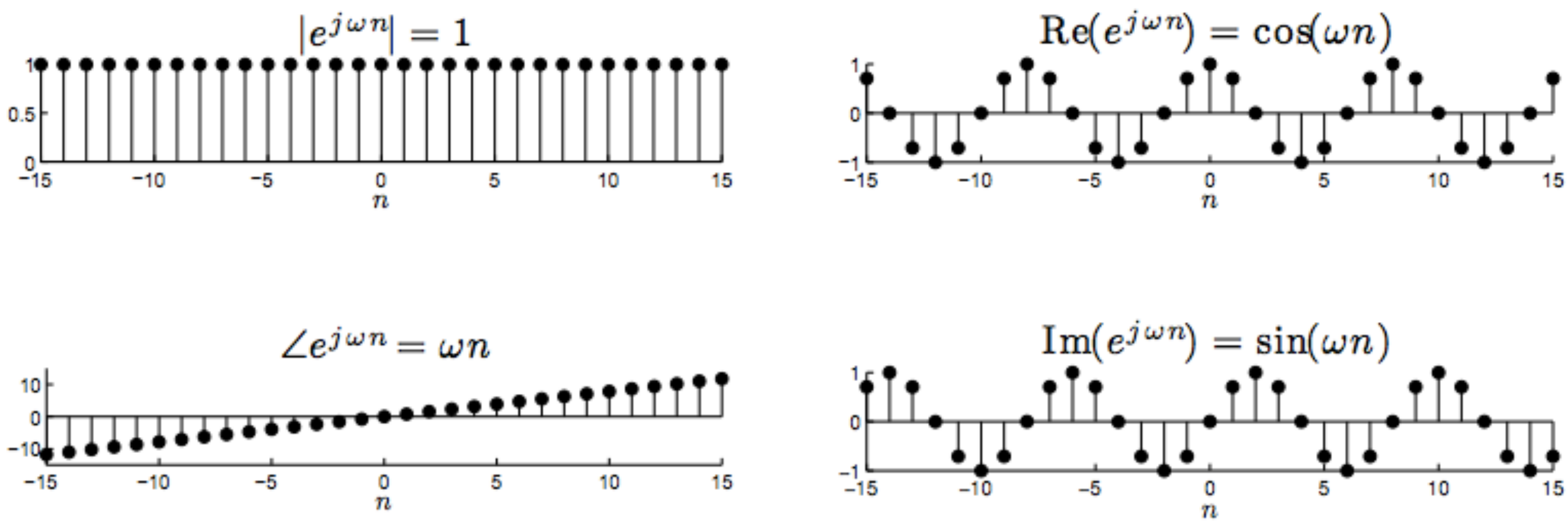
- $\cos(\pi n)$



Complex Sinusoid

- The complex-value sinusoid combines both the cos and sin terms using Euler's identity:

$$e^{j(\omega n+\phi)} = \cos(\omega n + \phi) + j\sin(\omega n + \phi)$$

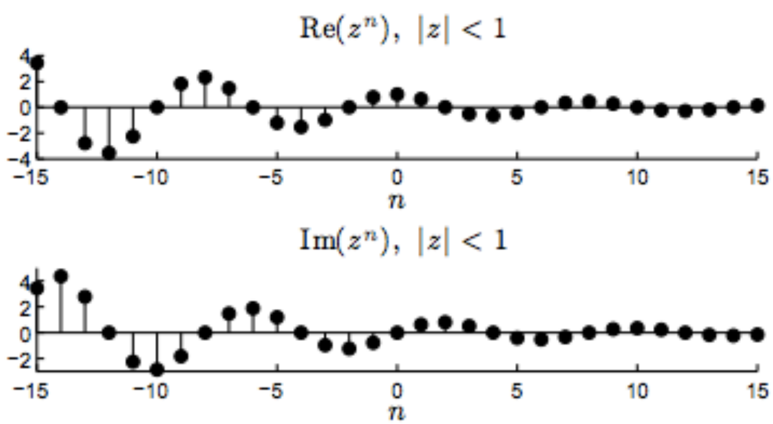


Complex Exponentials

$$z^n = \left(|z|\, e^{j\omega n}\right)^n = |z|^n\, e^{j\omega n}$$

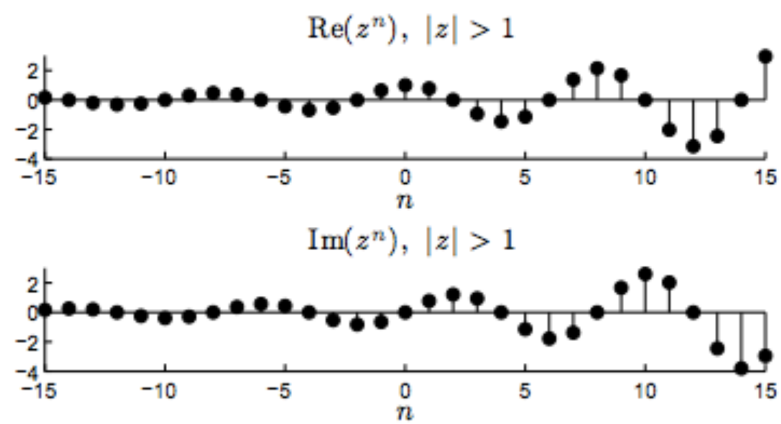
- $|z|^n$ is a real exponential envelope (a^n with $a = |z|$)
- $e^{j\omega n}$ is a complex sinusoid

$|z| < 1$



Bounded

$|z| > 1$



Unbounded

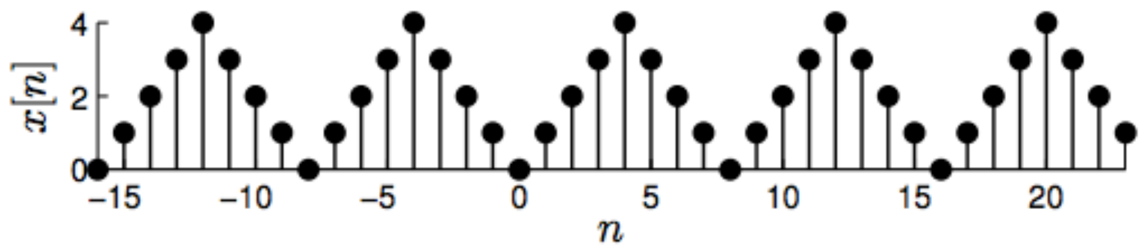
Signal Properties

Periodic Signals

DEFINITION

A discrete-time signal is **periodic** if it repeats with period $N \in \mathbb{Z}$:

$$x[n + mN] = x[n] \quad \forall m \in \mathbb{Z}$$



Notes:

- The period N must be an integer
- A periodic signal is infinite in length

DEFINITION

A discrete-time signal is **aperiodic** if it is not periodic

Discrete-Time Sinusoids

- Discrete-time sinusoids $e^{j(\omega n+\phi)}$ have two counterintuitive properties

- Both involve the frequency ω

- **Property #1**: Aliasing

- **Property #2**: Aperiodicity

Property #1: Aliasing of Sinusoids

- Consider two sinusoids with two different frequencies

$$\omega \quad \Rightarrow \quad x_1[n] = e^{j(\omega n+\phi)}$$

$$\omega + 2\pi \quad \Rightarrow \quad x_2[n] = e^{j((\omega+2\pi)n+\phi)}$$

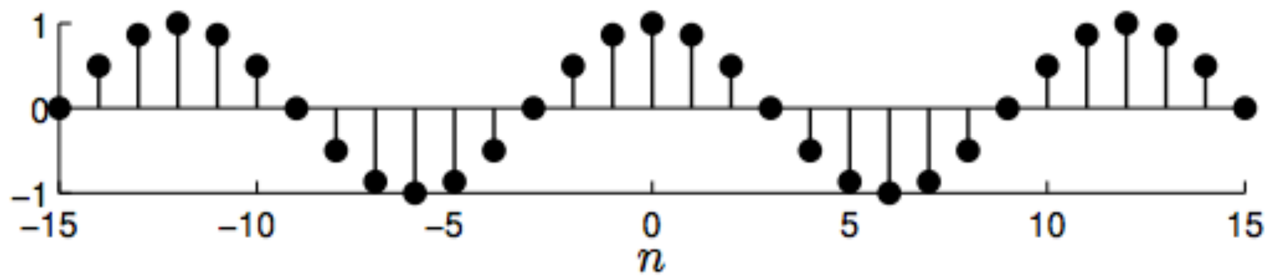
- But note that

$$x_2[n] = e^{j((\omega+2\pi)n+\phi)} = e^{j(\omega n+\phi)+j2\pi n} = e^{j(\omega n+\phi)}\ e^{j2\pi n} = e^{j(\omega n+\phi)} = x_1[n]$$

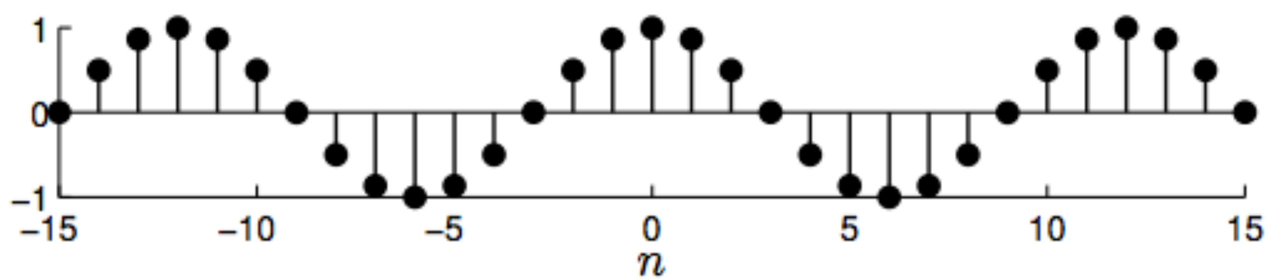
- The signals x_1 and x_2 have different frequencies but are **identical!**
- We say that x_1 and x_2 are aliases; this phenomenon is called aliasing
- Note: Any integer multiple of 2π will do; try with $x_3[n] = e^{j((\omega+2\pi m)n+\phi)},\ m \in \mathbb{Z}$

Aliasing Example

$$x_1[n] = \cos\left(\frac{\pi}{6}n\right)$$

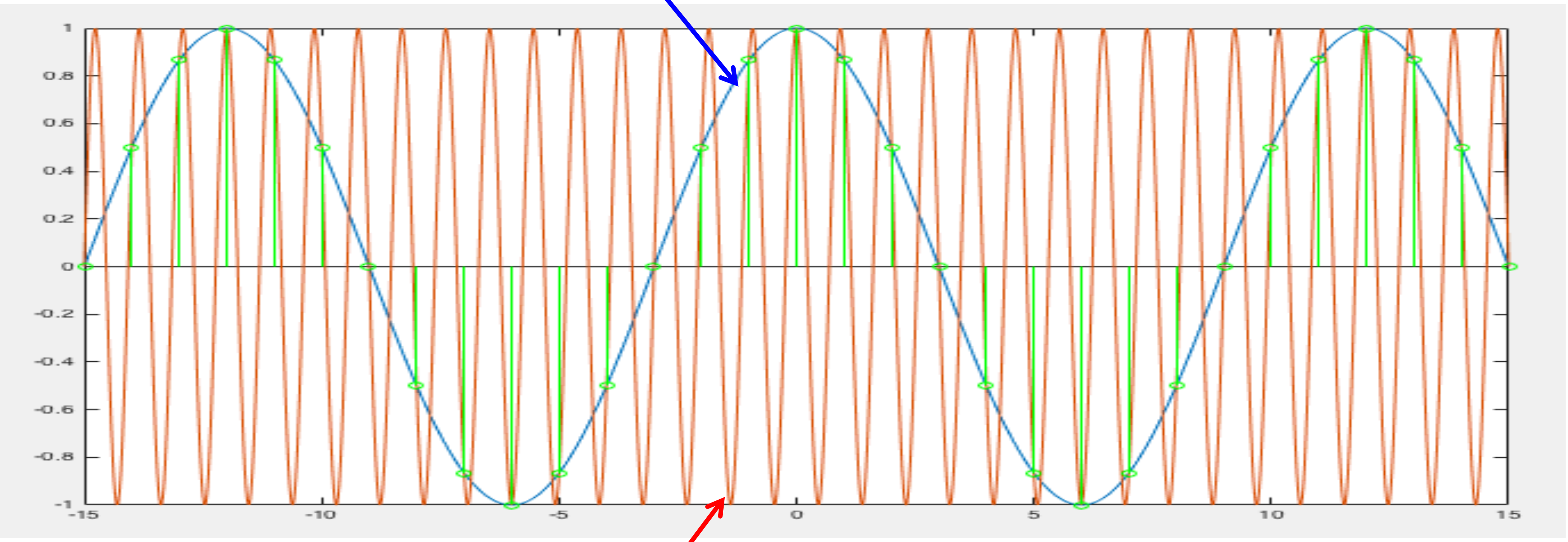


$$x_2[n] = \cos\left(\frac{13\pi}{6}n\right) = \cos\left(\left(\frac{\pi}{6} + 2\pi\right)n\right)$$



Aliasing Example

$x_1(t) = \cos\left(\frac{\pi}{6}t\right)$



$x_2(t) = \cos\left(\left(\frac{\pi}{6} + 2\pi\right)t\right) = \cos\left(\frac{13\pi}{6}t\right)$

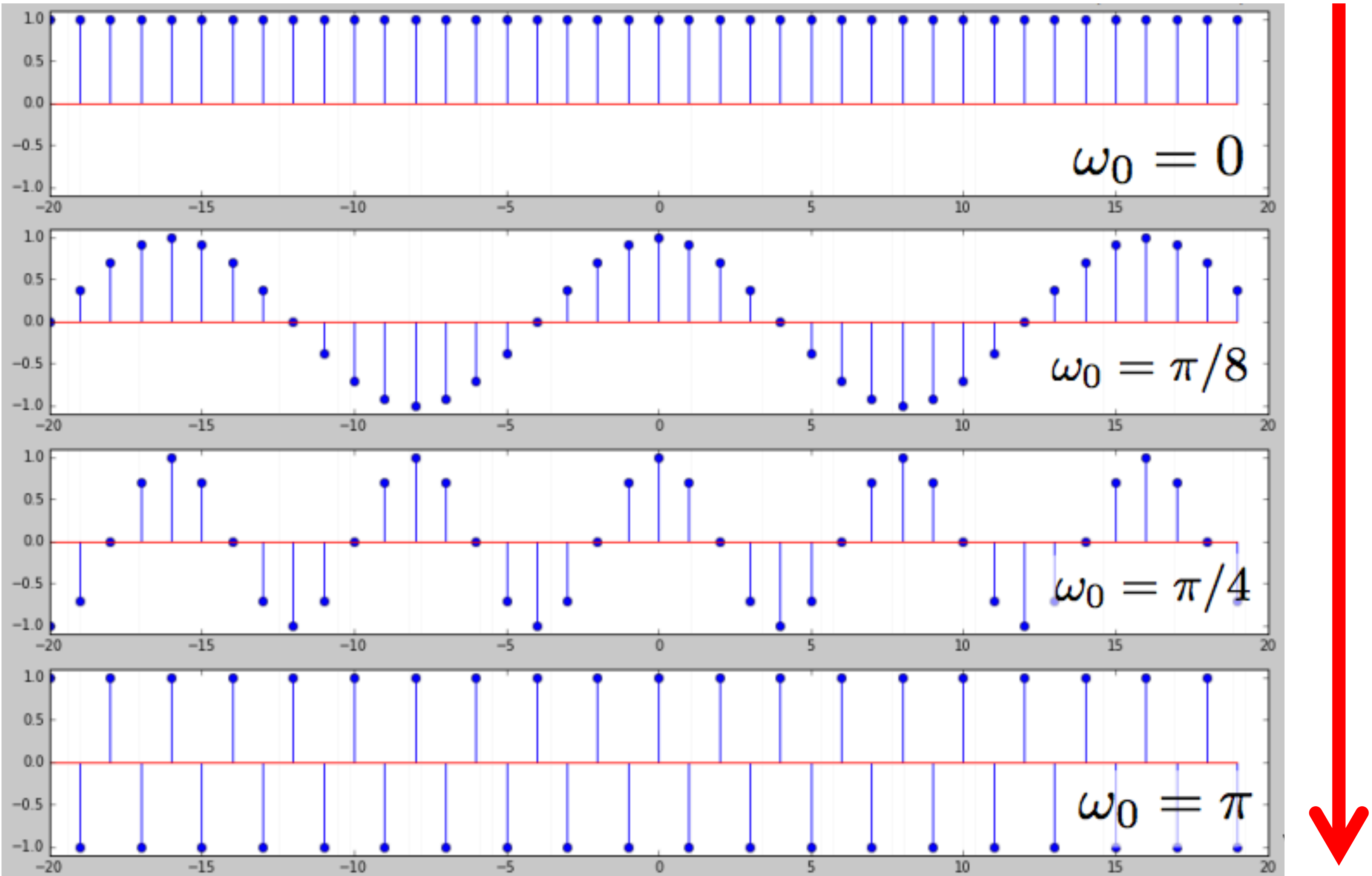
Video Example

- https://www.youtube.com/watch?v=ByTsISFXUoY

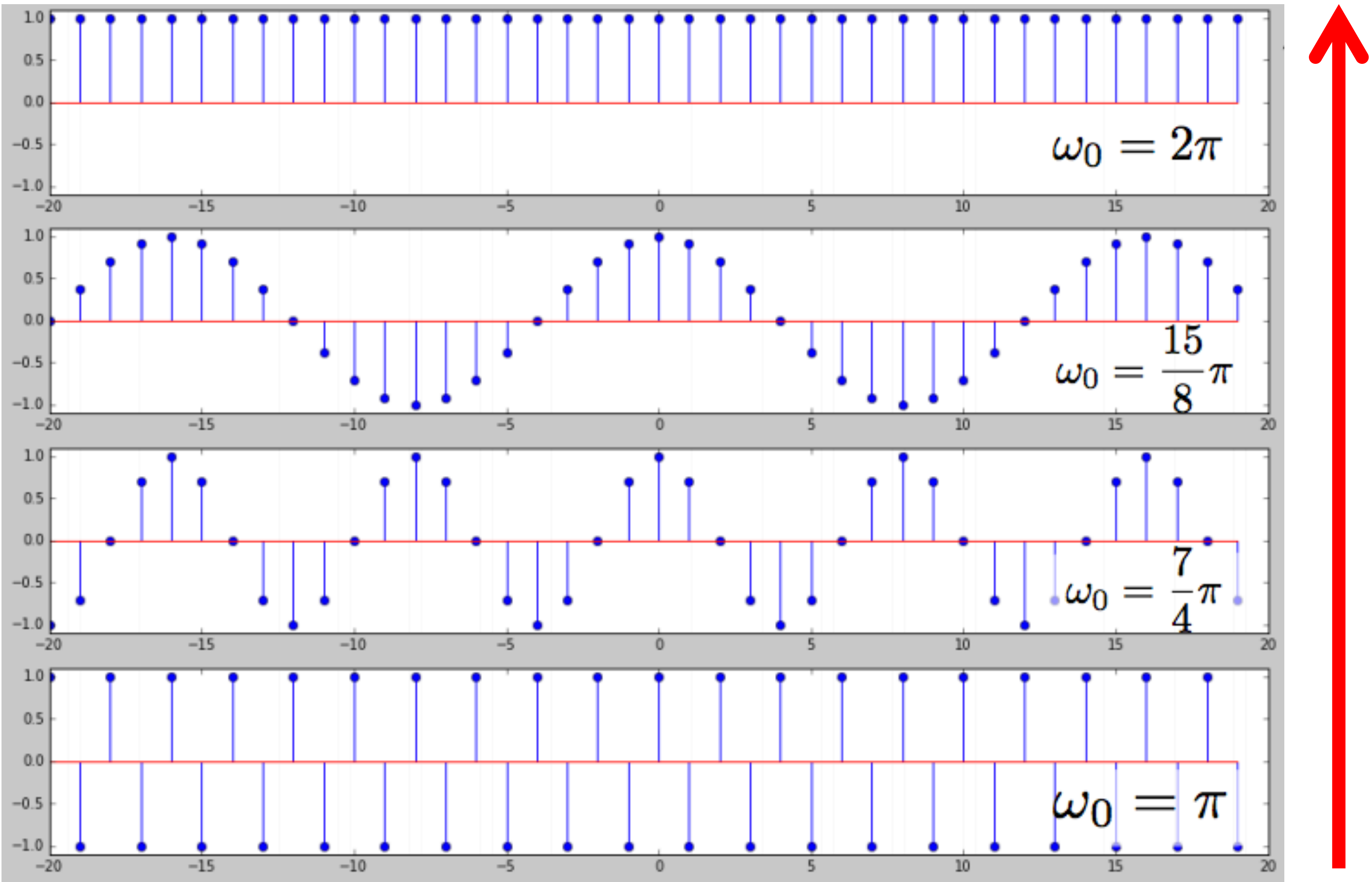
Which is higher in frequency?

- $\cos(\pi n)$ or $\cos(3\pi/2n)$?

Increasing Frequency



Decreasing Frequency



Property #2: Periodicity of Sinusoids

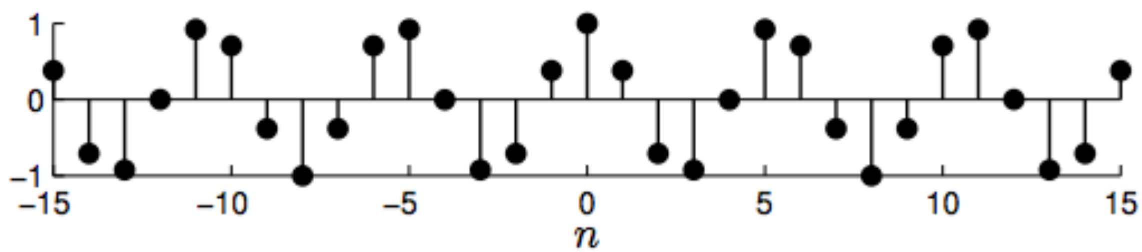
- Consider $x_1[n] = e^{j(\omega n+\phi)}$ with frequency $\omega = \frac{2\pi k}{N}$, $k, N \in \mathbb{Z}$ (harmonic frequency)

Property #2: Periodicity of Sinusoids

- Consider $x_1[n] = e^{j(\omega n + \phi)}$ with frequency $\omega = \frac{2\pi k}{N}, \quad k, N \in \mathbb{Z}$ (harmonic frequency)
- It is easy to show that x_1 is periodic with period N, since

$$x_1[n+N] = e^{j(\omega(n+N)+\phi)} = e^{j(\omega n + \omega N + \phi)} = e^{j(\omega n+\phi)}\, e^{j(\omega N)} = e^{j(\omega n+\phi)}\, e^{j(\frac{2\pi k}{N} N)} = x_1[n] \checkmark$$

- Ex: $x_1[n] = \cos(\frac{2\pi 3}{16} n), \quad N = 16$



- Note: x_1 is periodic with the (smaller) period of $\frac{N}{k}$ when $\frac{N}{k}$ is an integer

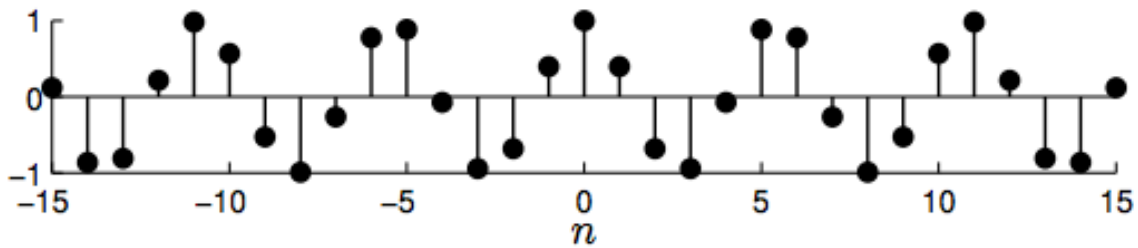
Aperiodicity of Sinusoids

- Consider $x_2[n] = e^{j(\omega n + \phi)}$ with frequency $\omega \neq \frac{2\pi k}{N}$, $\quad k, N \in \mathbb{Z}$ (not harmonic frequency)

- Is x_2 periodic?

$$x_2[n+N] = e^{j(\omega(n+N)+\phi)} = e^{j(\omega n + \omega N + \phi)} = e^{j(\omega n + \phi)}\, e^{j(\omega N)} \neq x_1[n] \quad \text{NO!}$$

- Ex: $x_2[n] = \cos(1.16\, n)$



- If its frequency ω is not harmonic, then a sinusoid oscillates but is not periodic!

Harmonic Sinusoids

$$e^{j(\omega n+\phi)}$$

- Semi-amazing fact: The **only** periodic discrete-time sinusoids are those with **harmonic frequencies**

$$\omega = \frac{2\pi k}{N}, \quad k, N \in \mathbb{Z}$$

- Which means that
  - **Most** discrete-time sinusoids are **not** periodic!
  - The harmonic sinusoids are somehow magical (they play a starring role later in the DFT)

Periodic or not?

- $\cos(5/7\pi n)$

- What are N and k? (I.e How many samples is one period?

Discrete-Time Fourier Transform (DTFT)

DTFT Definition

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

DTFT Definition

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Alternate

$$X(f) = \sum_{k=-\infty}^{\infty} x[k]e^{-j2\pi fk}$$

$$x[n] = \int_{-0.5f_s}^{0.5f_s} X(f)e^{j2\pi fn}df$$

z-Transform

- Define the **forward z-transform** of x[n] as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

- The core "basis functions" of the z-transform are the complex exponentials z^n with arbitrary $z \in C$; these are the eigenfunctions of LTI systems for infinite-length signals

- **Notation abuse alert:** We use X(•) to represent both the DTFT $X(e^{j\omega})$ and the z-transform X(z); they are, in fact, intimately related

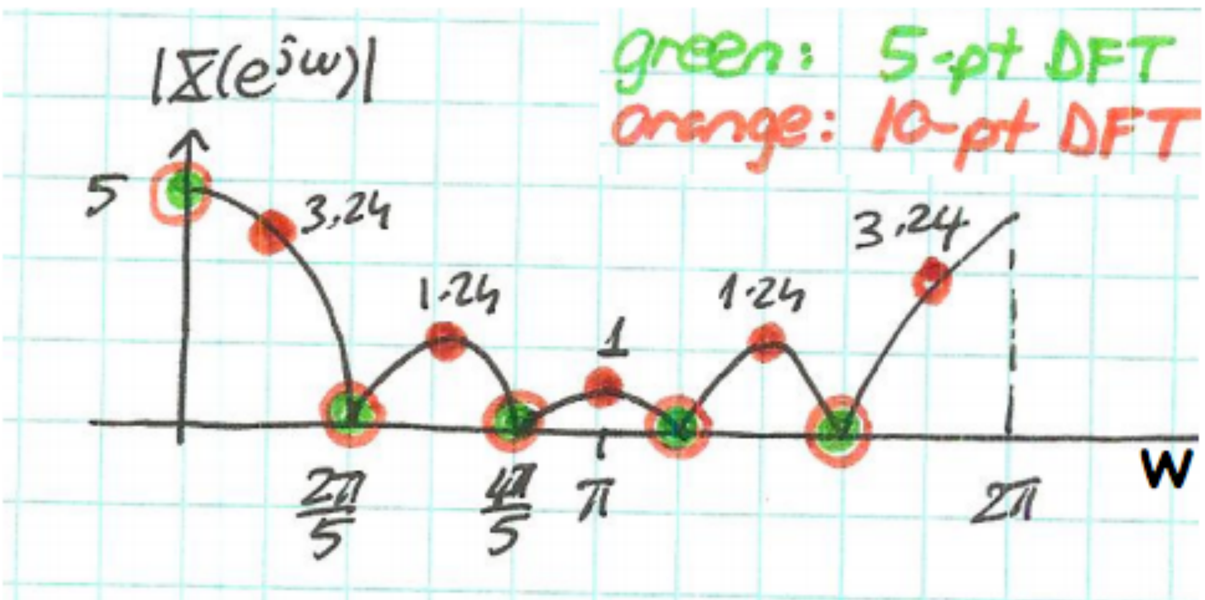
$$X_z(z)\big|_{z=e^{j\omega}} = X_z(e^{j\omega})$$

DFT/FFT vs DTFT

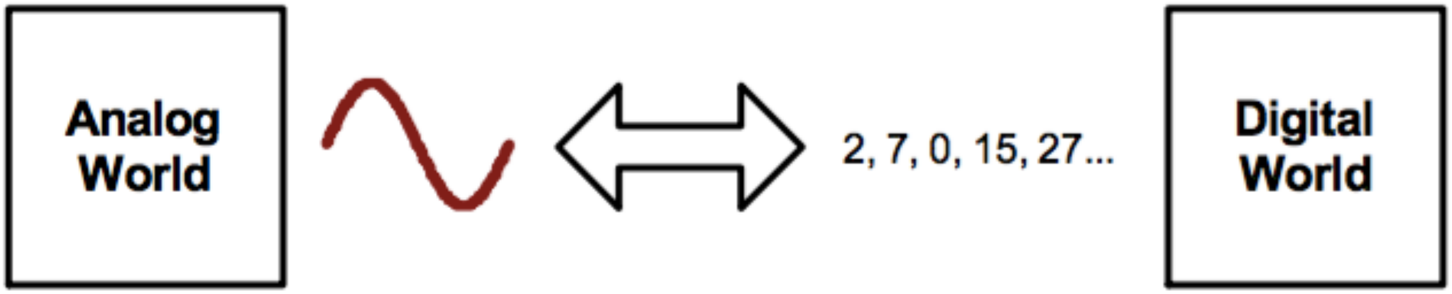
- Back to example

$$X[k] = \sum_{n=0}^{4} W_{10}^{nk}$$

$$= e^{-j\frac{4\pi}{10}k}\frac{\sin(\frac{\pi}{2}k)}{\sin(\frac{\pi}{10}k)}$$

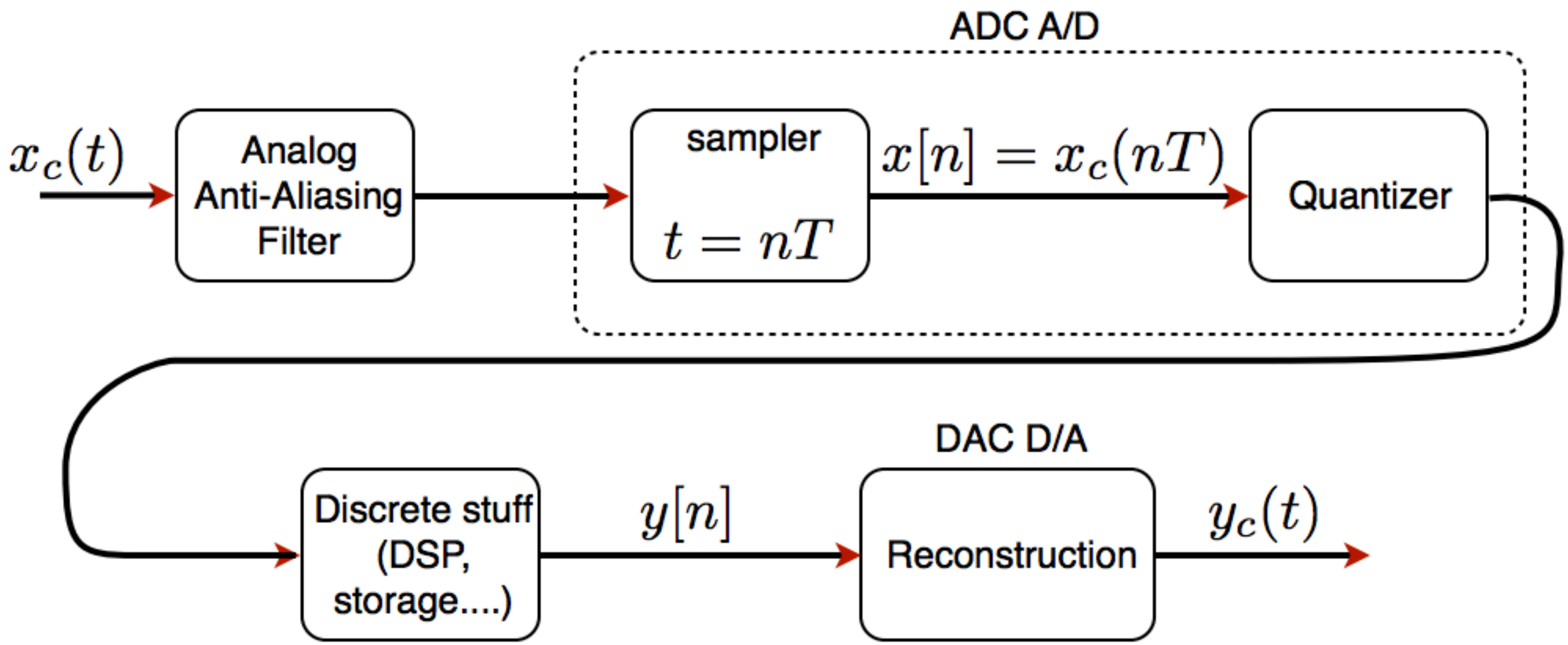


The Data Conversion Problem

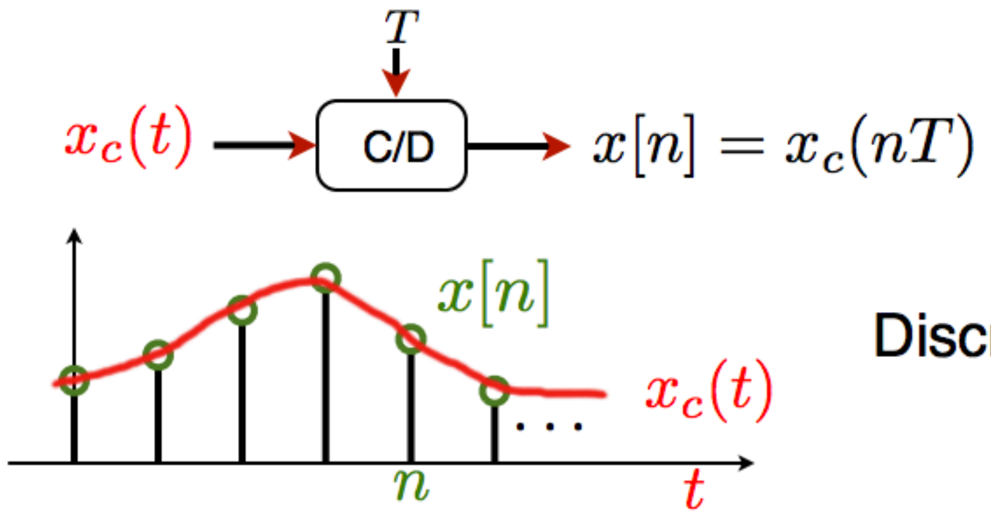


- Real world signals
  - Continuous time, continuous amplitude
- Digital abstraction
  - Discrete time, discrete amplitude
- Two problems
  - How to go discretize in time and amplitude
    - A/D conversion
  - How to "undescretize" in time and amplitude
    - D/A conversion

DSP System


ADC A/D
$x_c(t)$
Analog Anti-Aliasing Filter
sampler
$t = nT$
$x[n] = x_c(nT)$
Quantizer
DAC D/A
Discrete stuff (DSP, storage....)
$y[n]$
Reconstruction
$y_c(t)$

Ideal Sampling Model

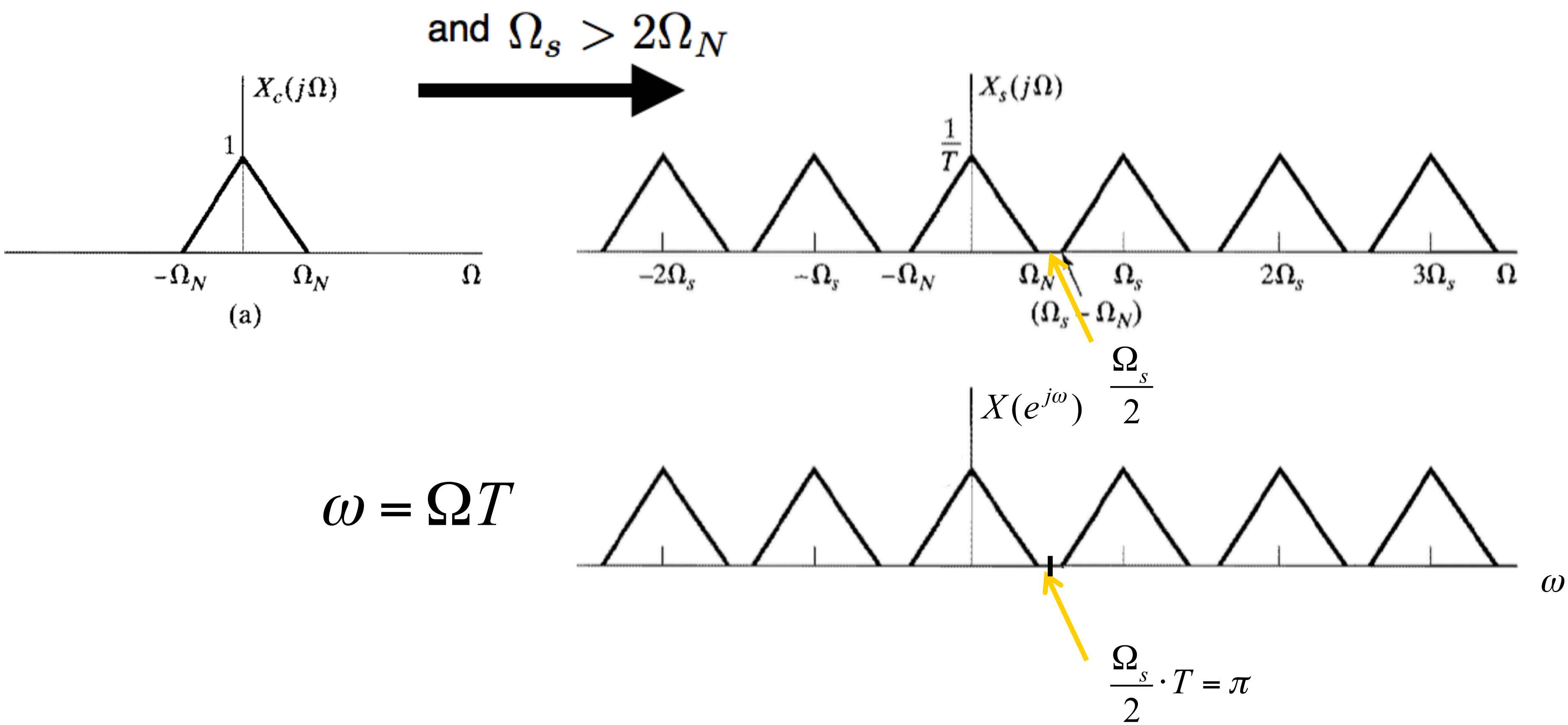


Discrete and Continuous

- Ideal continuous-to-discrete time (C/D) converter
  - T is the sampling period
  - f_s=1/T is the sampling frequency
  - Ω_s=2π/T

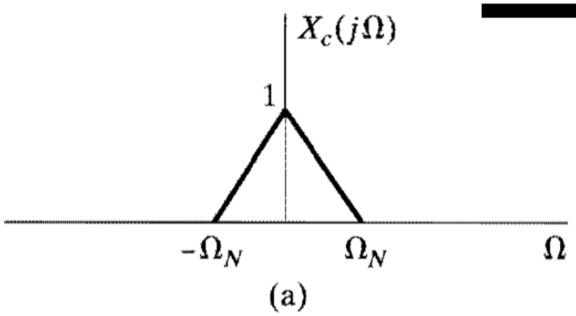
Frequency Domain Analysis

Sample greater than Nyquist

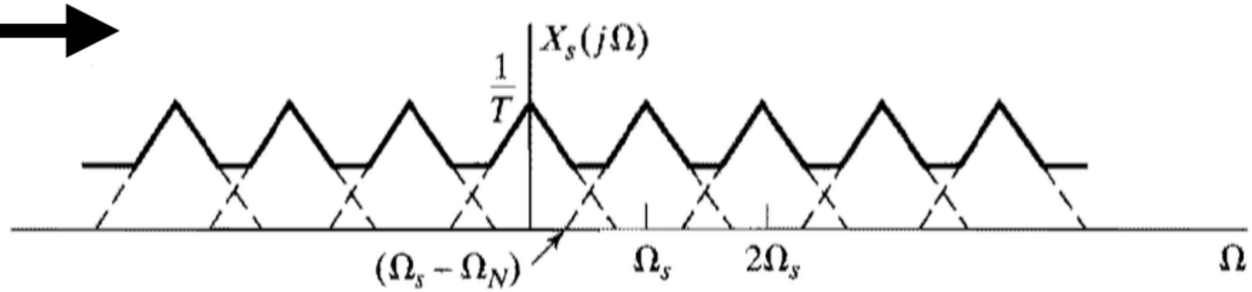


Frequency Domain Analysis w/Aliasing

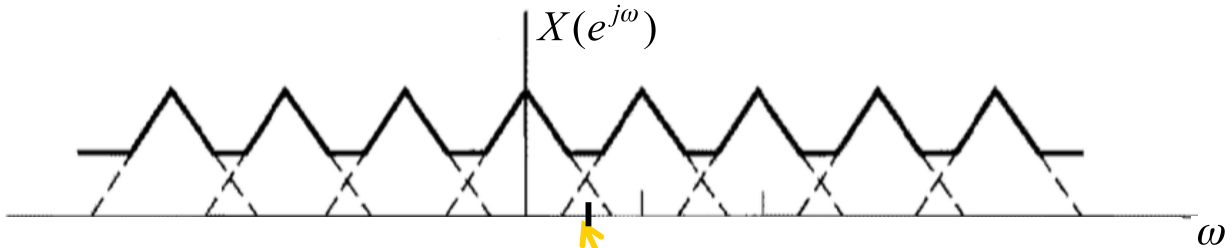
Sample less than Nyquist



and $\Omega_s < 2\Omega_N$

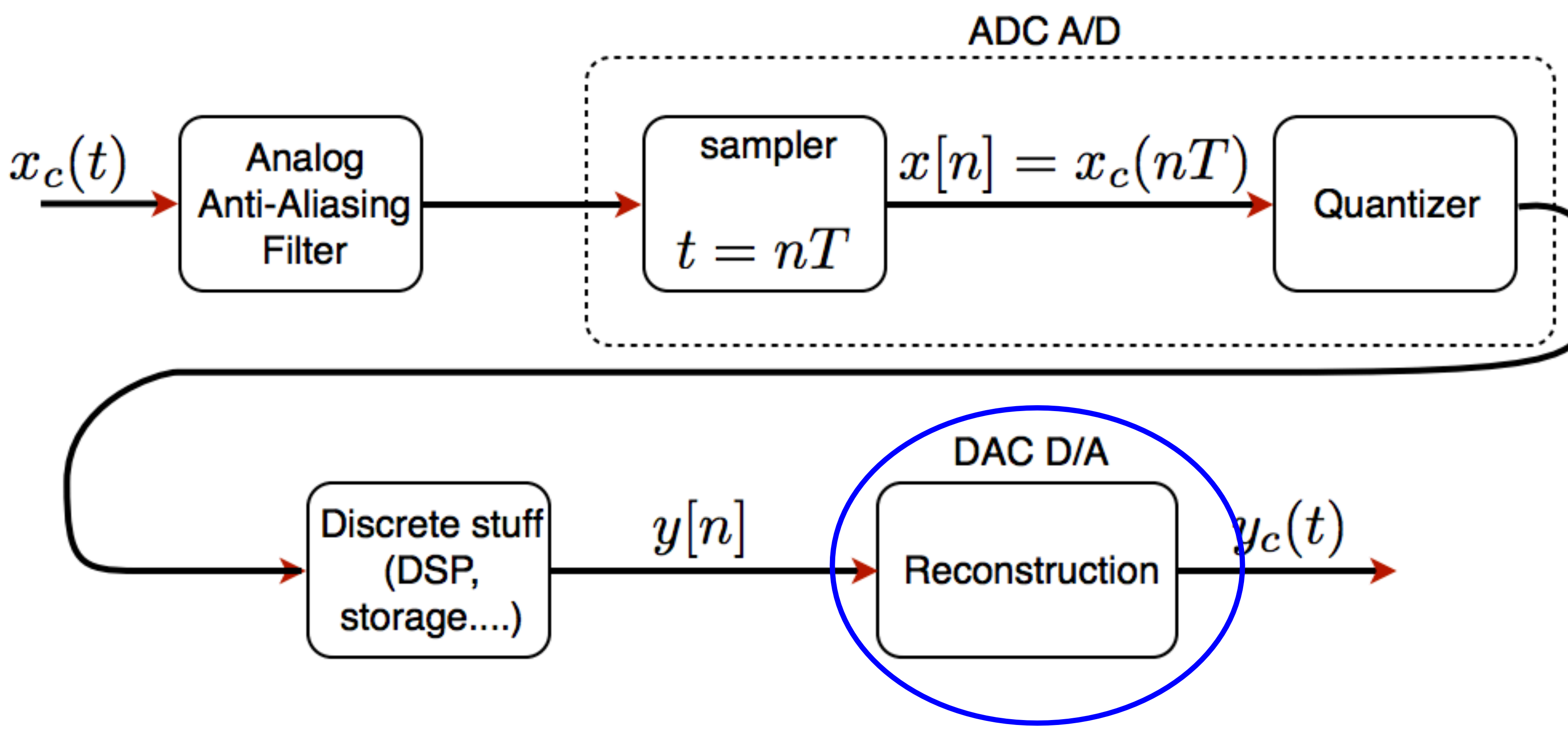


$$\omega = \Omega T$$



$$\frac{\Omega_s}{2} \cdot T = \pi$$

DSP System



ADC A/D
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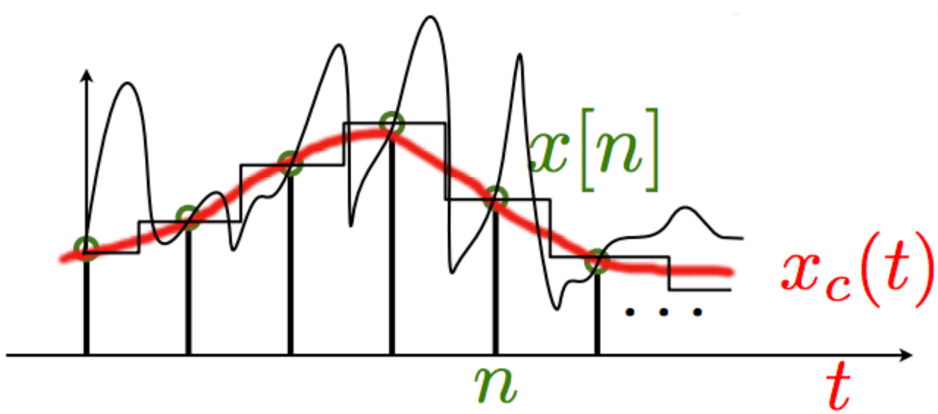
DAC D/A
Discrete stuff (DSP, storage....)
$y[n]$
Reconstruction
$y_c(t)$

Reconstruction of Bandlimited Signals

- Nyquist Sampling Theorem: Suppose $x_c(t)$ is bandlimited. I.e.

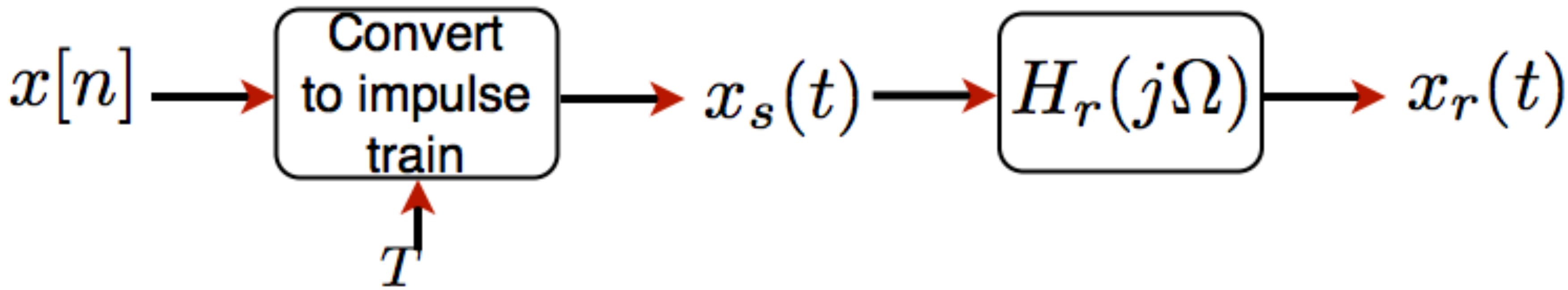
$$X_c(j\Omega) = 0 \;\; \forall \quad |\Omega| \geq \Omega_N$$

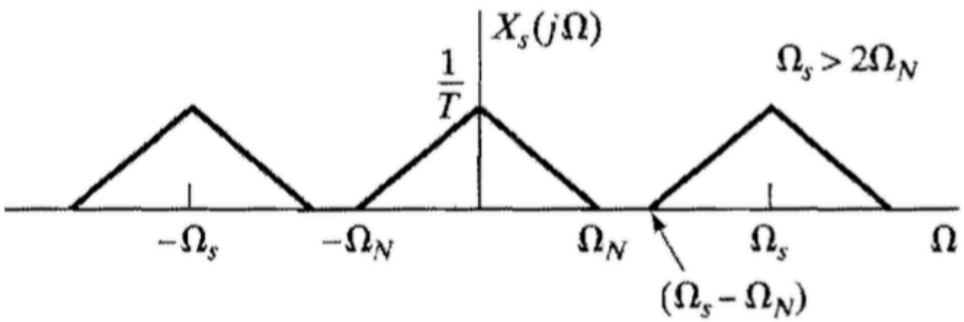
- If $\Omega_s \geq 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n]=x_c(nT)$
- Bandlimitedness is the key to uniqueness

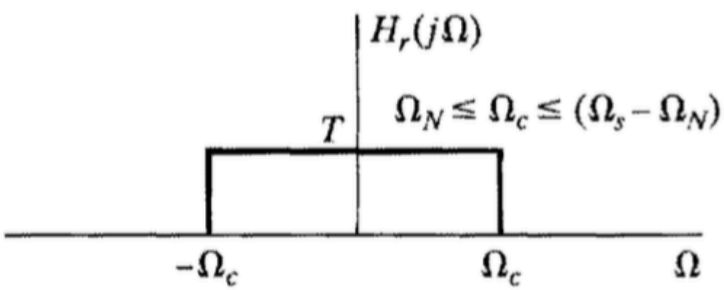


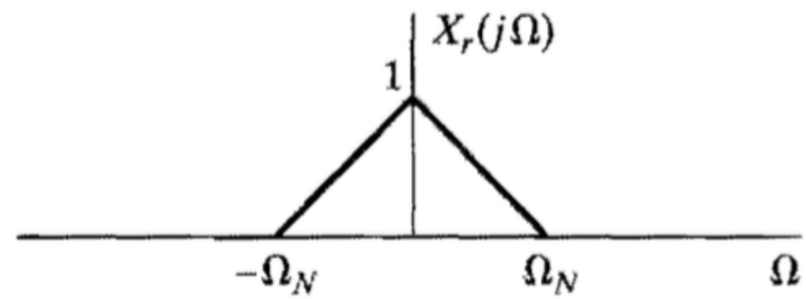
Mulitiple signals go through the samples, but only one is bandlimited within our sampling band

Reconstruction in Frequency Domain







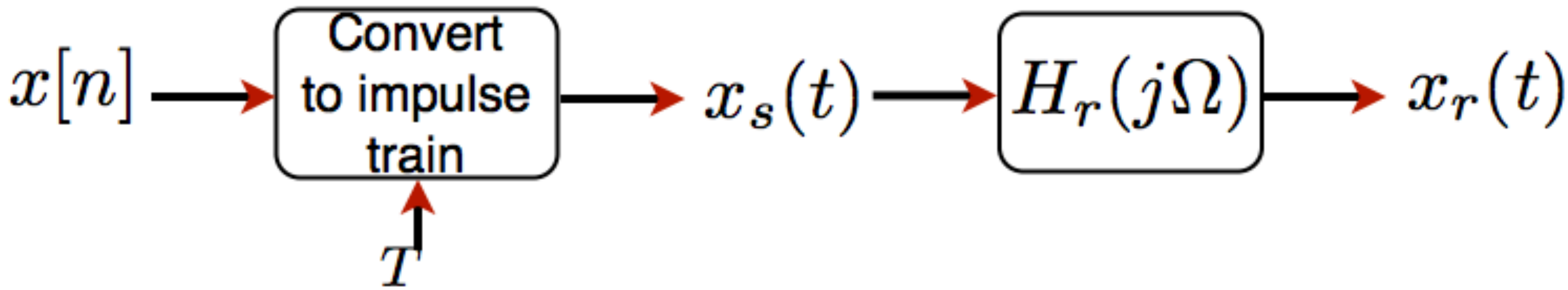


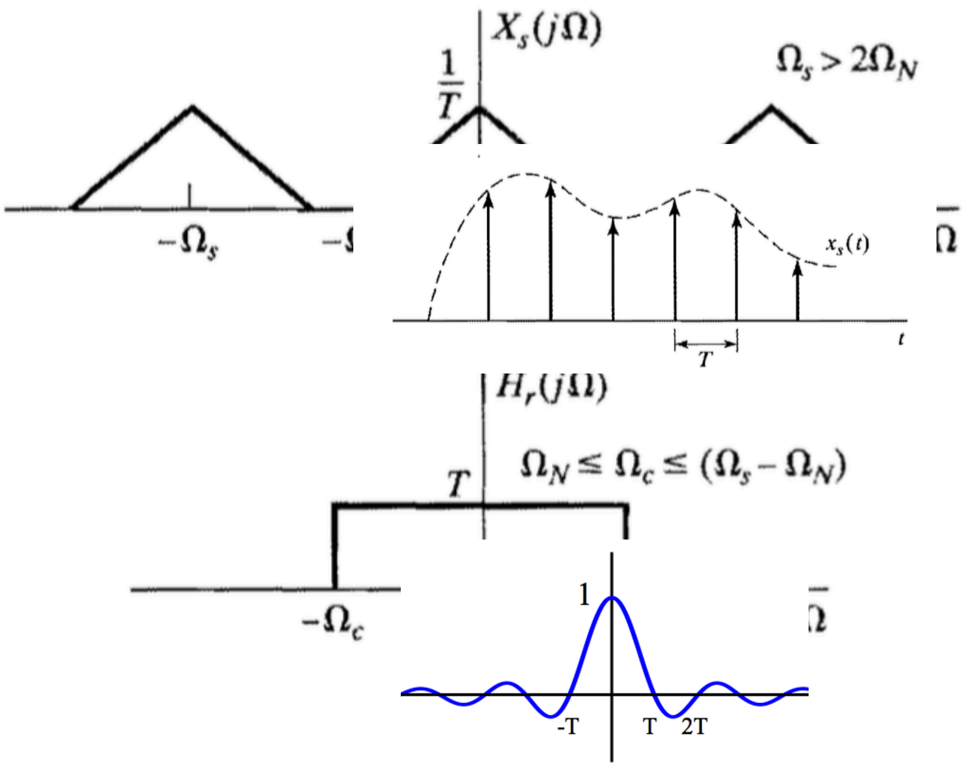
$2\Omega_N = \Omega_s$

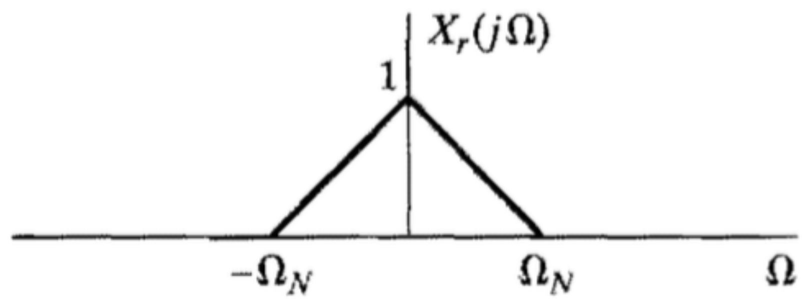
$\Omega_N = \Omega_C = \Omega_s/2$

Sample at Nyquist and filter at signal badnwidth

Reconstruction in Frequency Domain







$2\Omega_N = \Omega_s$

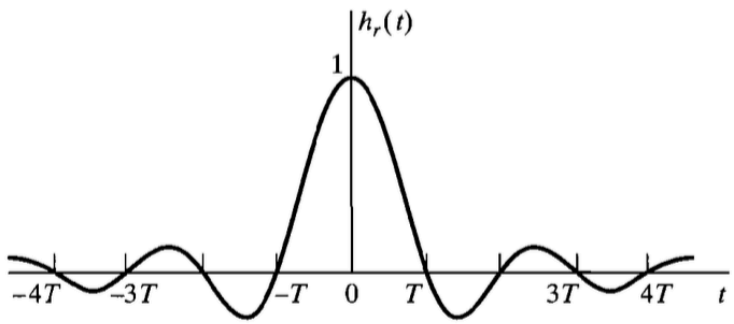
$\Omega_N = \Omega_C = \Omega_s/2$

Sample at Nyquist and filter at signal badnwidth

Reconstruction in Time Domain

$$x_r(t) = x_s(t) * h_r(t) = \left(\sum_n x[n]\delta(t - nT)\right) * h_r(t)$$

$$= \sum_n x[n]h_r(t - nT)$$



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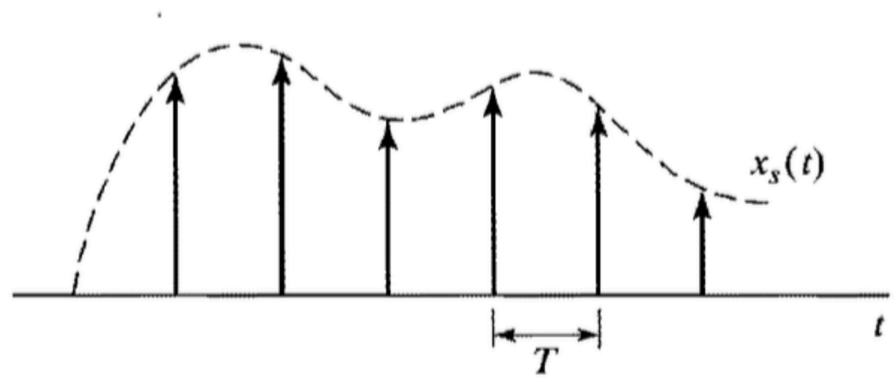

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Reconstruction in Time Domain

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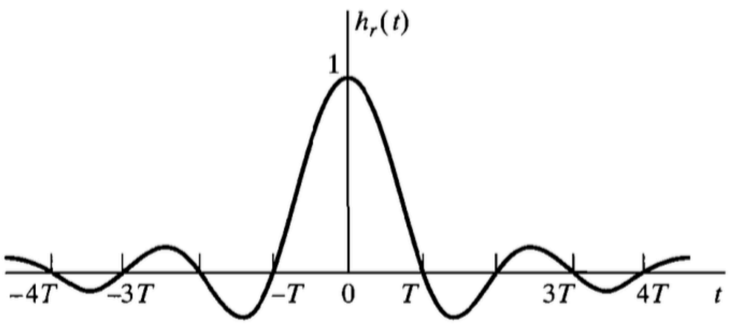

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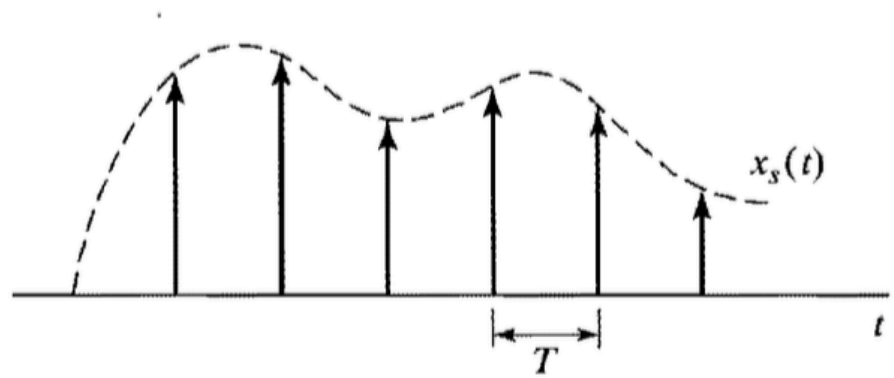

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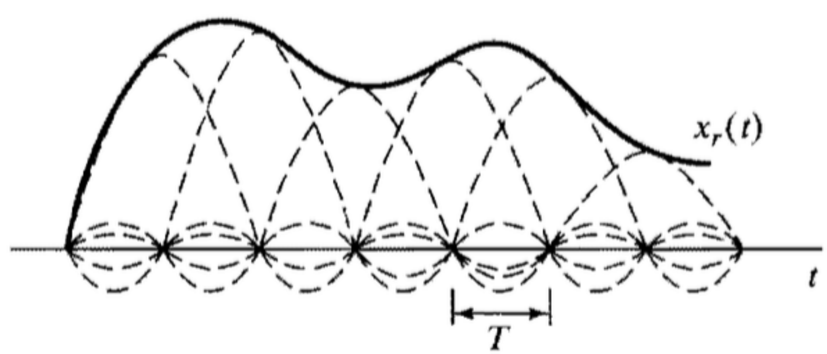
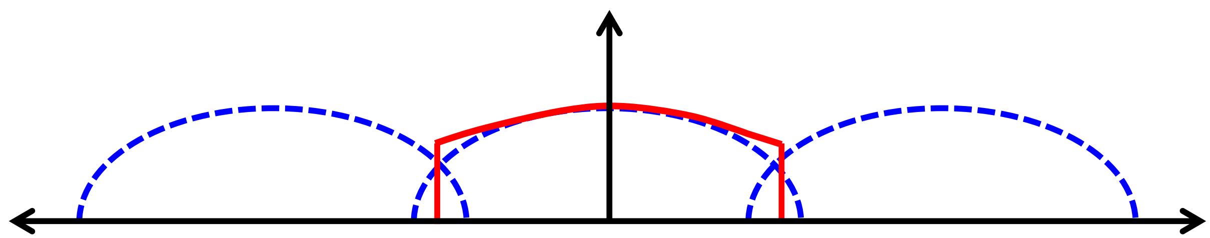


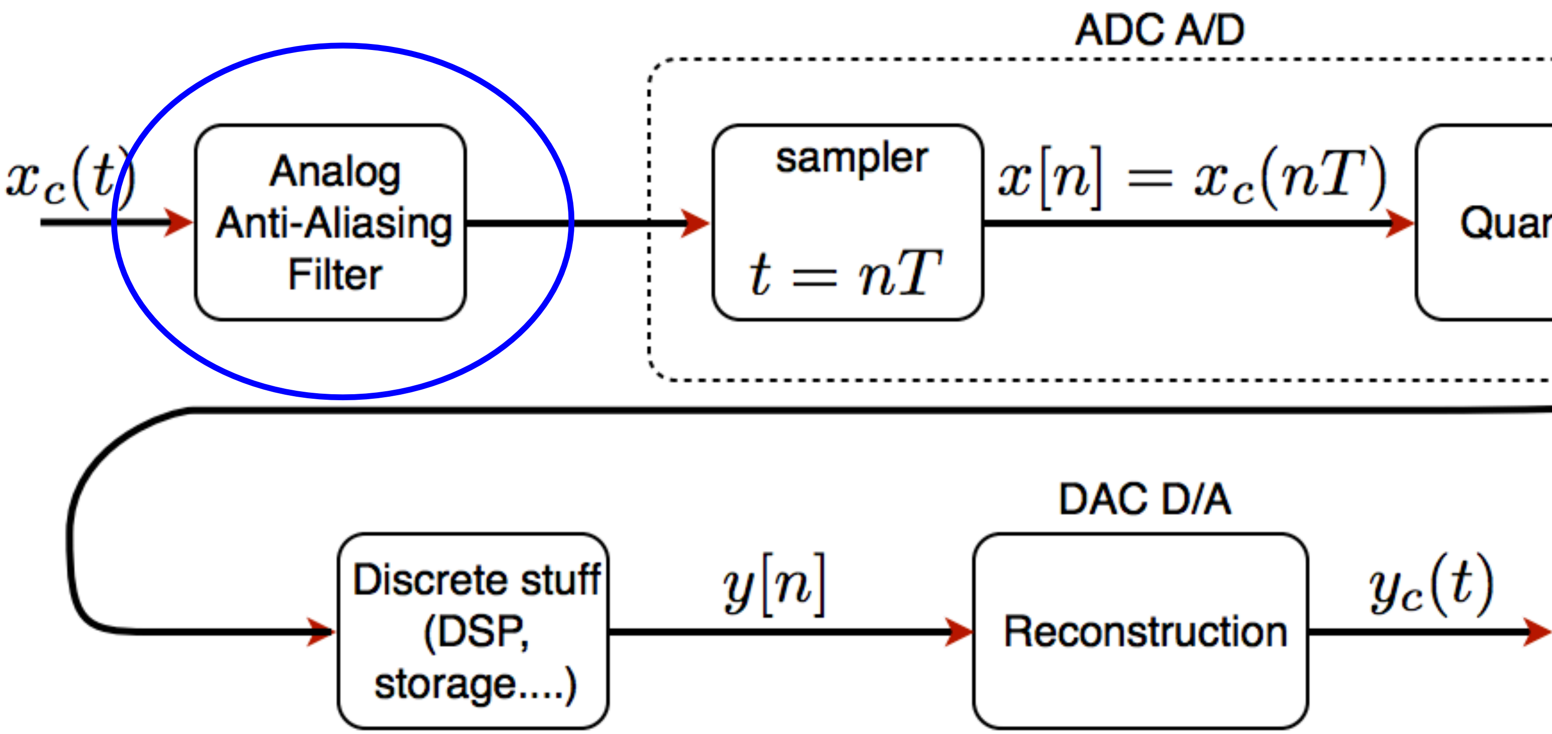
The sum of "sincs" gives $x_r(t)$ → unique signal that is bandlimited by sampling bandwidth

Aliasing

- If $\Omega_N > \Omega_s/2$, $x_r(t)$ is an aliased version of $x_c(t)$

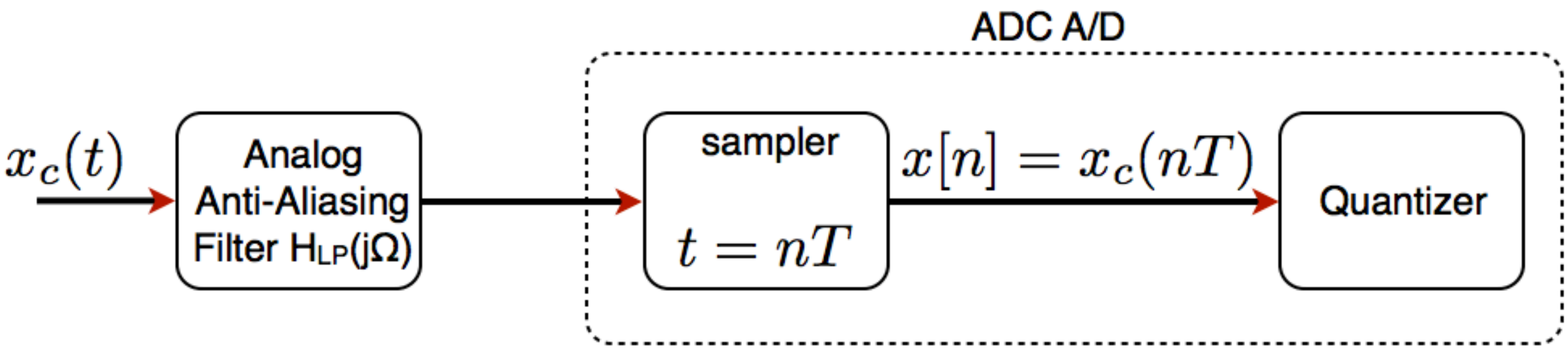


DSP System

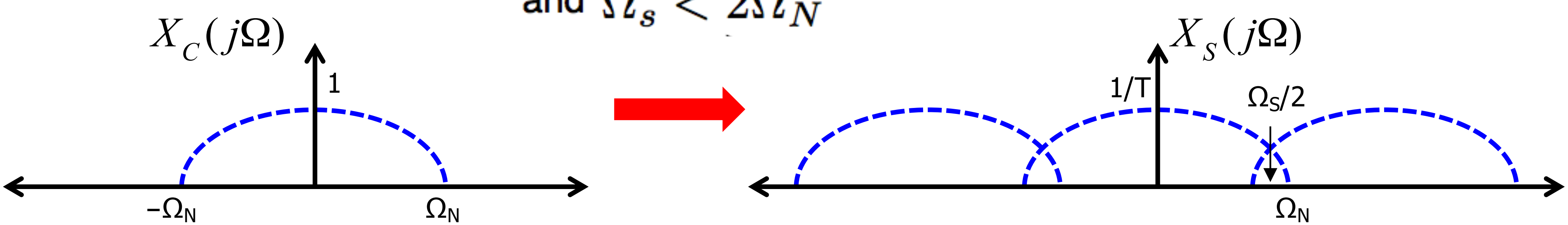


ADC A/D
$x_c(t)$
Analog
Anti-Aliasing
Filter
sampler
$t = nT$
$x[n] = x_c(nT)$
Quantizer
DAC D/A
Discrete stuff
(DSP,
storage....)
$y[n]$
Reconstruction
$y_c(t)$

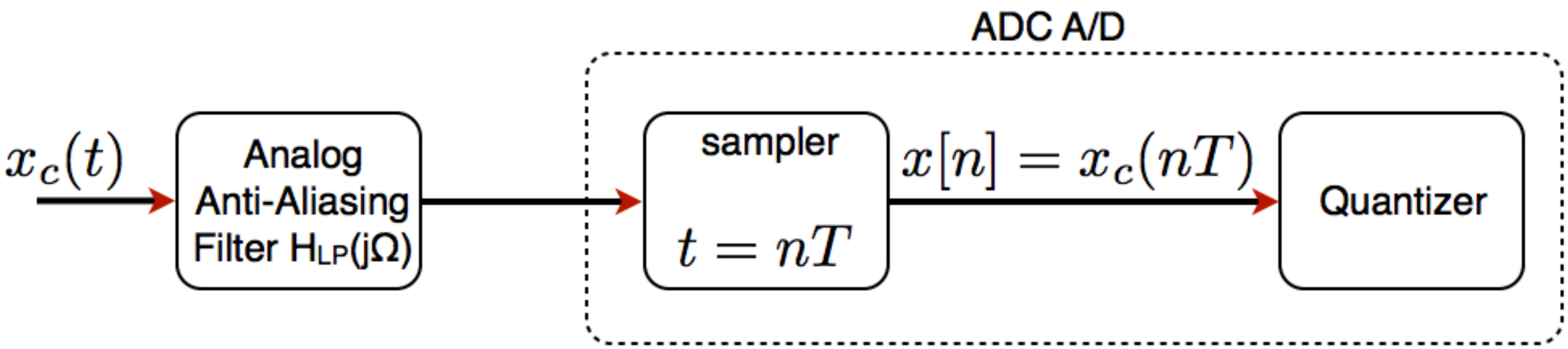
Anti-Aliasing Filter



and $\Omega_s < 2\Omega_N$



Anti-Aliasing Filter



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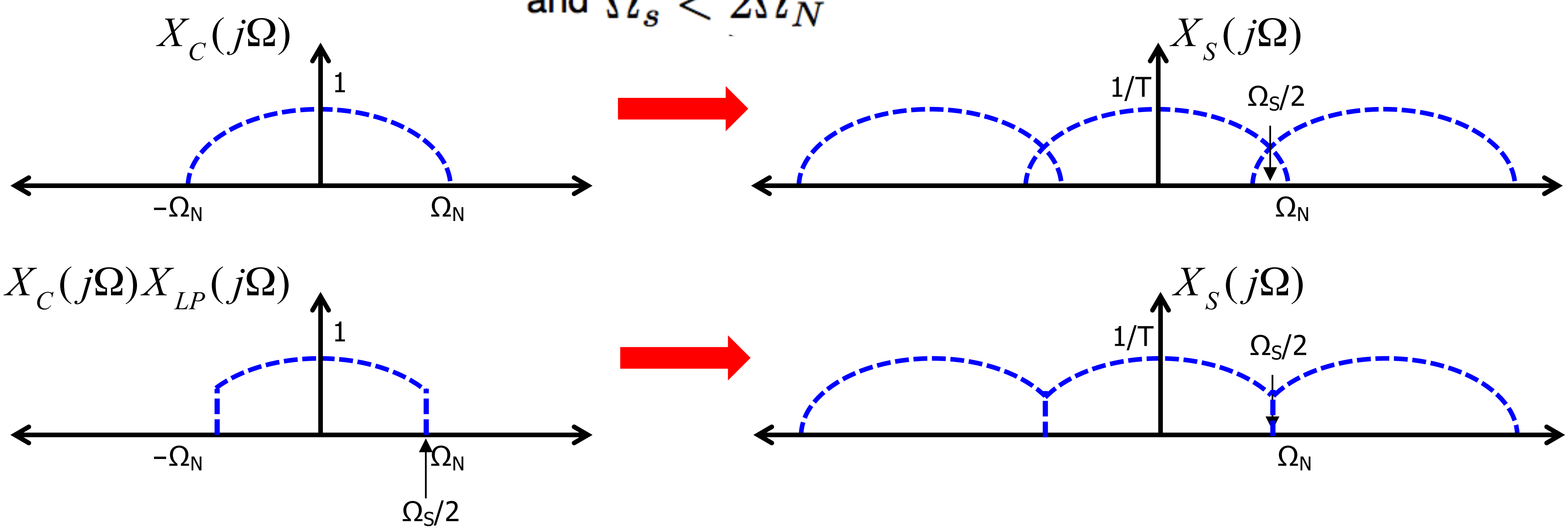
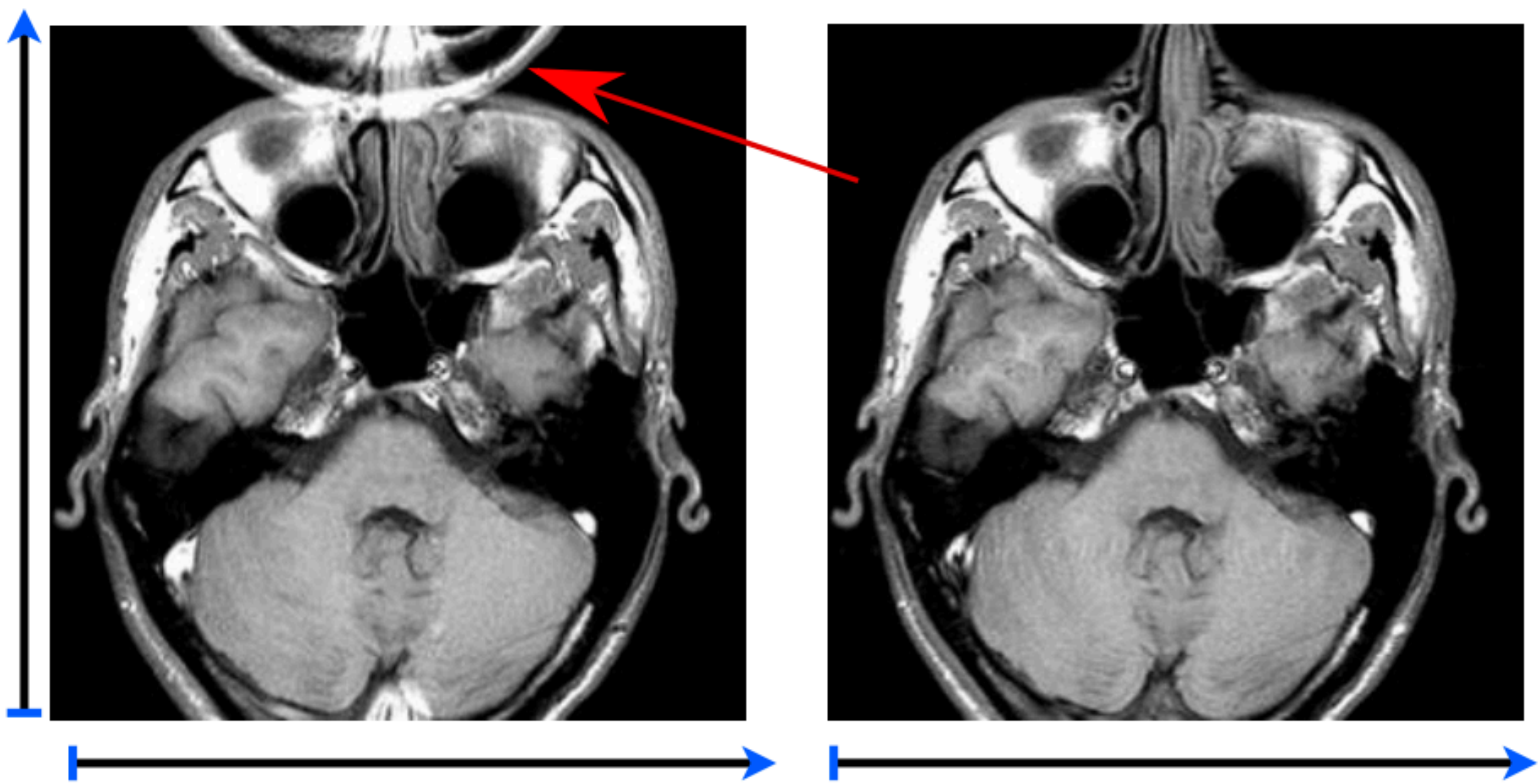


MRI aliasing example



MRI anti-aliasing example

MRI anti-aliasing example

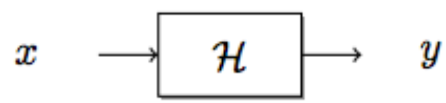
Discrete-Time Systems

Discrete Time Systems

DEFINITION

A discrete-time **system** $\mathcal{H}$ is a transformation (a rule or formula) that maps a discrete-time input signal x into a discrete-time output signal y

$$y = \mathcal{H}\{x\}$$

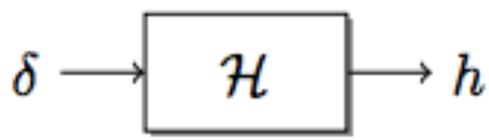


- Systems manipulate the information in signals
- Examples:
  - A speech recognition system converts acoustic waves of speech into text
  - A radar system transforms the received radar pulse to estimate the position and velocity of targets
  - A fMRI system transforms measurements of electron spin into voxel-by-voxel estimates of brain activity
  - A 30 day moving average smooths out the day-to-day variability in a stock price

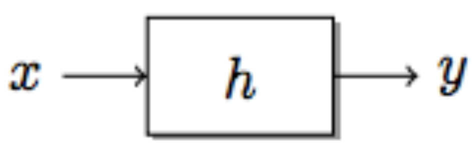
LTI Systems

> **DEFINITION**
>
> A system $\mathcal{H}$ is **linear time-invariant** (LTI) if it is both linear and time-invariant

- LTI system can be completely characterized by its impulse response



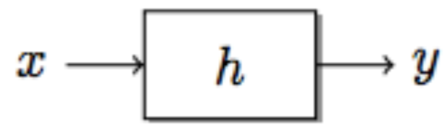
- Then the output for an arbitrary input is a sum of weighted, delay impulse responses



$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

$$y[n] = x[n] * h[n]$$

Convolution



- Convolution formula:

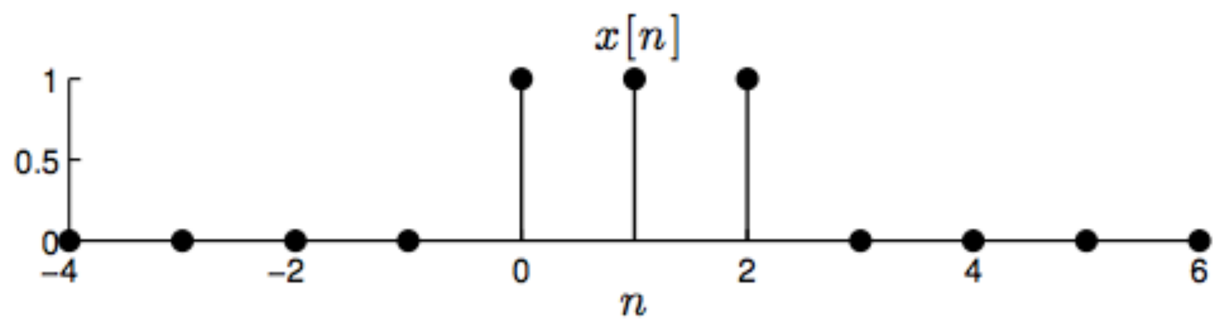
$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- Convolution method:
  - 1) Time reverse the impulse response and shift it *n* time steps to the right
  - 2) Compute the inner product between the shifted impulse response and the input vector
  - Repeat for every *n*

Convolution Example

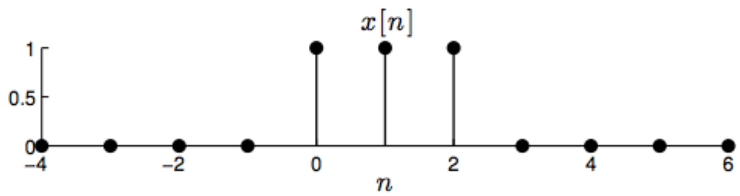
$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- Convolve a unit pulse with itself



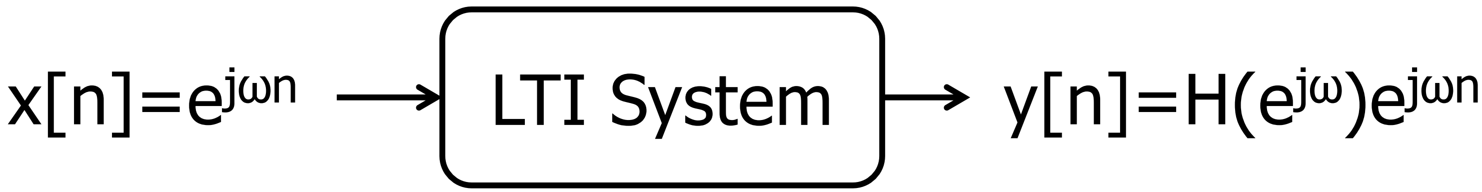
Convolution Example

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$



LTI System Frequency Response

- Fourier Transform of impulse response



$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

Eigenvalue (frequency response)

- $x[n]=e^{j\omega n}$

$$y[n]=\sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

$$=\sum_{k=-\infty}^{\infty} e^{j\omega(n-k)}h[k]$$

$$=e^{j\omega n}\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

$$=H(e^{j\omega})e^{j\omega n}$$

$$H(e^{j\omega})=\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

- Describes the change in amplitude and phase of signal at frequency ω
- Frequency response
- Complex value
  - Re and Im
  - Mag and Phase

DT Frequency Response

- $H(e^{j(\omega+2\pi)})$?

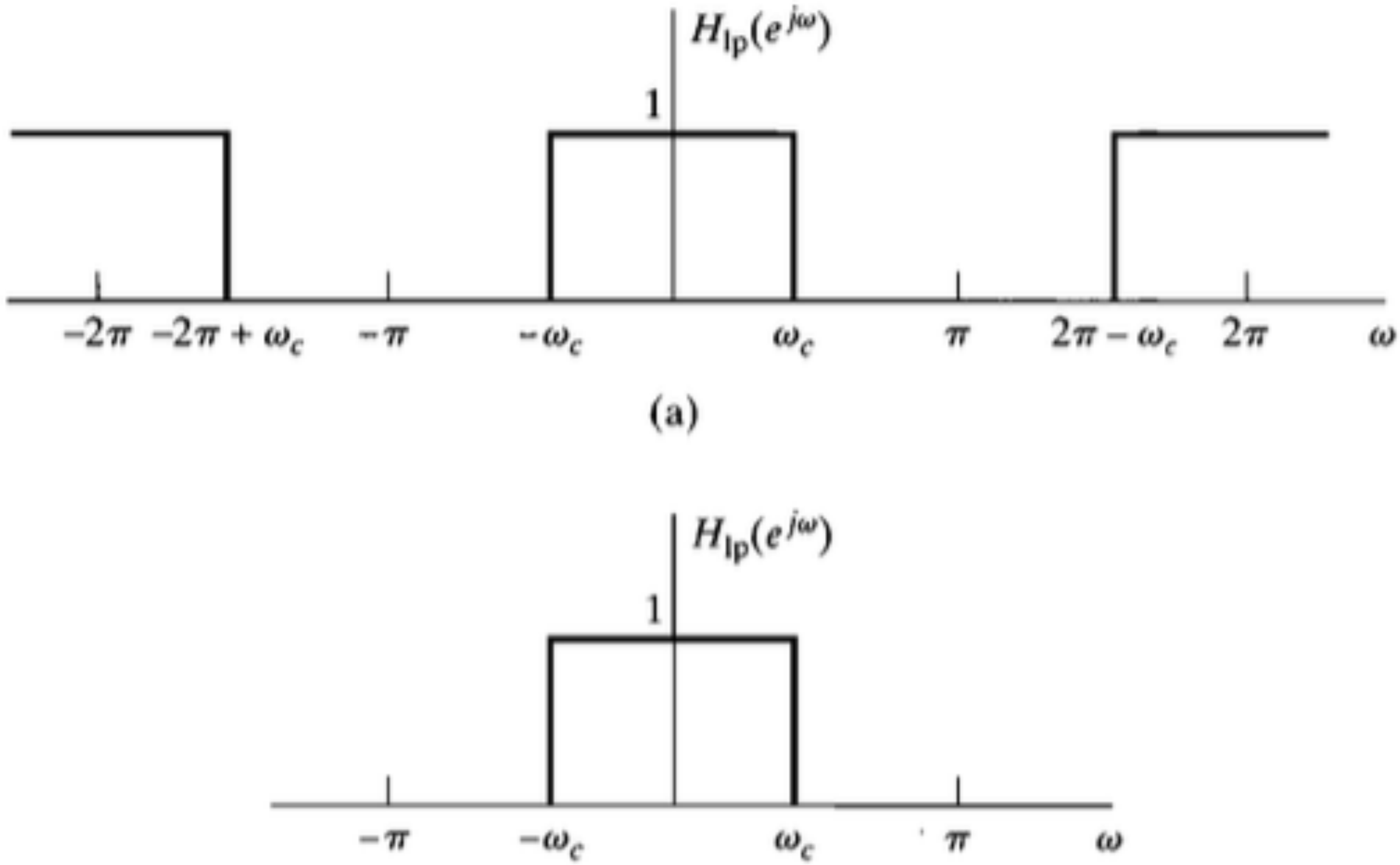
$$H(e^{j(\omega+2\pi)n}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j(\omega+2\pi)k}$$

$$= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}e^{-j2\pi k}$$

$$= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

$$= H(e^{j\omega n})$$

Periodicity of Low Pass Freq Response



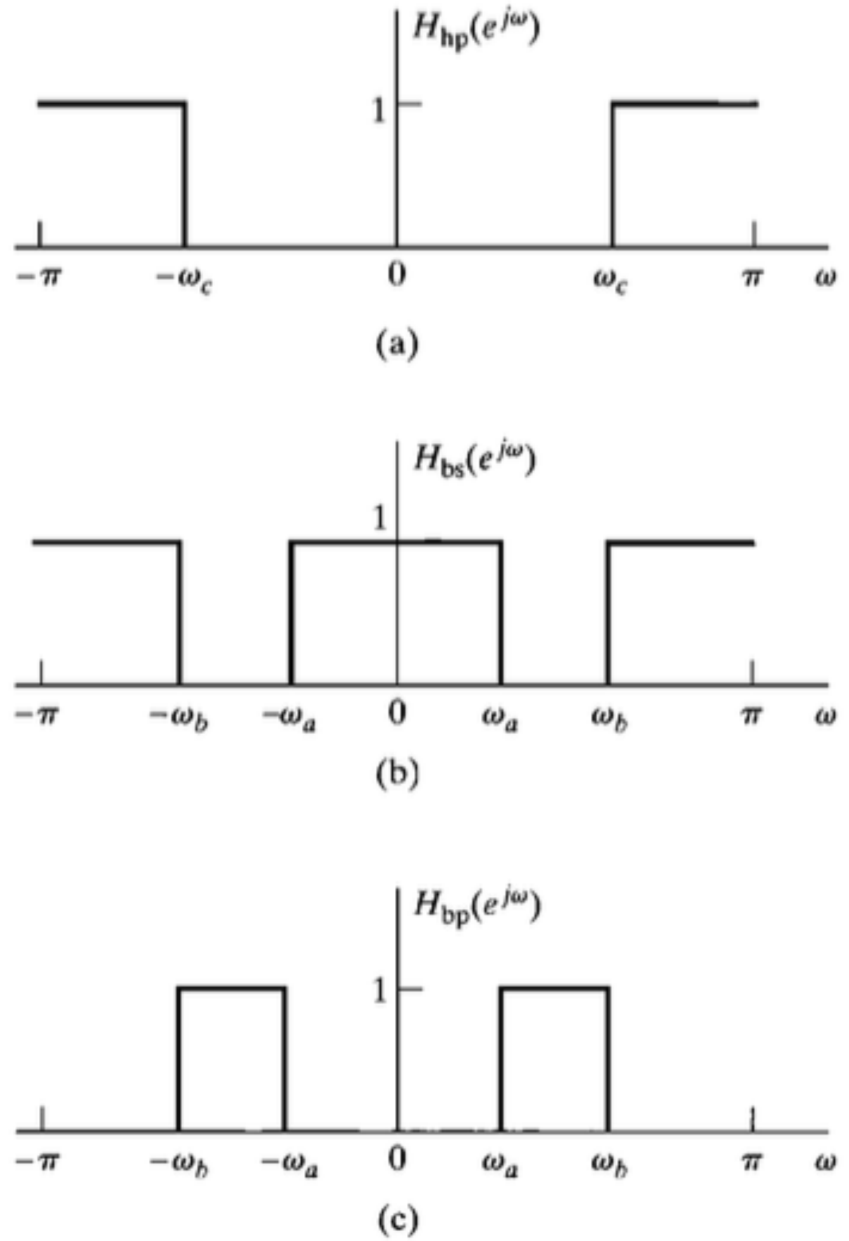
(b)

Other Filters

High-pass

Band-stop

Band-pass

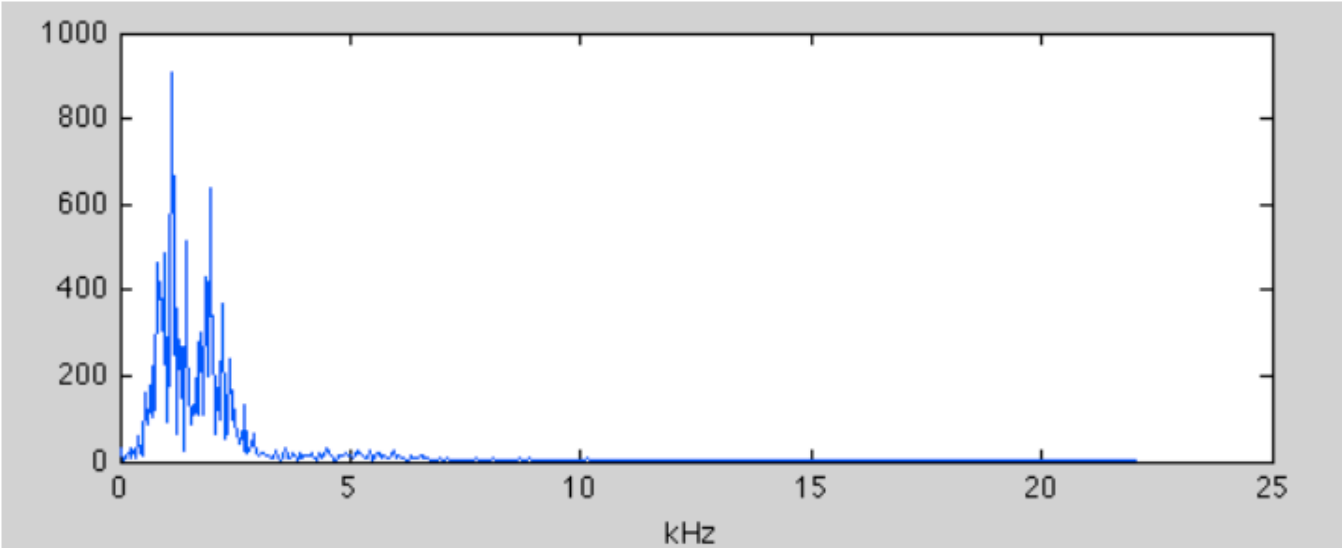


Non-Linear System Example

- Median Filter
  - $y[n]=MED\{x[n-k], \ldots x[n+k]\}$
  - Let $k=1$
  - $y[n]=MED\{x[n-1], x[n], x[n+1]\}$

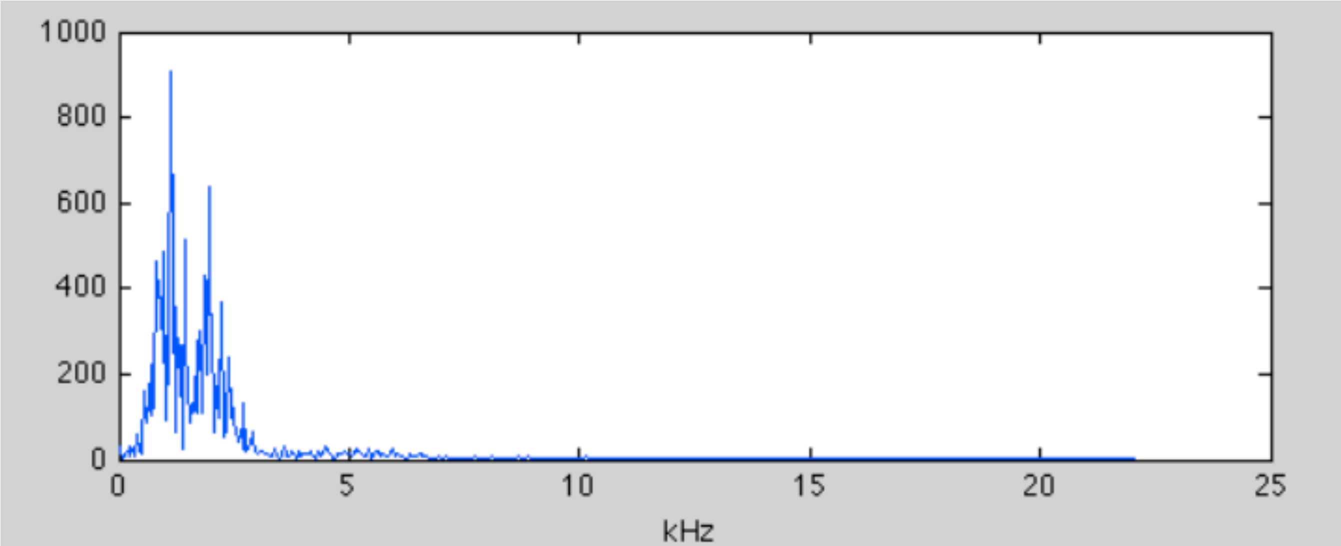
Spectrum of Speech

Speech

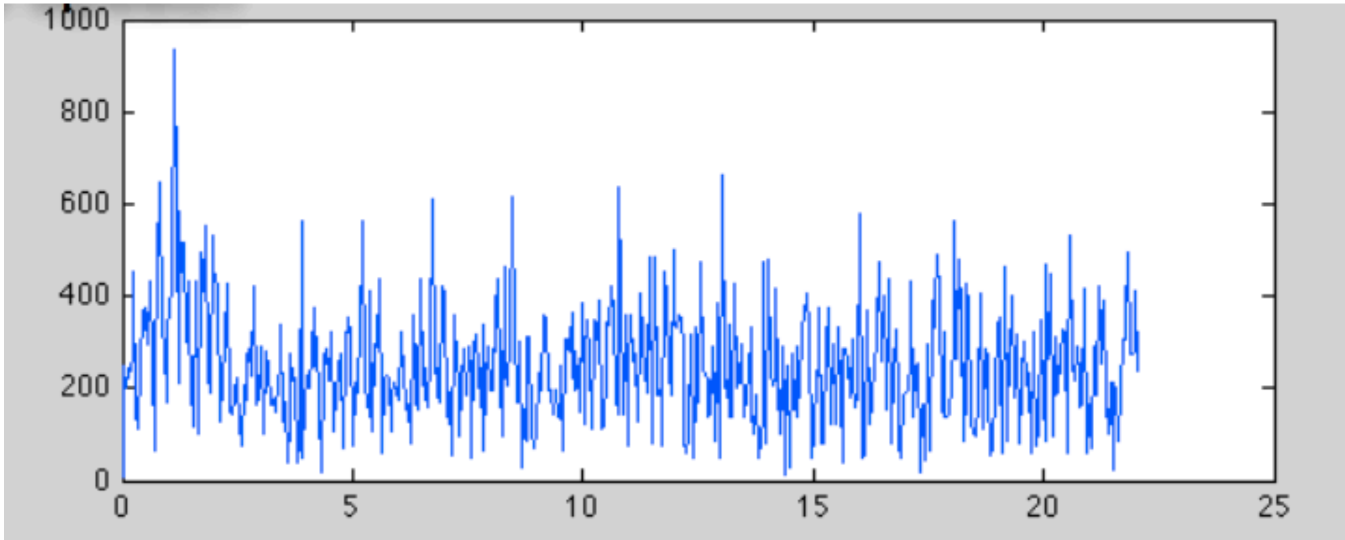


Spectrum of Speech

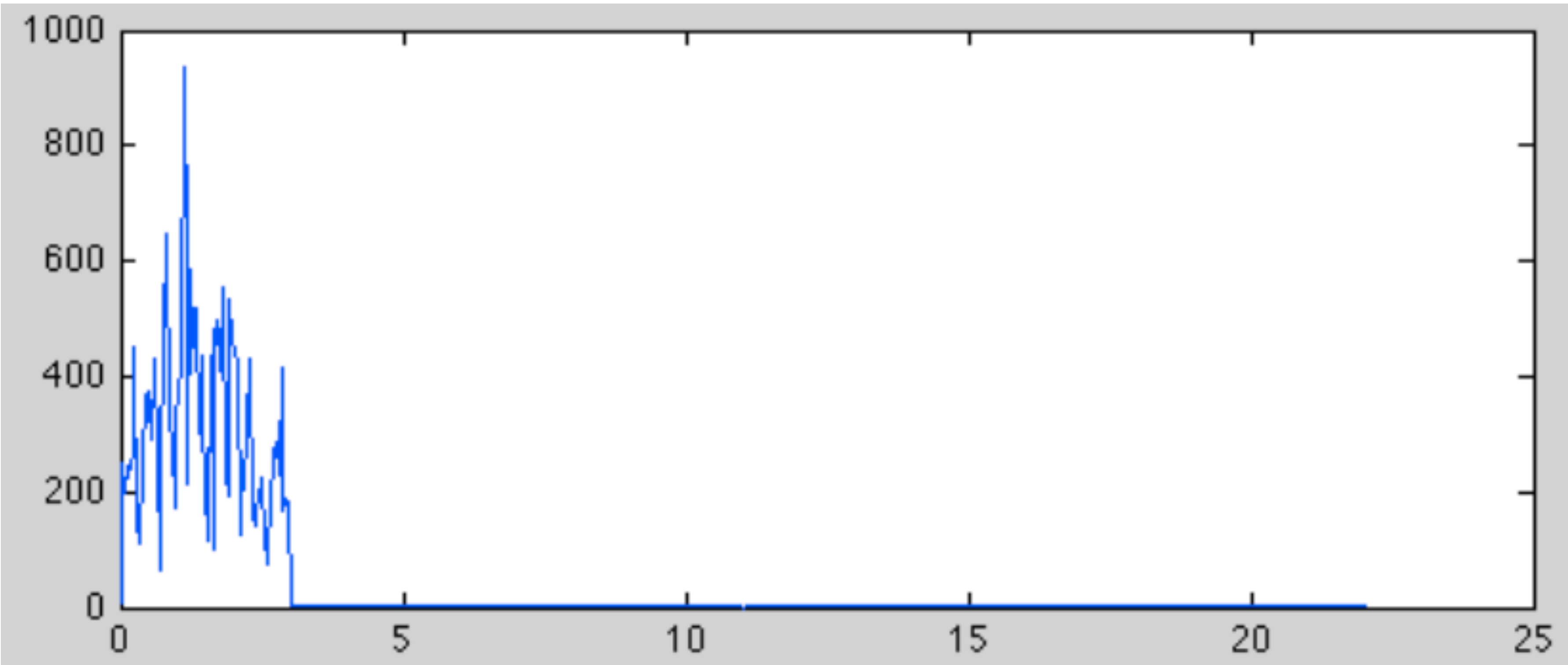
Speech



Corrupted Speech

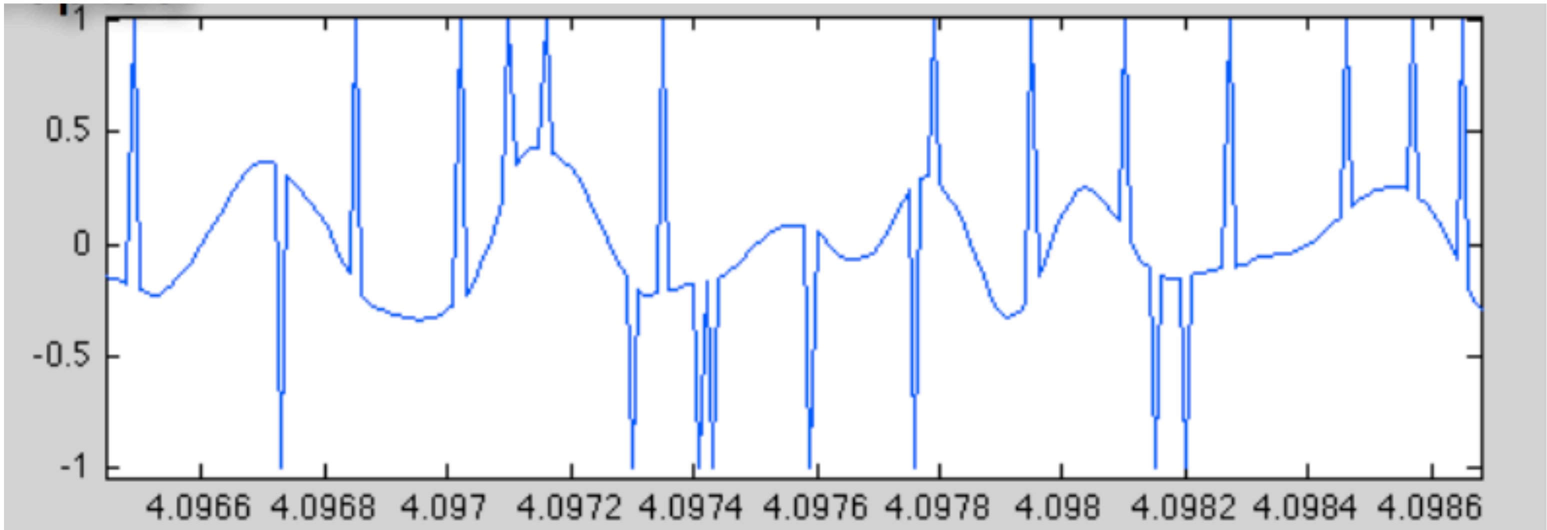


Low Pass Filtering



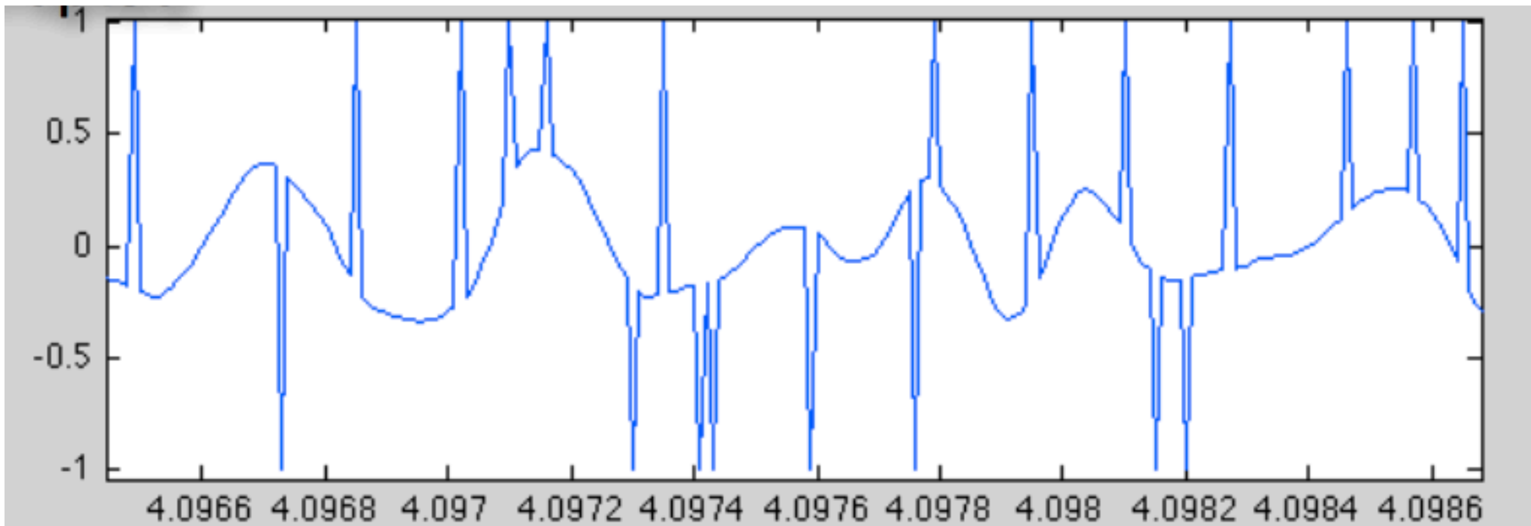
Speech in Time

Corrupted Speech

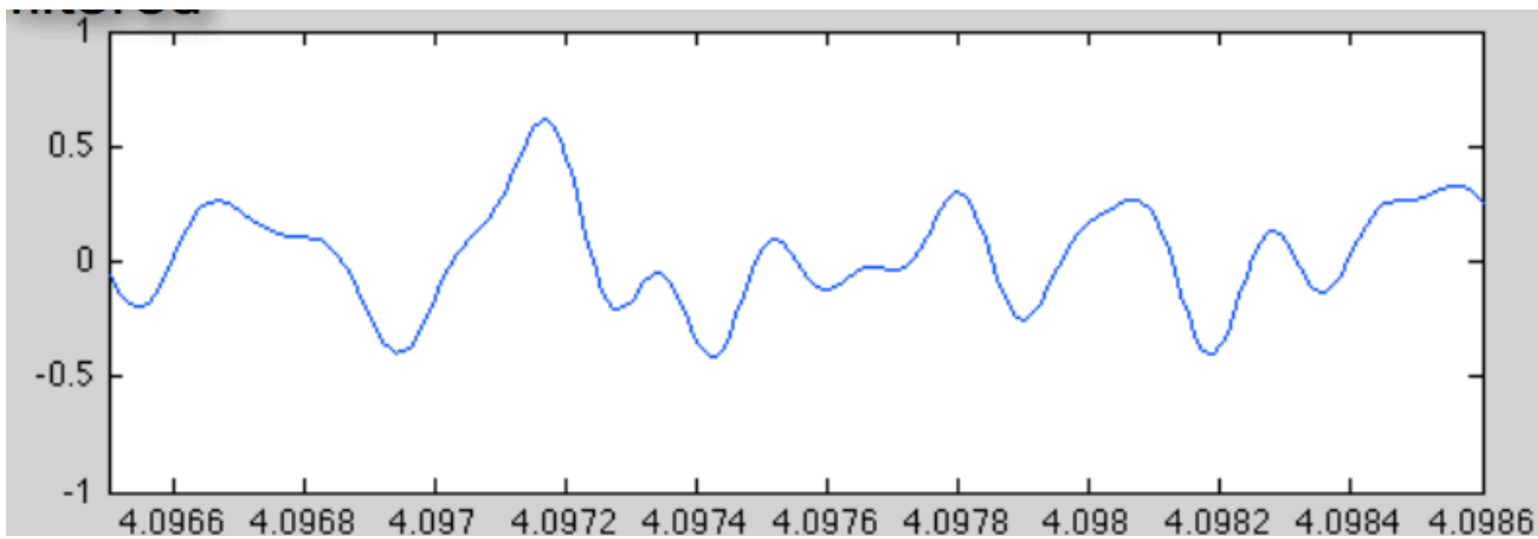


Low Pass Filtering

Corrupted Speech

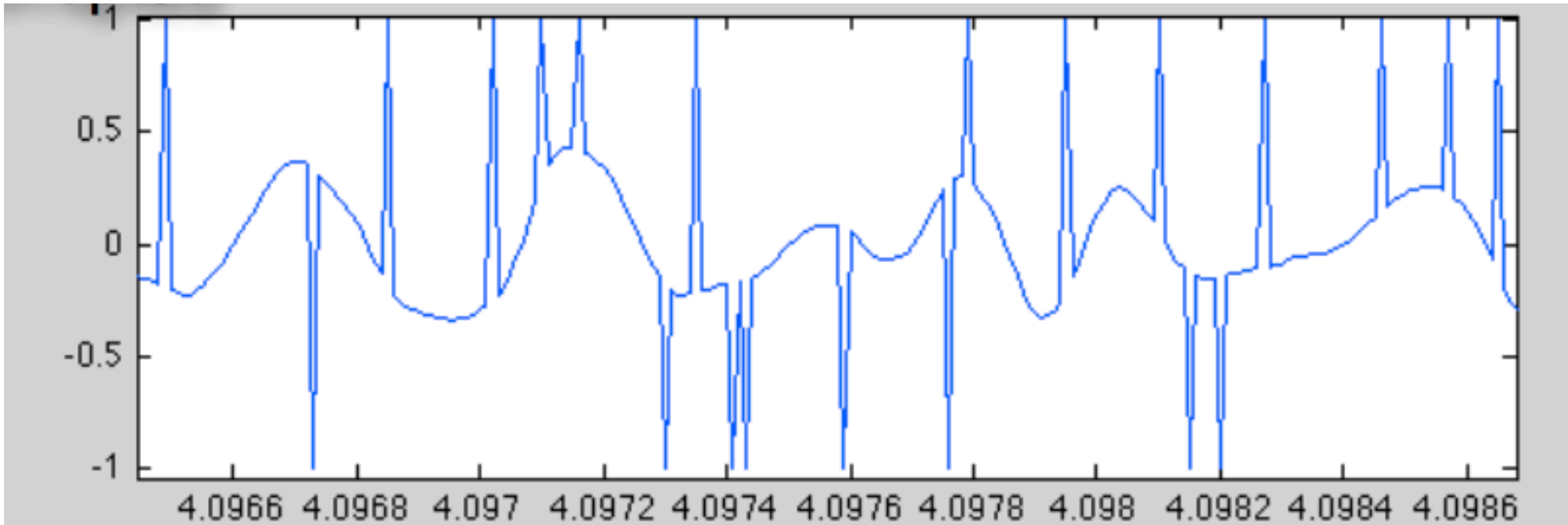


LP-Filtered Speech

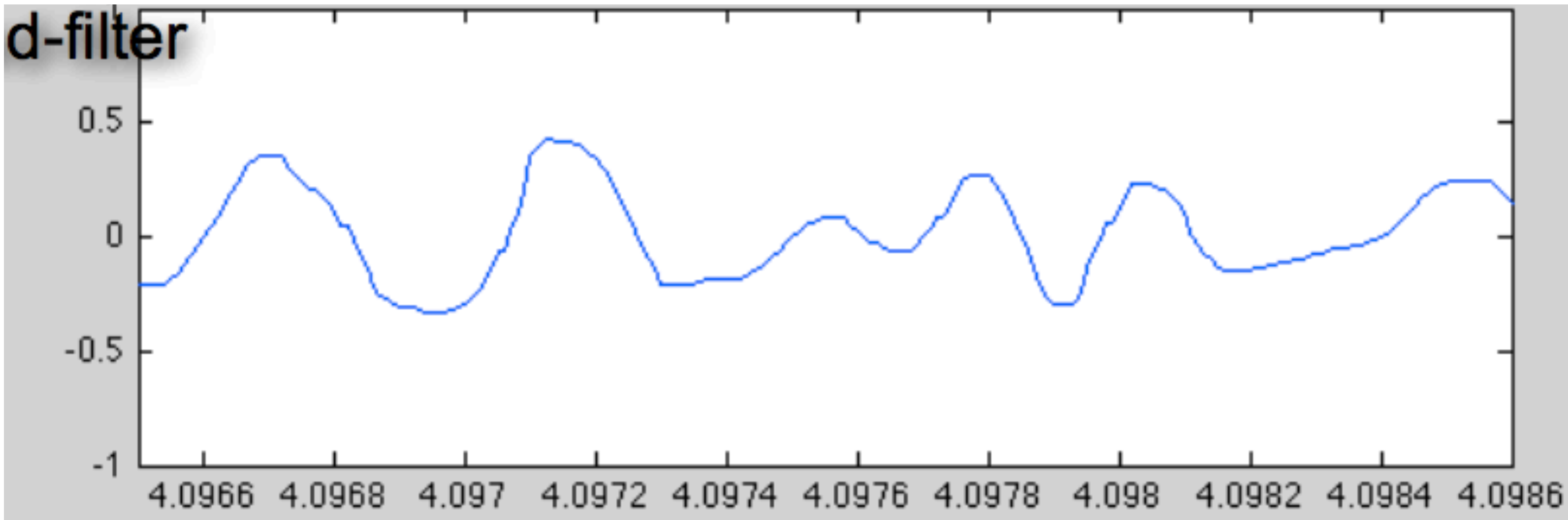


Median Filtering

Corrupted Speech



Med-Filter Speech



Big Ideas

- Continuous Time Signals
  - Represent signals in time and frequency
- Discrete Time Signals
  - Sample at Nyquist to avoid aliasing
- DTFT/Z-Transform
  - Represent signals in time and frequency
  - Find frequency content of signal
- Sampling and Reconstruction
  - Must sample at greater than the Nyquist rate
  - Actually oversample most of the time
- Discrete Time Systems
  - LTI Systems are predictable and practical to implement
  - Convolution to evaluate output in time domain
  - Frequency Response to evaluate output in frequency domain
    - Put a frequency in, get the same frequency out

Admin

- Lab next week
  - Design PCB
  - Watch video before lab to get acquainted with Altium