ESE 3400: Medical Devices Lab

Lec 7: November 1, 2023

Digital Filters and Spectral Analysis

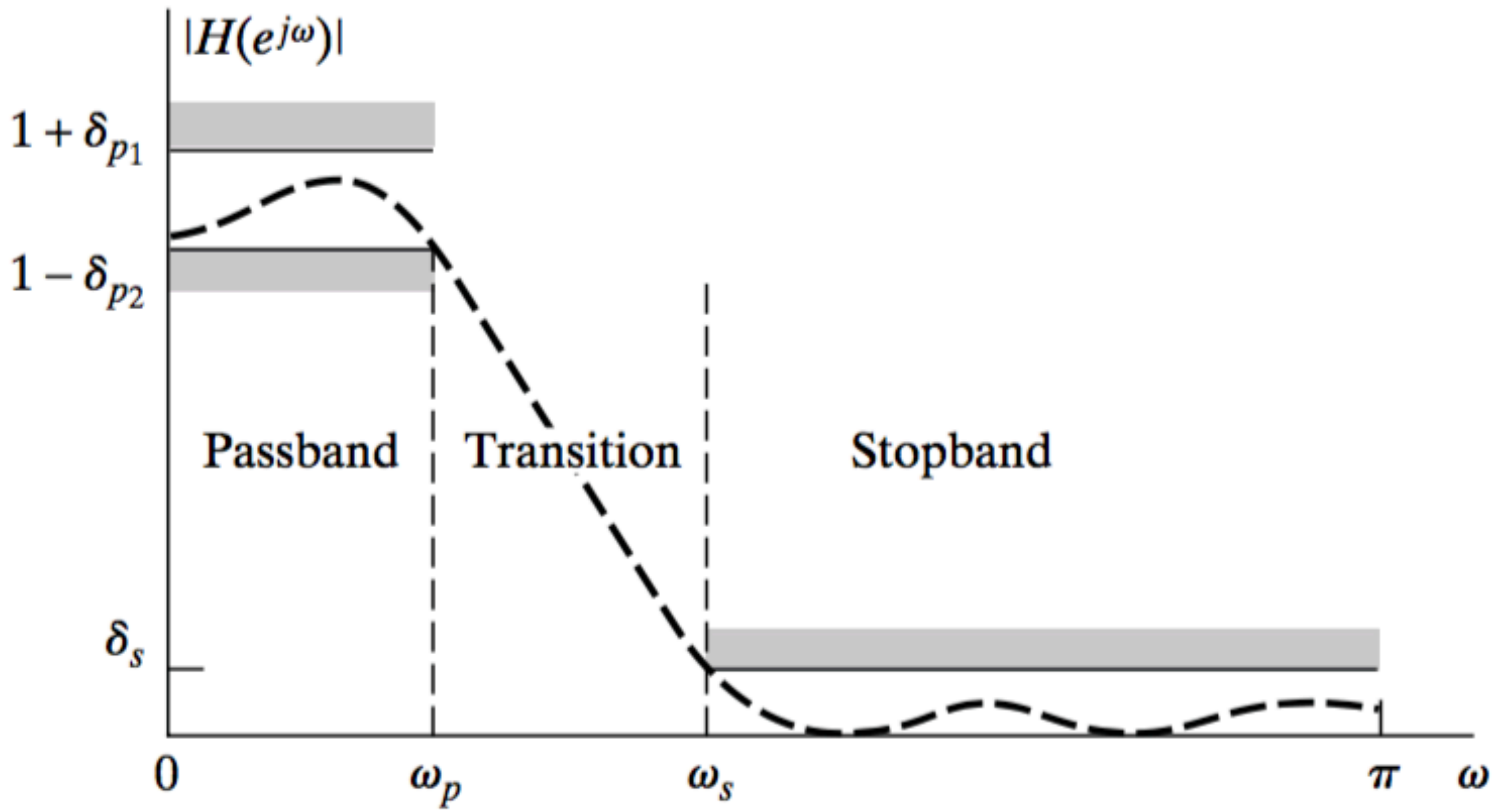
Linear Filter Design

- Used to be an art
  - Now, lots of tools to design optimal filters
- For DSP there are two common classes
  - Infinite impulse response IIR
  - Finite impulse response FIR
- We will focus on FIR designs

What is a Linear Filter?

- Attenuates certain frequencies
- Passes certain frequencies
- Affects both phase and magnitude

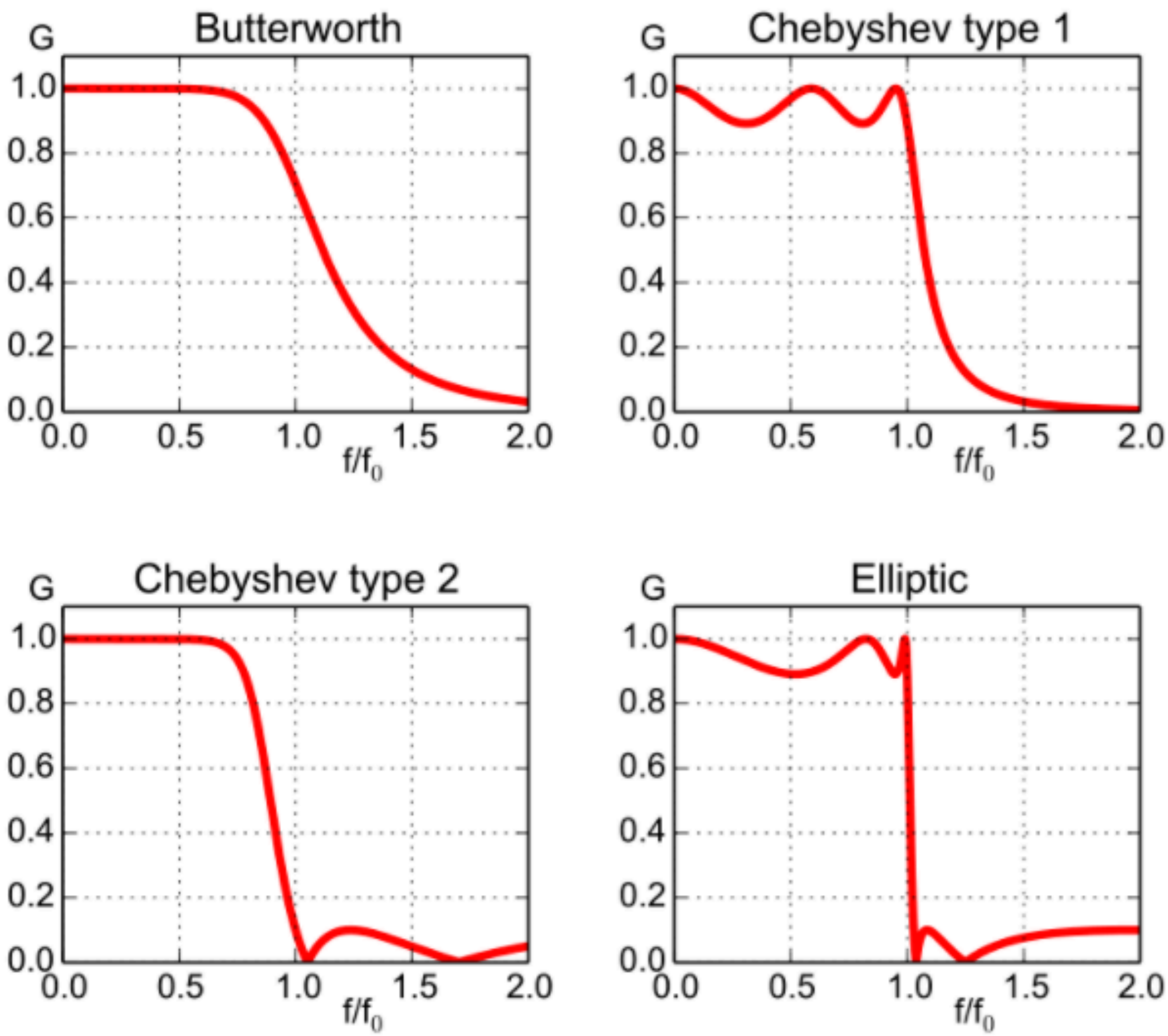
Filter Specifications



CT Filters

- Butterworth
  - Monotonic in pass and stop bands
- Chebyshev, Type I
  - Equiripple in pass band and monotonic in stop band
- Chebyshev, Type II
  - Monotonic in pass band and equiripple in stop band
- Elliptic
  - Equiripple in pass and stop bands

Comparisons



DTFT Definition

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

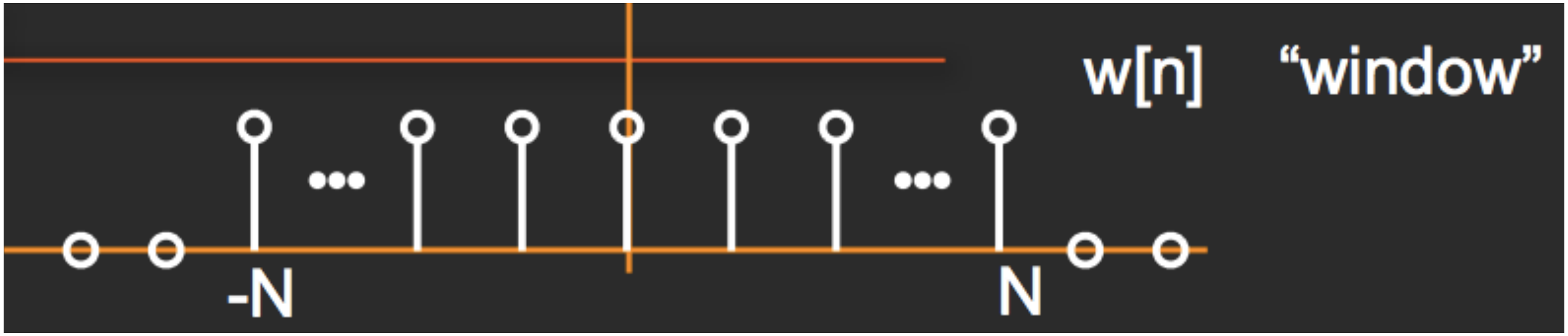
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Alternate

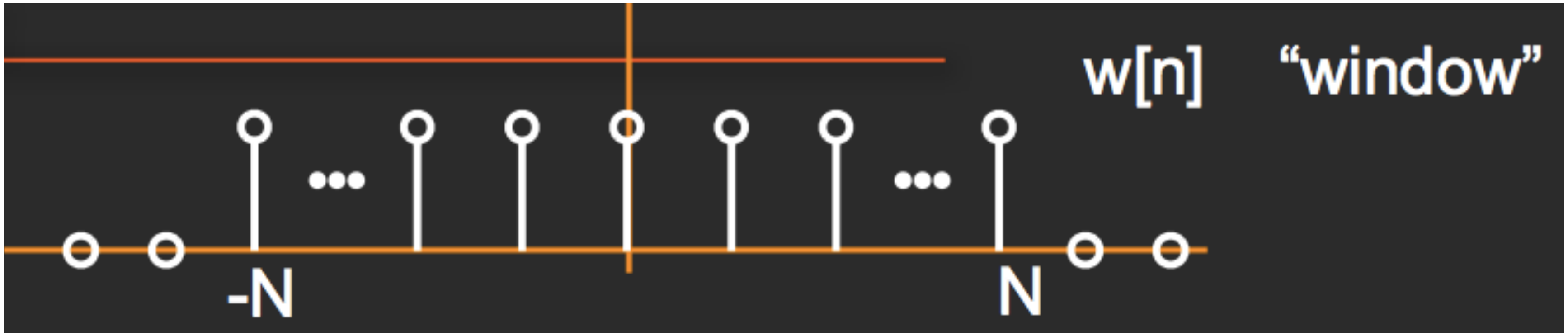
$$X(f) = \sum_{k=-\infty}^{\infty} x[k]e^{-j2\pi fk}$$

$$x[n] = \int_{-0.5f_s}^{0.5f_s} X(f)e^{j2\pi fn}df$$

Example: Window DTFT



Example: Window DTFT



$$W(e^{j\omega}) = \sum_{k=-\infty}^{\infty} w[k]e^{-j\omega k}$$

$$= \sum_{k=-N}^{N} e^{-j\omega k}$$

Example: Window DTFT

$$W(e^{j\omega}) = \sum_{k=-N}^{N} e^{-j\omega k}$$

Useful sum: $1 + p + p^2 + \ldots + p^M = \dfrac{1 - p^{M+1}}{1 - p}$

Example: Window DTFT

$$W(e^{j\omega}) = \sum_{k=-N}^{N} e^{-j\omega k}$$

$$= e^{j\omega N} + e^{j\omega(N-1)} + \ldots + e^{j\omega 0} + \ldots e^{-j\omega(N-1)} + e^{-j\omega N}$$

$$= e^{-j\omega N}(1 + e^{j\omega} + \ldots + e^{j\omega N} + \ldots + e^{j\omega(2N-1)} + e^{j\omega 2N})$$

Useful sum: $1 + p + p^2 + \ldots + p^M = \dfrac{1 - p^{M+1}}{1 - p}$

Example: Window DTFT

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Useful sum: $1 + p + p^2 + \ldots + p^M = \dfrac{1 - p^{M+1}}{1 - p}$

$$p = e^{j\omega} \qquad M = 2N$$

$$W(e^{j\omega}) = e^{-j\omega N} \frac{1 - e^{j\omega(2N+1)}}{1 - e^{j\omega}}$$

Example: Window DTFT

$$W(e^{j\omega}) = e^{-j\omega N}\frac{1-e^{j\omega(2N+1)}}{1-e^{j\omega}}$$

Example: Window DTFT

$$W(e^{j\omega}) = e^{-j\omega N}\frac{1-e^{j\omega(2N+1)}}{1-e^{j\omega}}$$

$$= \frac{e^{-j\omega N}-e^{j\omega(N+1)}}{1-e^{j\omega}}$$

$$= \frac{e^{-j\omega N}-e^{j\omega(N+1)}}{1-e^{j\omega}} \times \frac{e^{-j\omega/2}}{e^{-j\omega/2}}$$

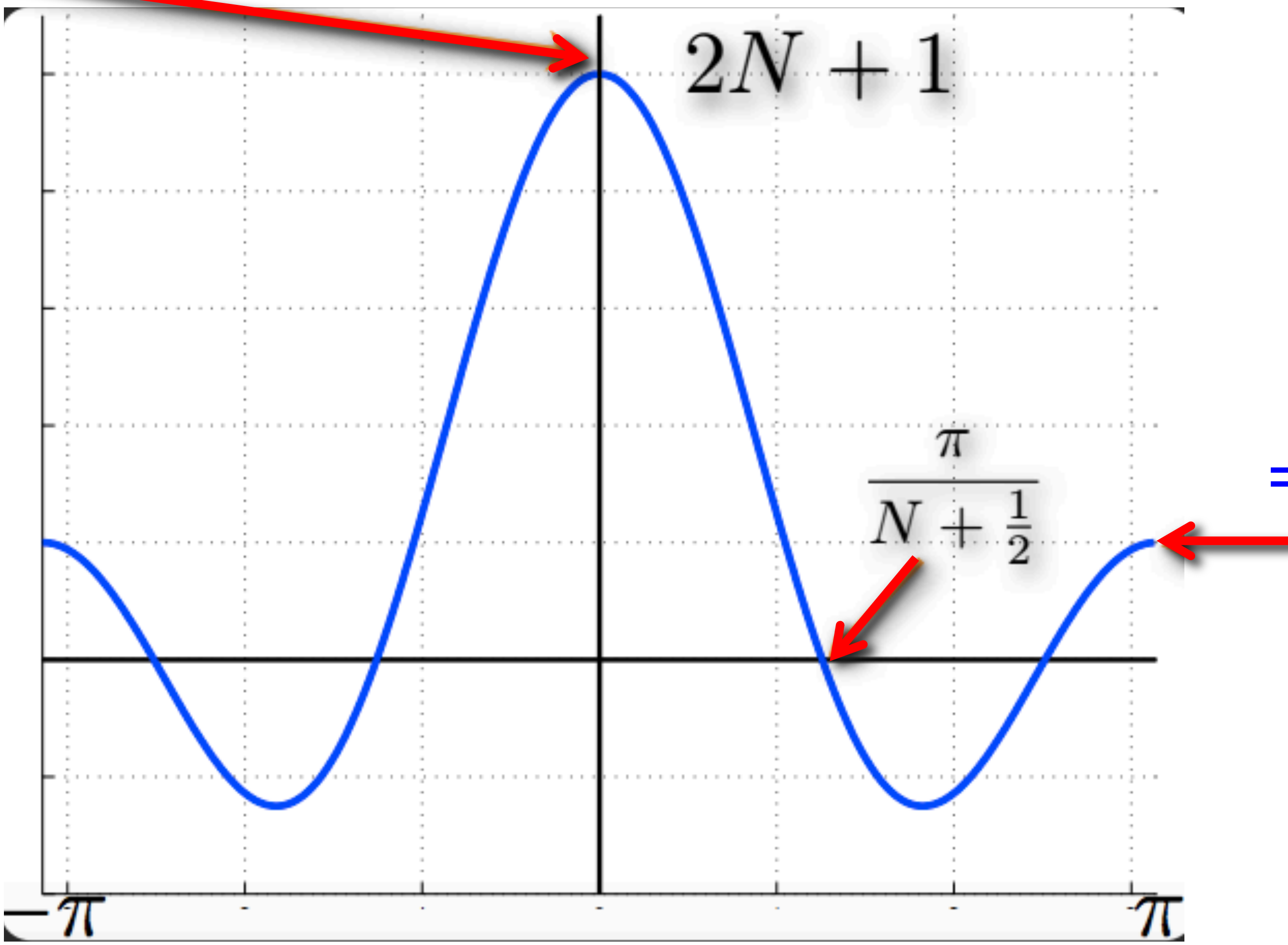
$$= \frac{e^{-j\omega(N+1/2)}-e^{j\omega(N+1/2)}}{e^{-j\omega/2}-e^{j\omega/2}} = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Periodic sinc

Example: Window DTFT

$$W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Also, Σw[n]



=1 why?

Plot for N=2

Example: Window DTFT

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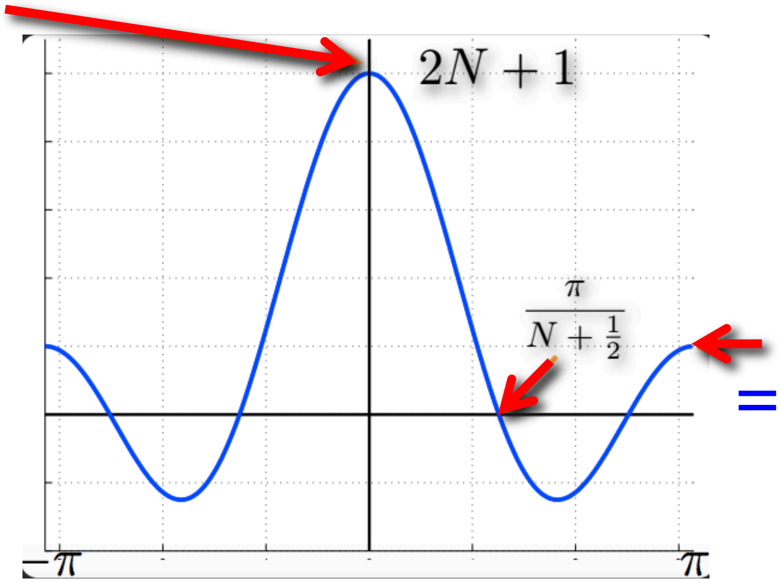

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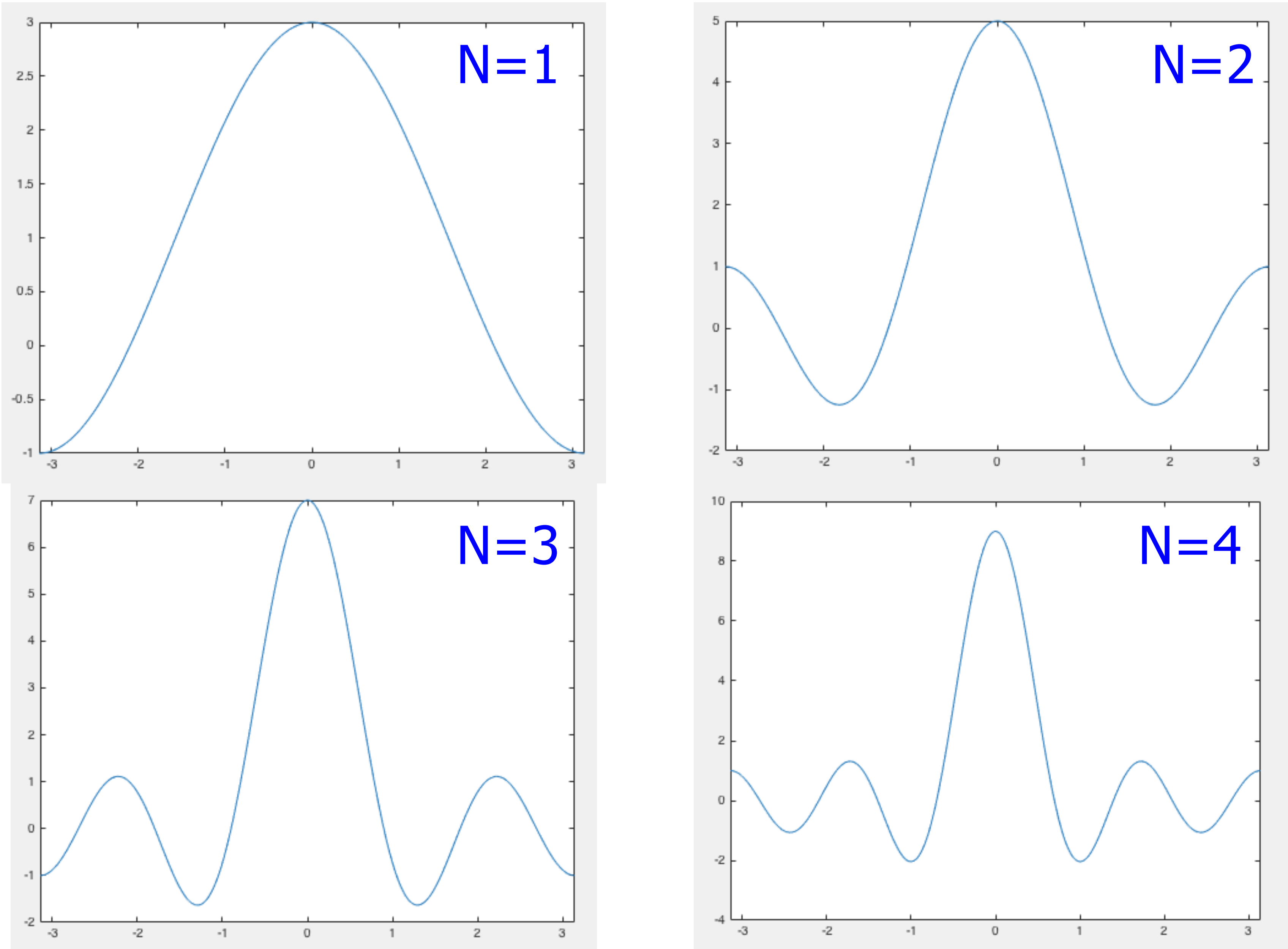


Commonly Used Windows

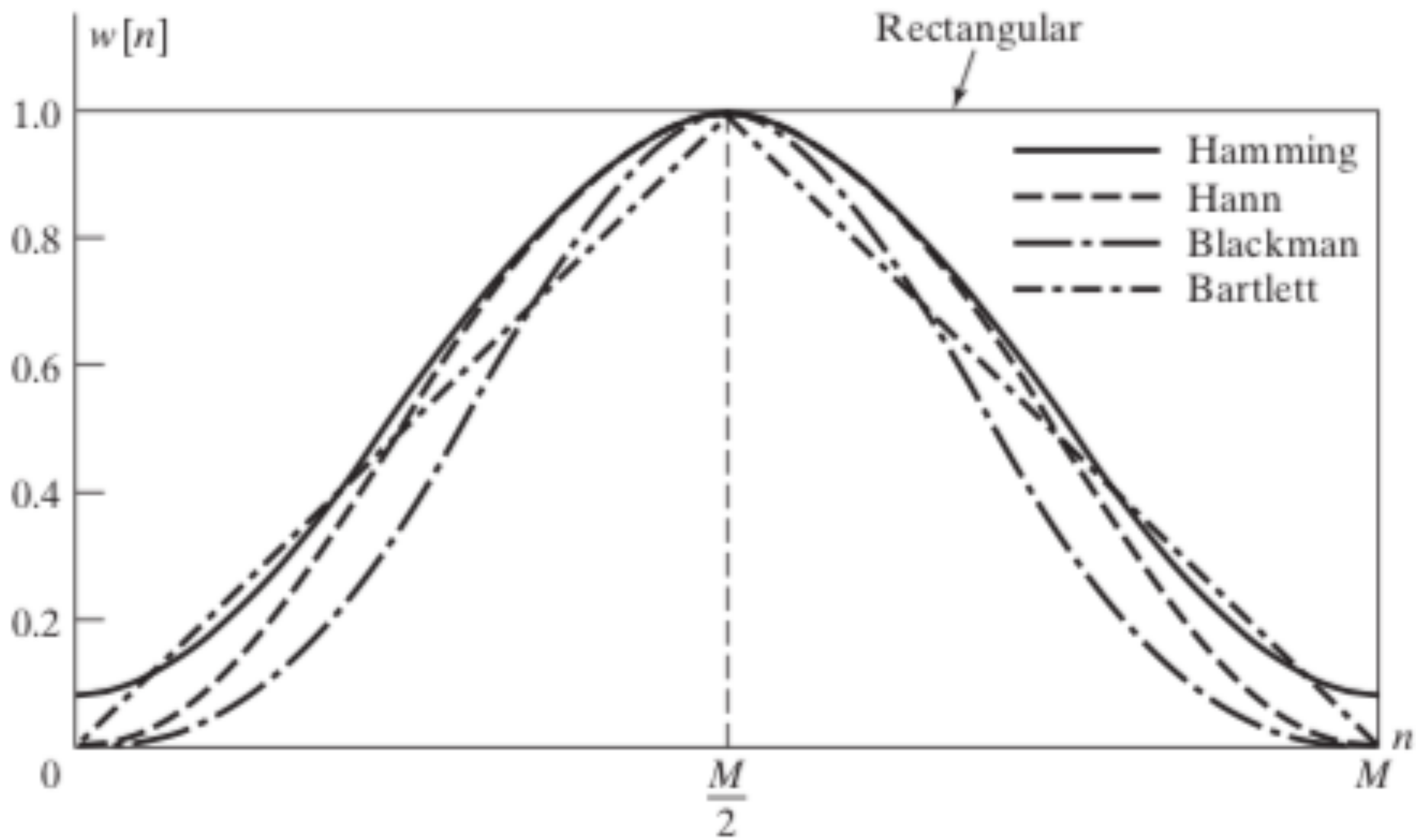
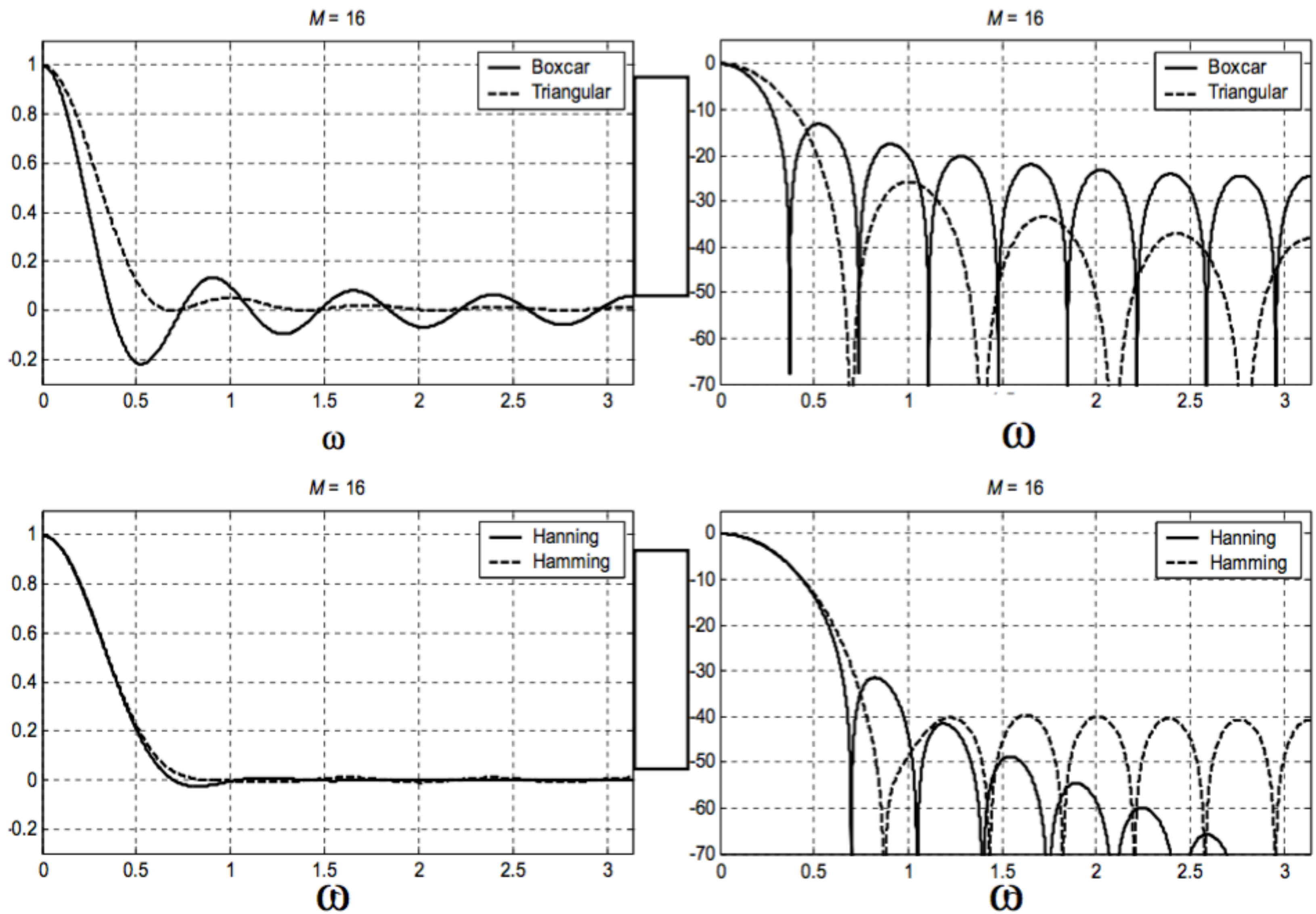
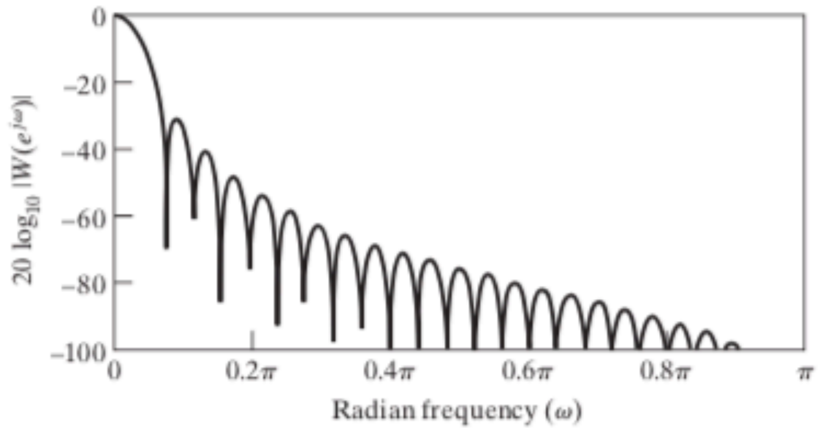


Figure 7.29 Commonly used windows.

Tradeoff – Ripple vs. Transition Width

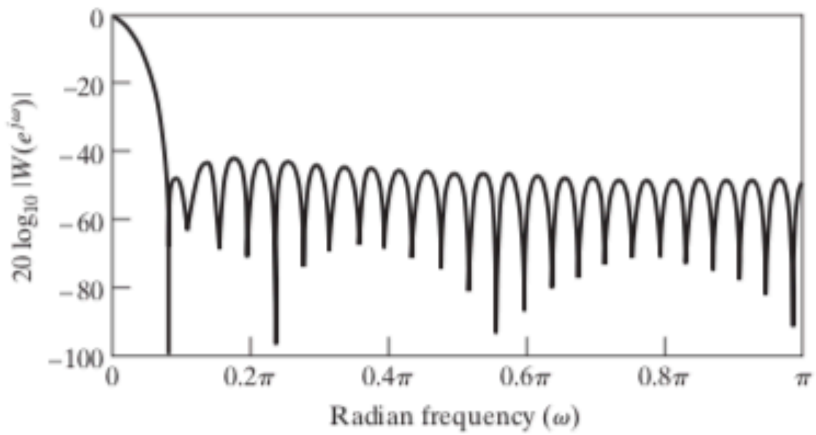


Hann



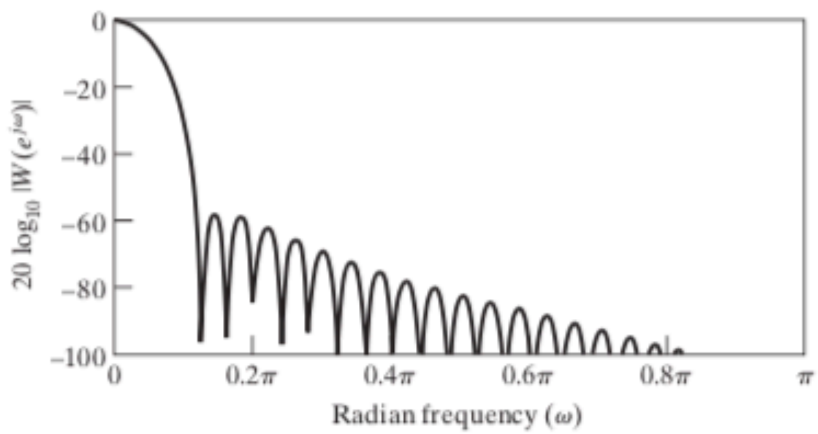
(c)

Hamming



(d)

Blackman



(e)

Kaiser Window

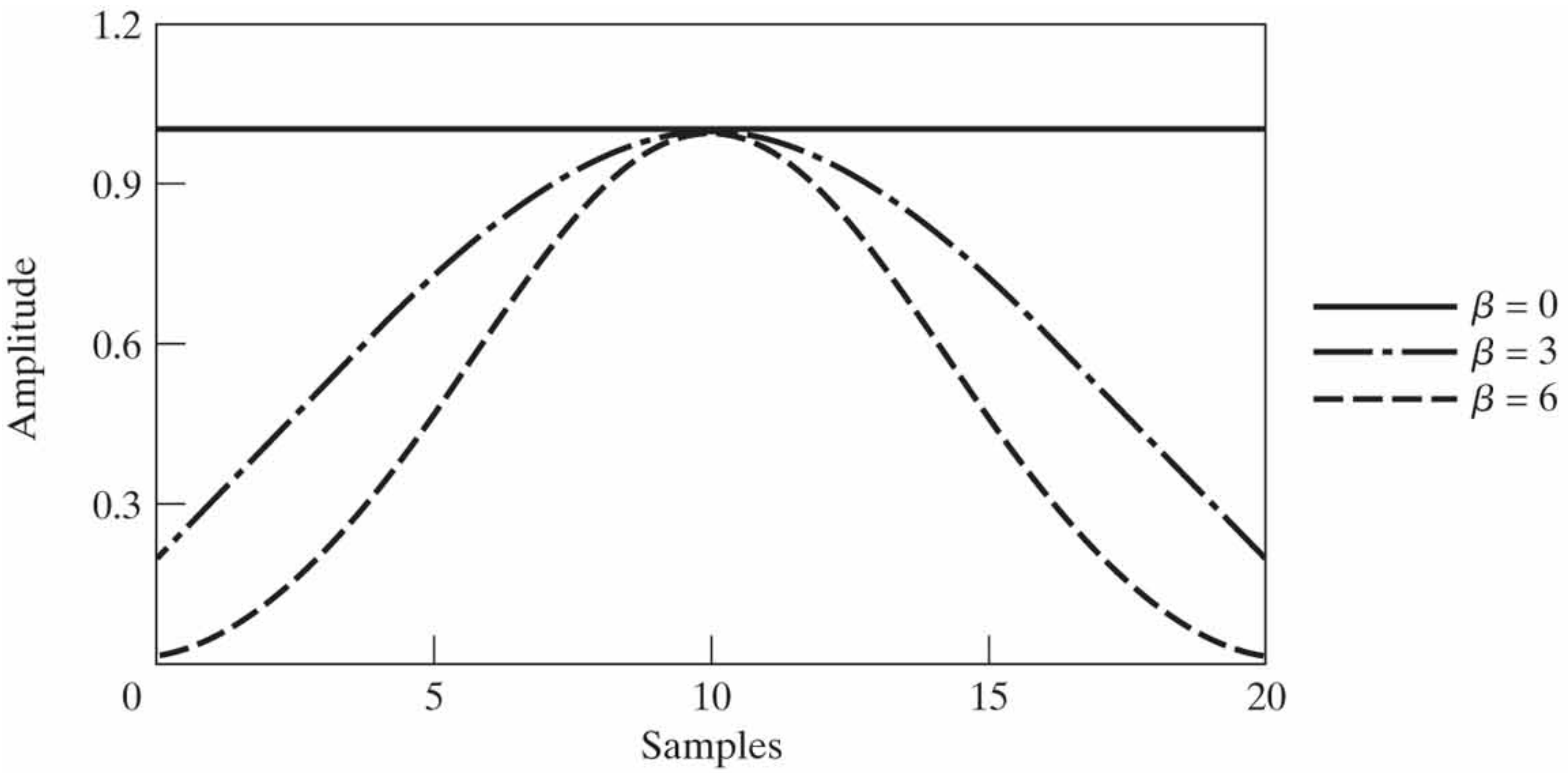
- Near optimal window quantified as the window maximally concentrated around $\omega=0$

$$w[n] = \begin{cases} \dfrac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \le n \le M, \\ 0, & \text{otherwise,} \end{cases}$$

- Two parameters – M and β
- $\alpha=M/2$
- $I_0(x)$ – zeroth order Bessel function of the first kind

Kaiser Window

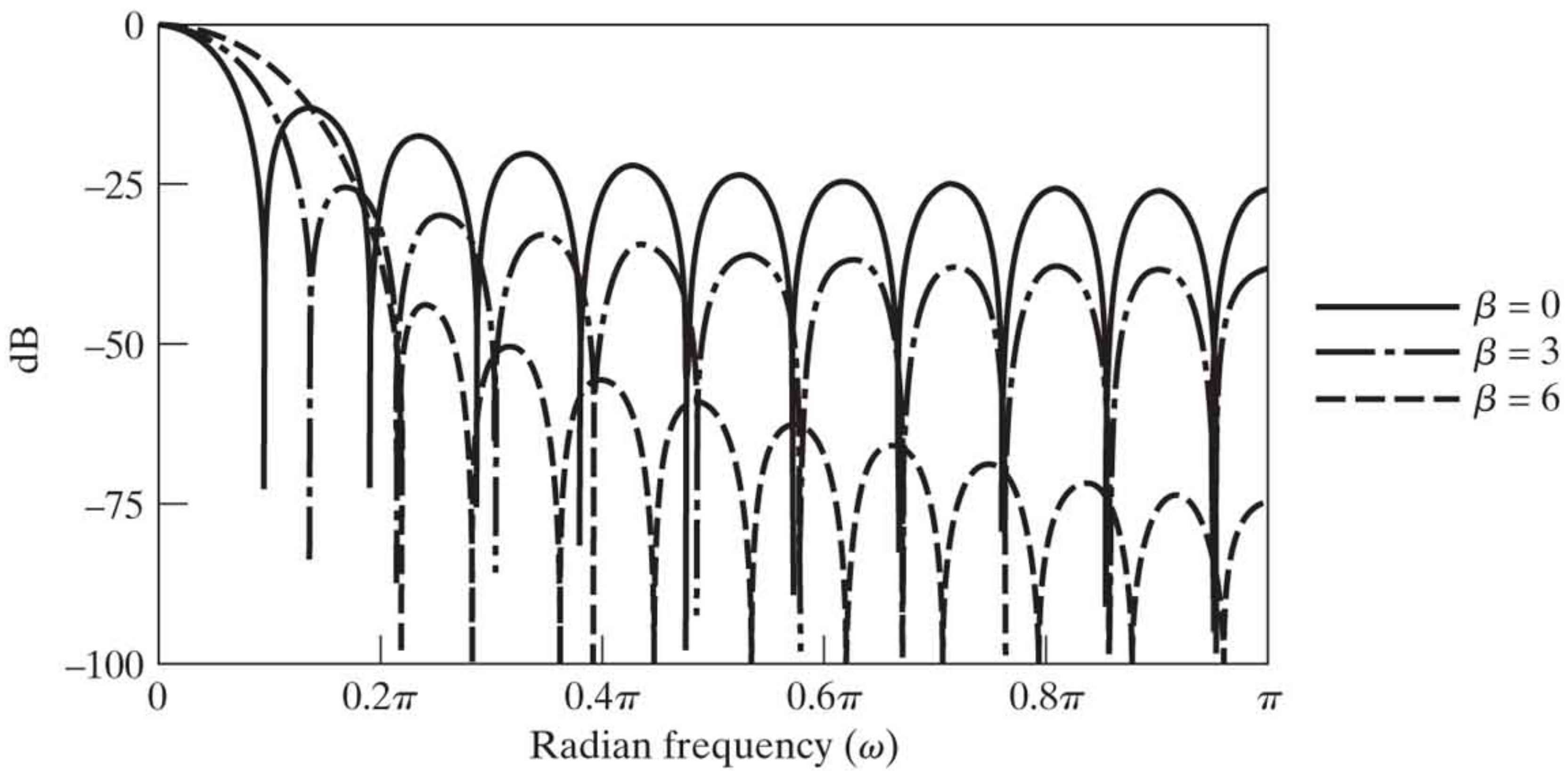
- M=20



(a)

Kaiser Window

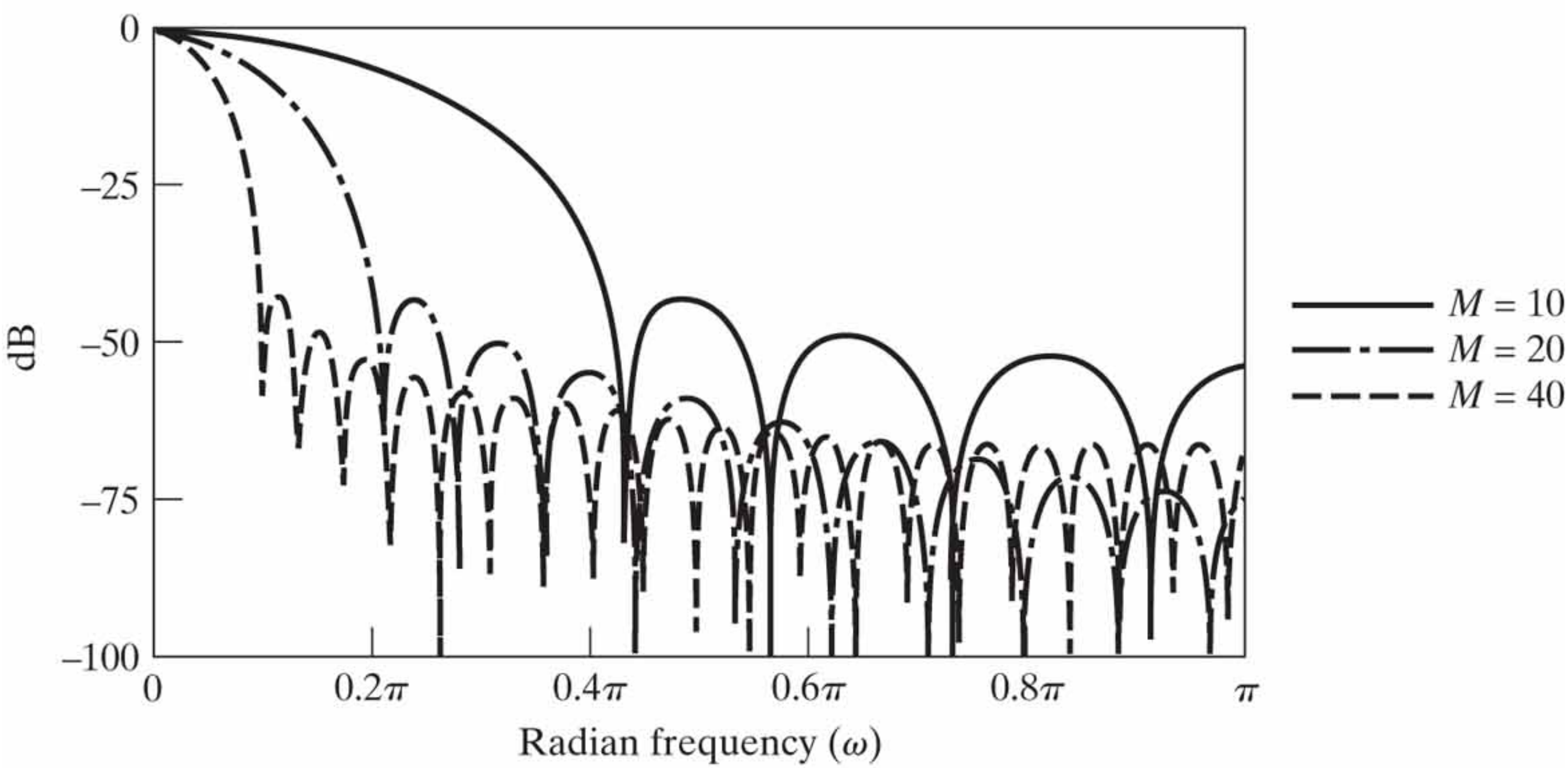
- M=20



(b)

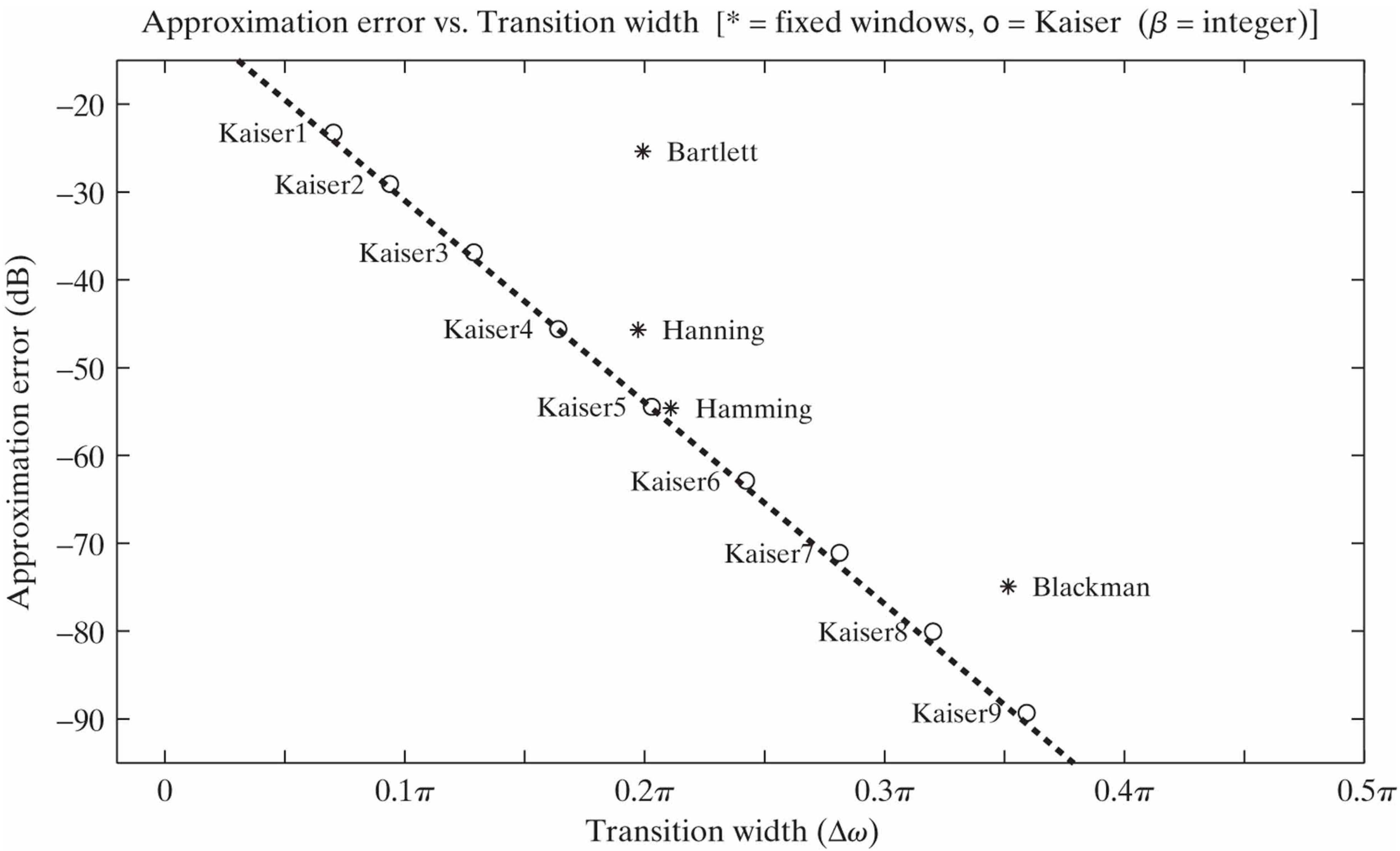
Kaiser Window

- β=6



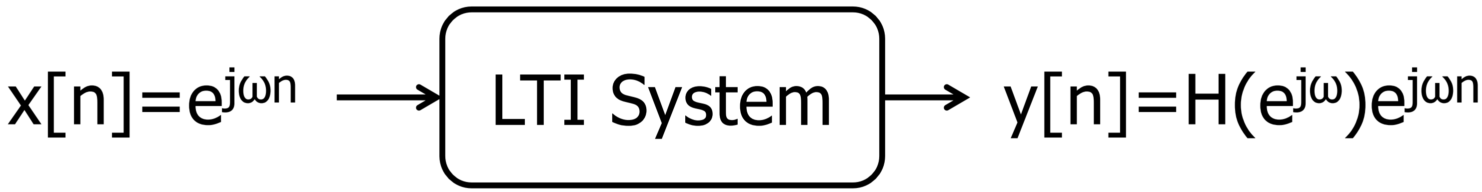
(c)

Approximation Error



LTI System Frequency Response

- (DT)Fourier Transform of impulse response



$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

Example: Moving Average

- Moving Average Filter
  - Causal: $M_1=0$, $M_2=M$

$$y[n]=\frac{x[n-M]+\ldots+x[n]}{M+1}$$

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  - Causal: $M_1=0$, $M_2=M$

$$y[n] = \frac{x[n-M] + \ldots + x[n]}{M+1}$$

Impulse response

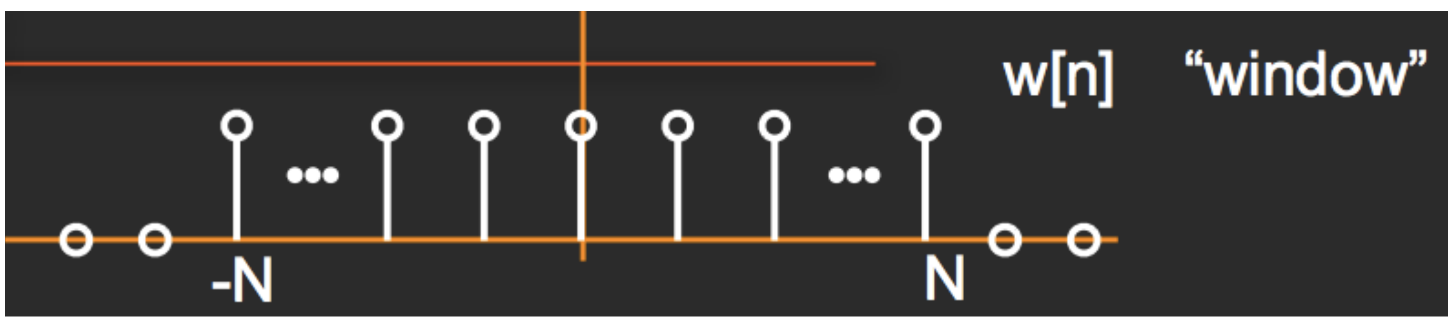
Example: Moving Average

- Moving Average Filter
  - Causal: $M_1=0$, $M_2=M$

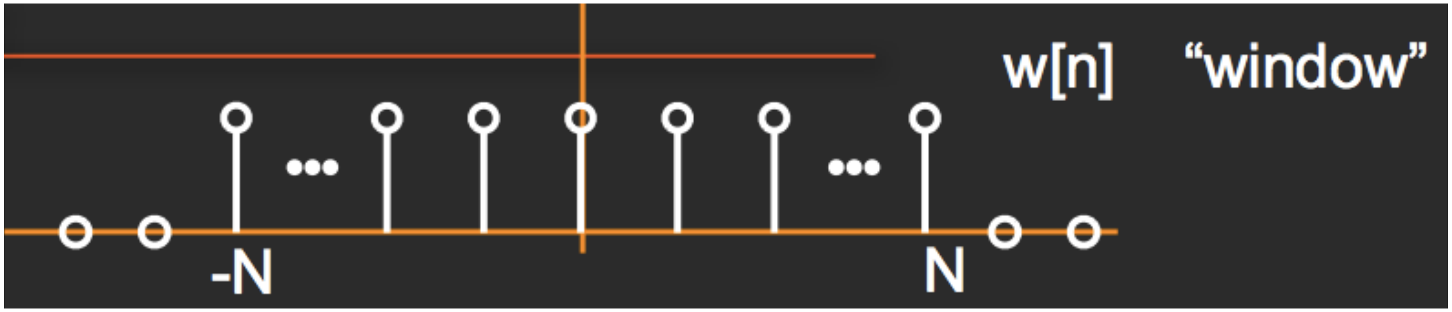
$$y[n]=\frac{x[n-M]+\ldots+x[n]}{M+1}$$

Impulse response

Scaled &Time Shifted window

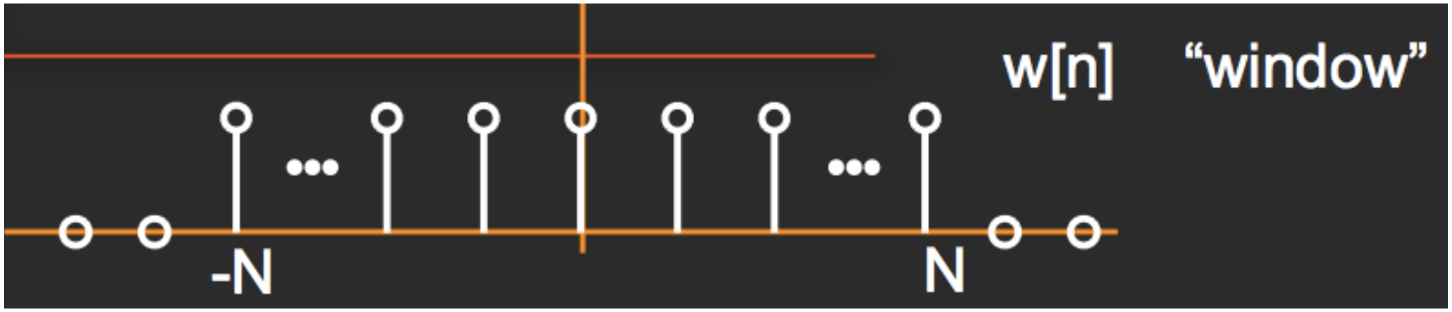


Example: Moving Average



$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

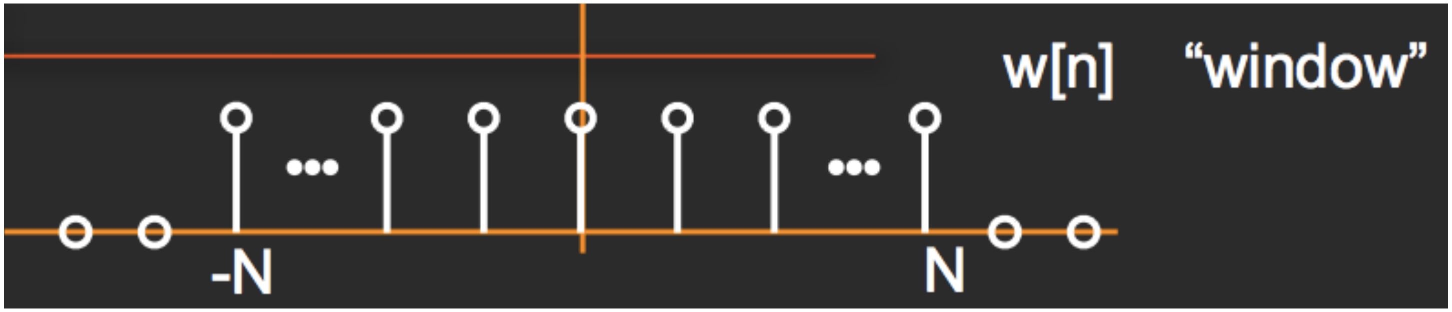
Example: Moving Average



$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

$$h[n] =$$

Example: Moving Average



$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

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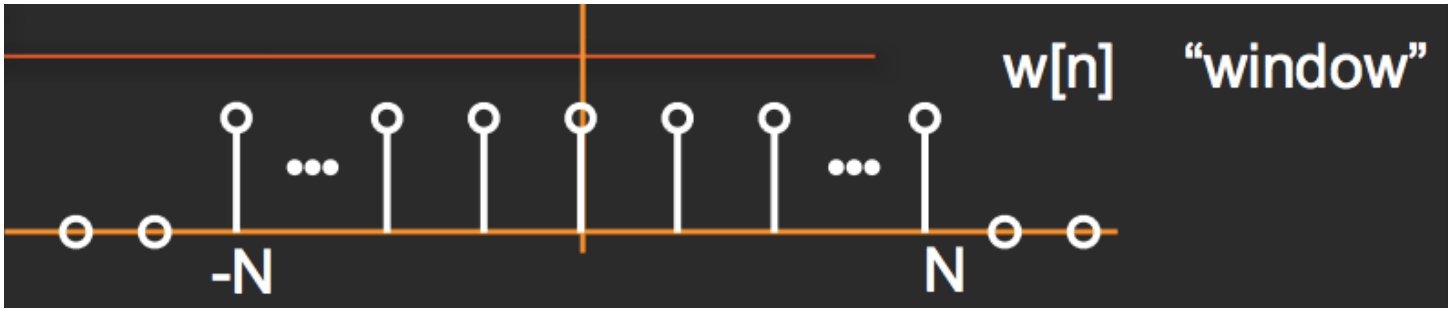
Example: Moving Average



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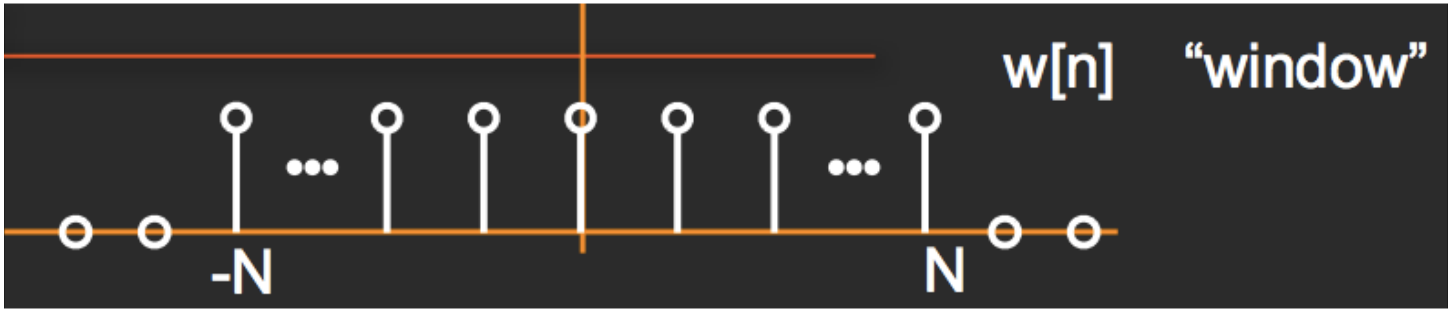
$$x[n - n_d] \leftrightarrow e^{-j\omega n_d} X(e^{j\omega})$$

Example: Moving Average



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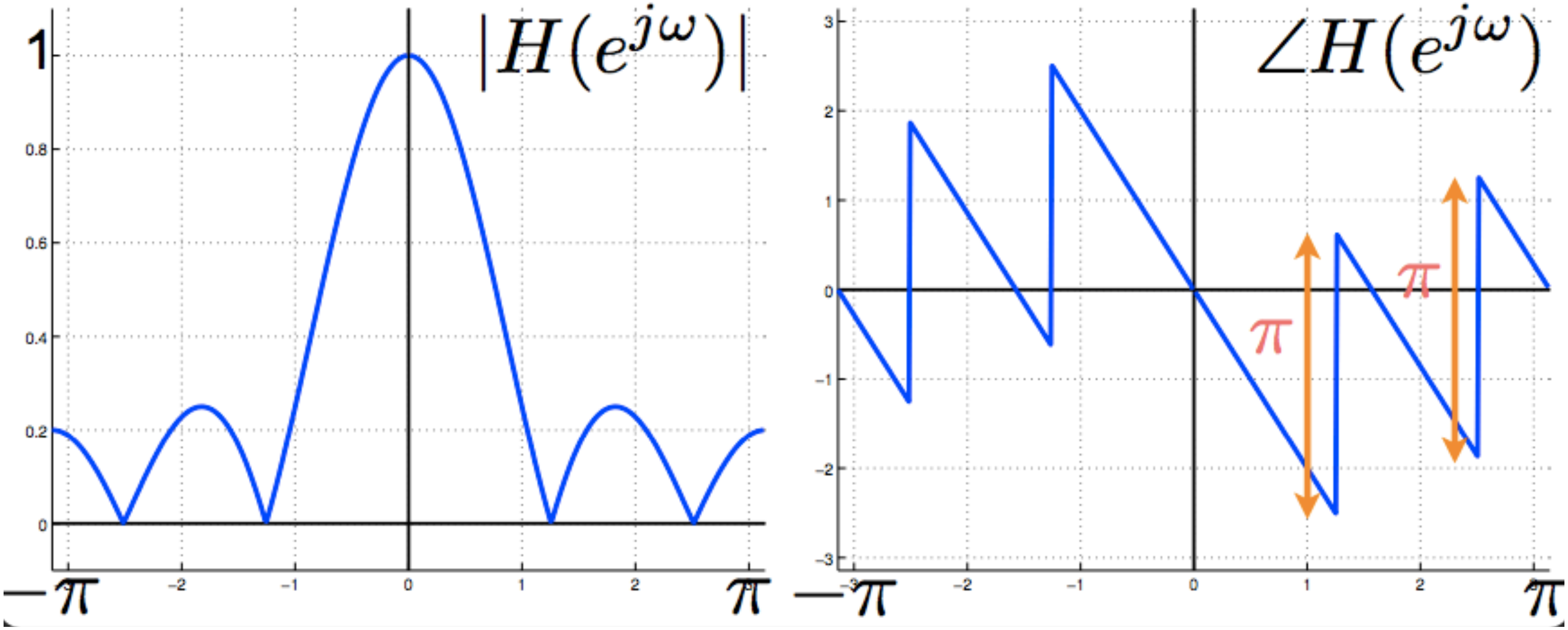
$$h[n] = \frac{1}{M+1} w[n - M/2] \leftrightarrow H(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin\left((M/2+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Example: Moving Average



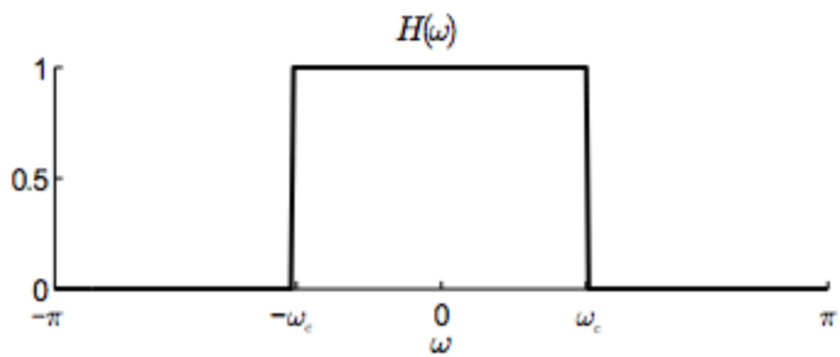
M=4
(N=2)

$$H(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin\left((M/2+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Example: Ideal Low-Pass Filter

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm\pi$)

$$H(\omega) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$



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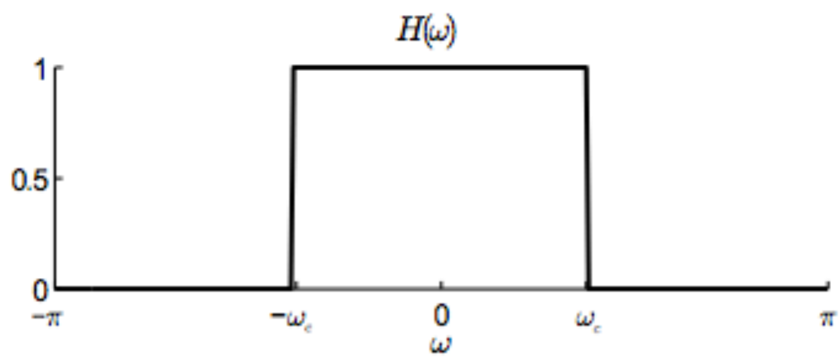

- Compute the impulse response $h[n]$ given this $H(\omega)$
- Apply the inverse DTFT

$$h[n] = \int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n} \frac{d\omega}{2\pi} = \int_{-\omega_c}^{\omega_c} e^{j\omega n} \frac{d\omega}{2\pi} = \left. \frac{e^{j\omega n}}{2\pi jn} \right|_{-\omega_c}^{\omega_c} = \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2\pi jn} = \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n}$$

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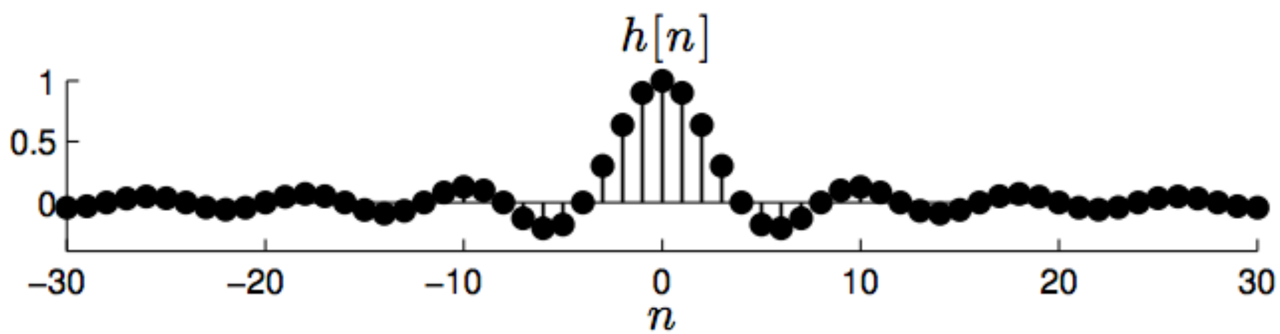

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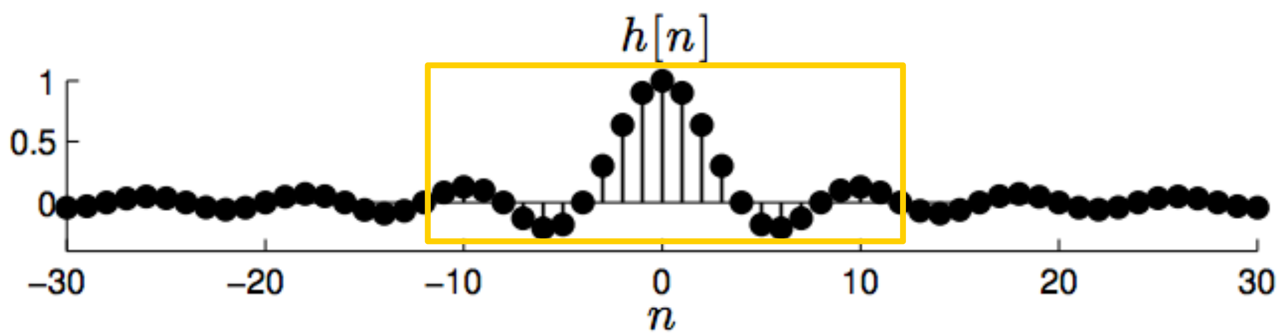
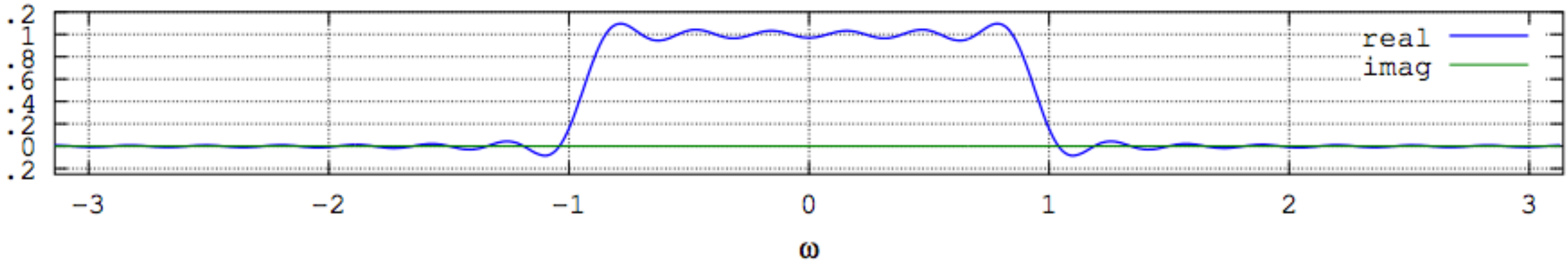
$$h[n] \;=\; 2\omega_c \frac{\sin(\omega_c n)}{\omega_c n}$$



Truncate and shift

$$h_{LP}[n] = w_N[n-N] \cdot h[n-N]$$

Example: Practical LP Filter



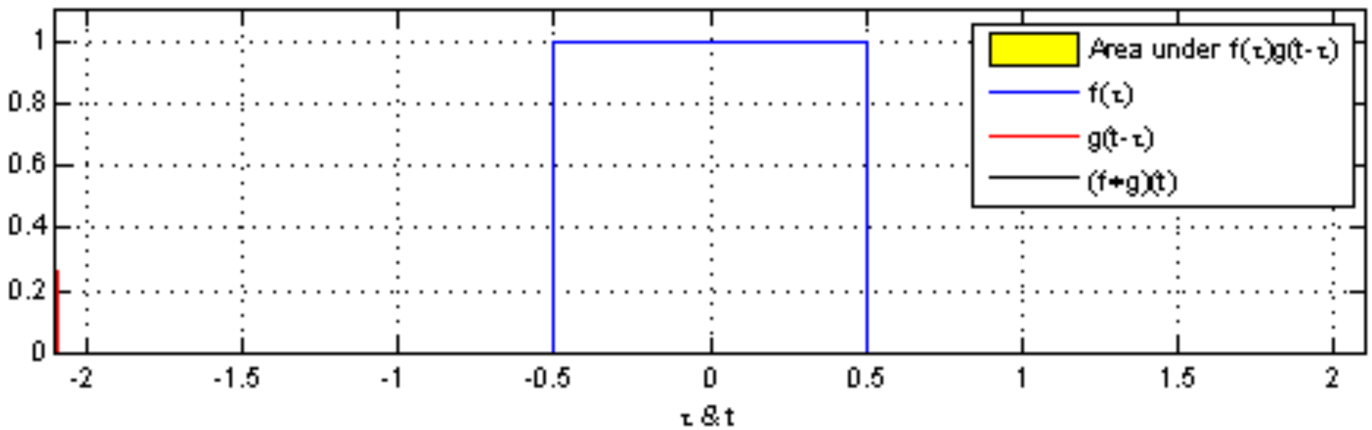
- Pass band smeared and rippled
  - Smearing determined by width of main lobe
  - Rippling determined by size of side lobes

FIR Design by Windowing

- With multiplication in time property,

$$h_{LP}[n] = w_N[n-N] \cdot h[n-N]$$

$$H(e^{j\omega}) = H_d(e^{j\omega}) * W(e^{j\omega})$$



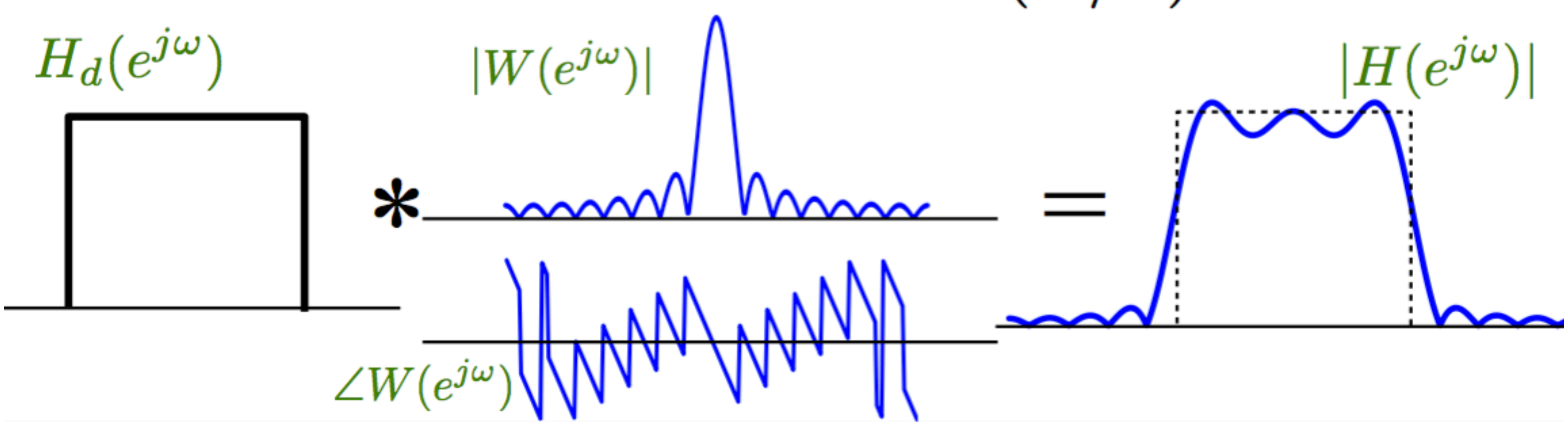
FIR Design by Windowing

- With multiplication in time property,

$$H(e^{j\omega}) = H_d(e^{j\omega}) * W(e^{j\omega})$$

- For Boxcar (rectangular) window

$$W(e^{j\omega}) = e^{-j\omega\frac{M}{2}} \frac{\sin(w(M+1)/2)}{\sin(w/2)}$$



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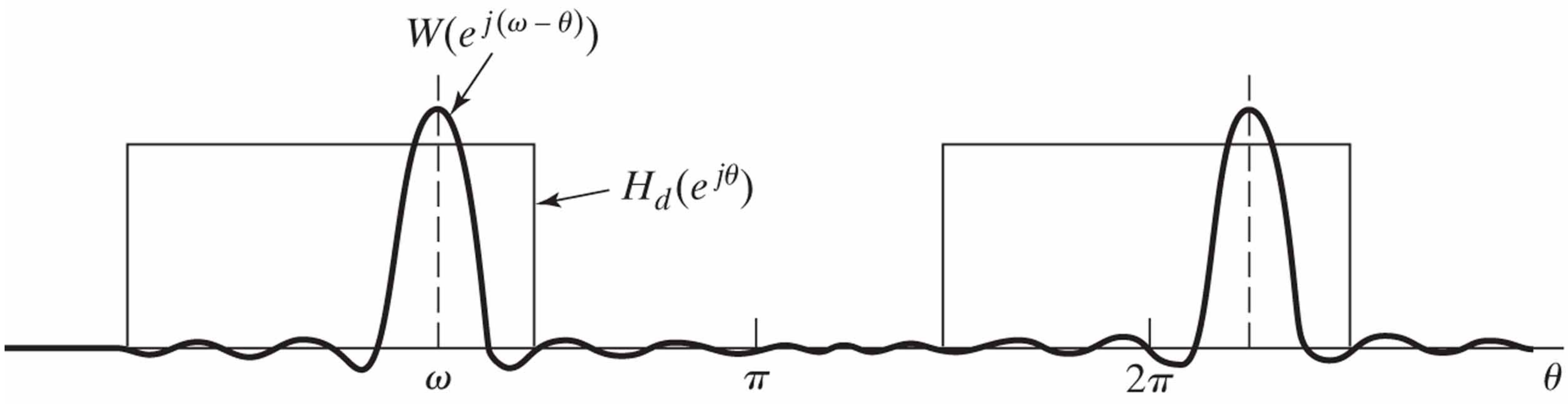
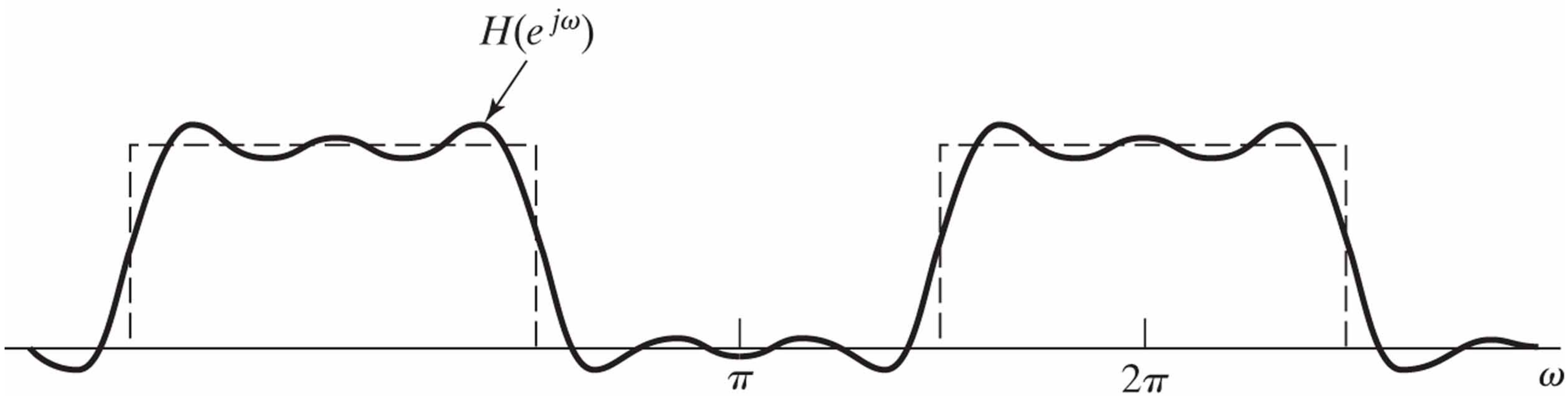
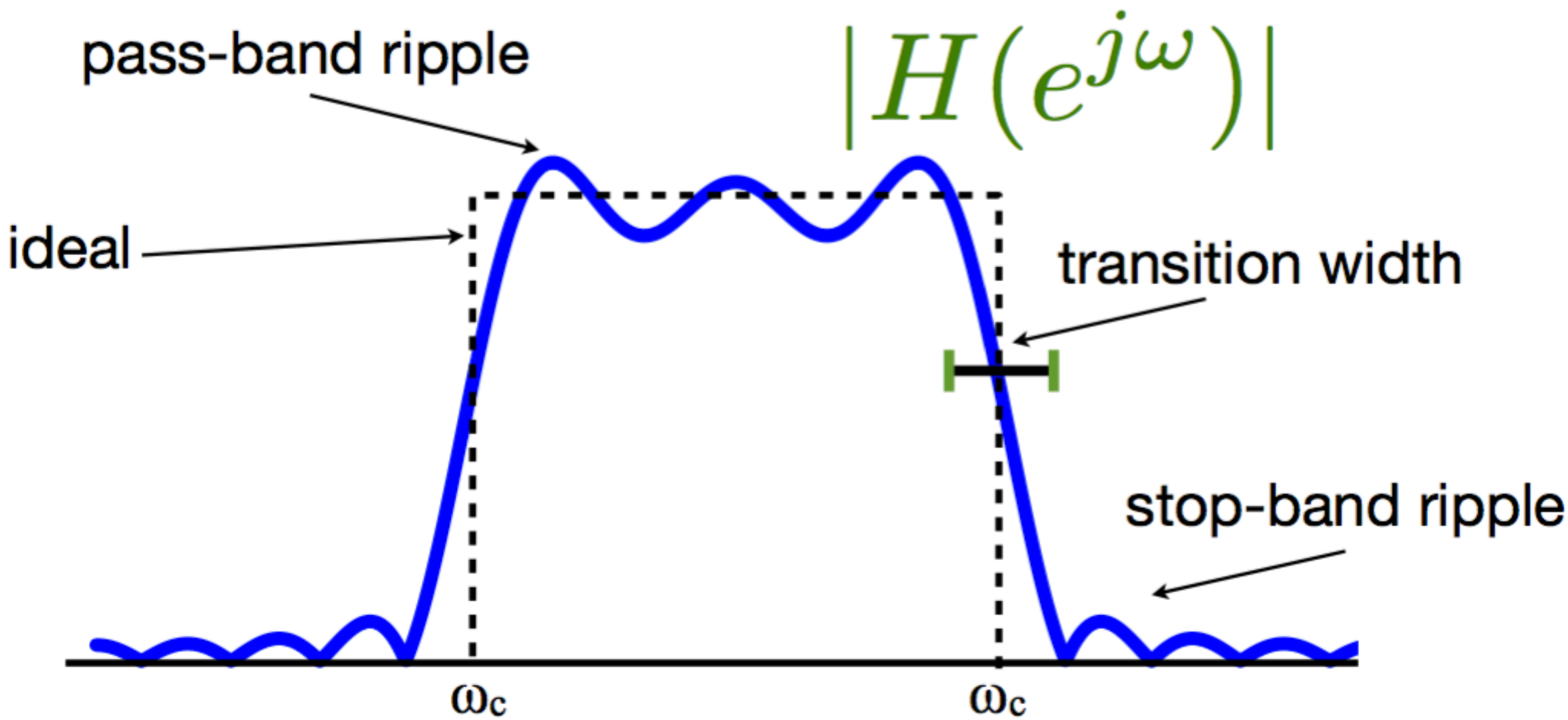


(a)



(b)

FIR Design by Windowing



FIR Filter Design

- Choose a desired frequency response $H_d(e^{j\omega})$
  - non causal (zero-delay), and infinite imp. response
  - If derived from C.T, choose T and use:

$$H_d(e^{j\omega}) = H_c(j\frac{\Omega}{T})$$

- Window:
  - Length M+1 ⇔ affects transition width
  - Type of window ⇔ transition-width/ ripple

FIR Filter Design

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- Window:
  - Length M+1 ⇔ affects transition width
  - Type of window ⇔ transition-width/ ripple
  - Modulate to shift impulse response
    - Force causality

$$H_d(e^{j\omega})e^{-j\omega\frac{M}{2}}$$

FIR Filter Design

- Determine truncated impulse response $h_1[n]$

$$h_1[n] = \begin{cases} \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{-j\omega\frac{M}{2}}e^{j\omega n} & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

- Apply window

$$h_w[n] = w[n]h_1[n]$$

- Check:
  - Compute $H_w(e^{j\omega})$, if does not meet specs increase M or change window

Example: FIR Low-Pass Filter Design

$$H_d(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Choose M ⇒ Window length and set

$$H_1(e^{j\omega}) = H_d(e^{j\omega})e^{-j\omega\frac{M}{2}}$$

$$h_1[n] = \begin{cases} \frac{\sin(\omega_c(n-M/2))}{\pi(n-M/2)} & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\omega_c}{\pi}\text{sinc}(\frac{\omega_c}{\pi}(n - M/2))$$

Example: FIR Low-Pass Filter Design

- The result is a windowed sinc function

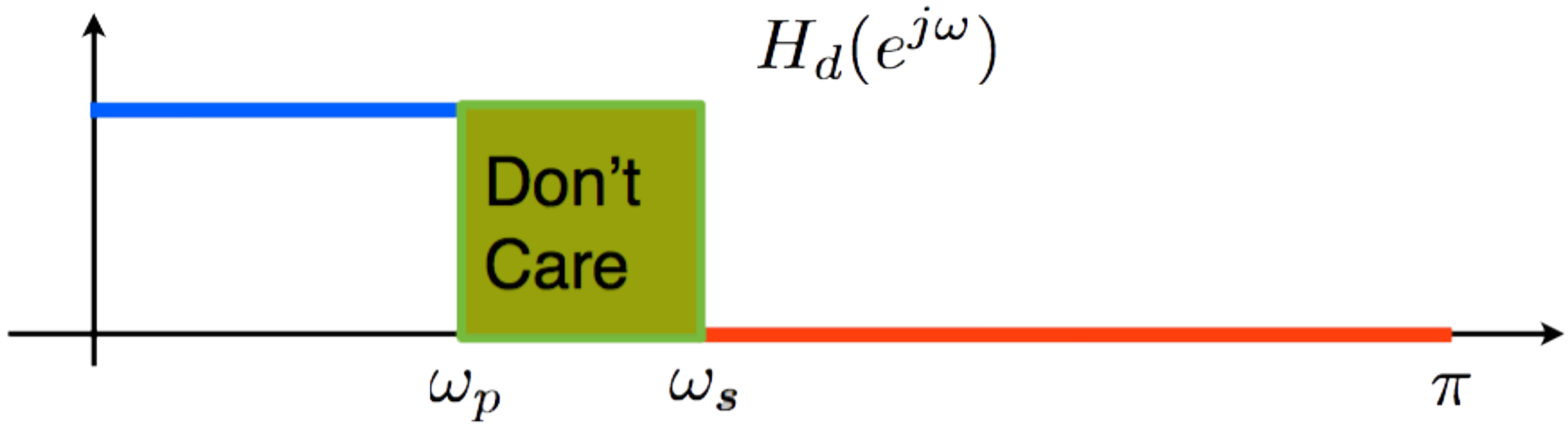
$$h_w[n] = w[n]h_1[n]$$

$$\frac{\omega_c}{\pi}\text{sinc}(\frac{\omega_c}{\pi}(n - M/2))$$

Optimal Filter Design

- Window method
  - Design Filters heuristically using windowed sinc functions
  - Choose order and window type
  - Check DTFT to see if filter specs are met
- Optimal design
  - Design a filter h[n] with $H(e^{j\omega})$
  - Approximate $H_d(e^{j\omega})$ with some optimality criteria - or satisfies specs.

Optimality – Least Squares



- Least Squares:

$$\text{minimize} \quad \int_{\omega \in \text{care}} |H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

- Variation: Weighted Least Squares:

$$\text{minimize} \quad \int_{-\pi}^{\pi} W(\omega) |H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

Spectral Analysis Using the DFT

- Two important tools:
  - Applying a window → reduced artifacts
  - Zero-padding → increases spectral sampling

Parameter	Symbol	Units
Sampling interval	T	s
Sampling frequency	$\Omega_s = \frac{2\pi}{T}$	rad/s
Window length	L	unitless
Window duration	$L \cdot T$	s
DFT length	$N \geq L$	unitless
DFT duration	$N \cdot T$	s
Spectral resolution	$\frac{\Omega_s}{L} = \frac{2\pi}{L \cdot T}$	rad/s
Spectral sampling interval	$\frac{\Omega_s}{N} = \frac{2\pi}{N \cdot T}$	rad/s

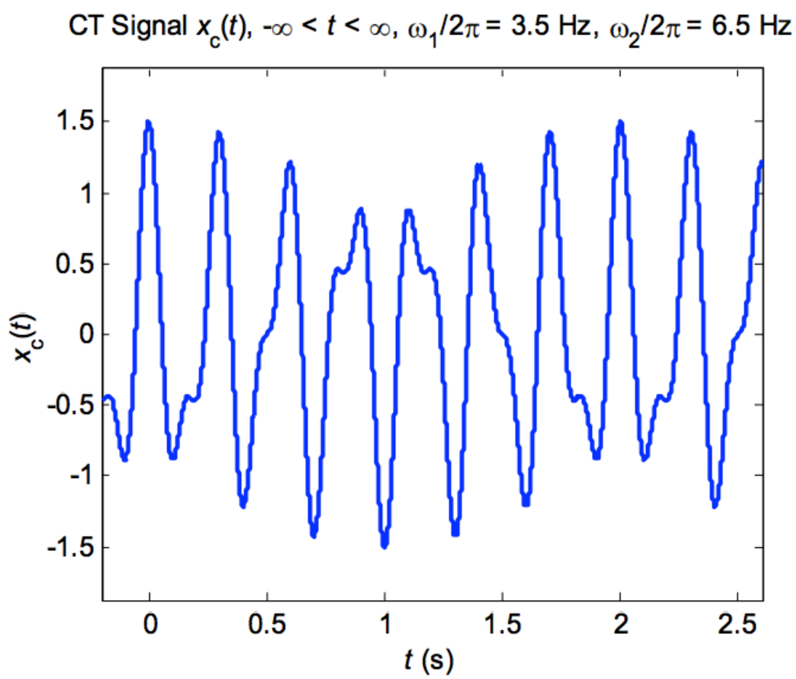
CT Signal Example

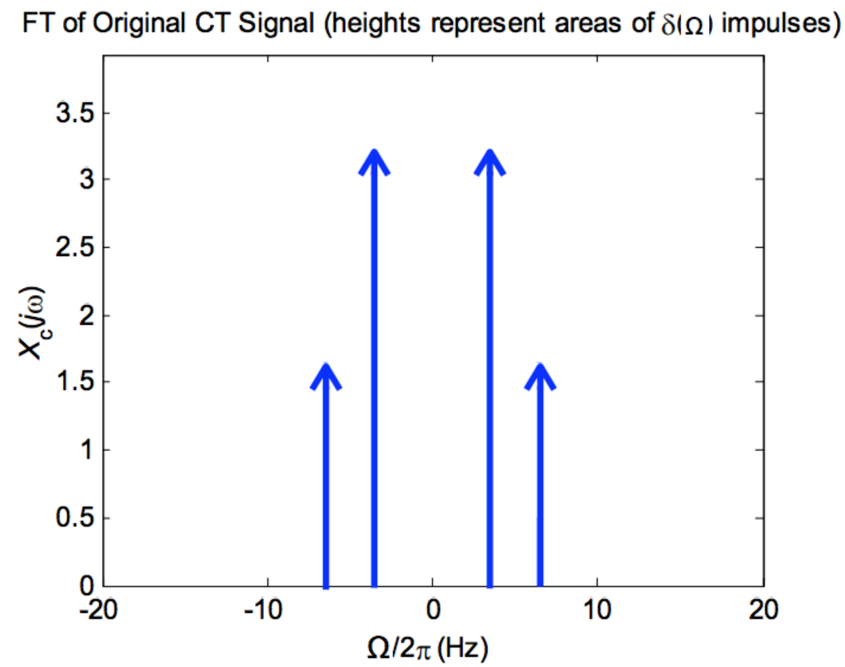
$$x_c(t) = A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t)$$
$$X_c(j\Omega) = A_1\pi\left[\delta(\Omega-\omega_1)+\delta(\Omega+\omega_1)\right] + A_2\pi\left[\delta(\Omega-\omega_2)+\delta(\Omega+\omega_2)\right]$$

CT Signal Example

$$x_c(t) = A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t)$$

$$X_c(j\Omega) = A_1 \pi \left[\delta(\Omega - \omega_1) + \delta(\Omega + \omega_1) \right] + A_2 \pi \left[\delta(\Omega - \omega_2) + \delta(\Omega + \omega_2) \right]$$





Sampled CT Signal Example

- If we sample the signal over an infinite time duration, we would have:

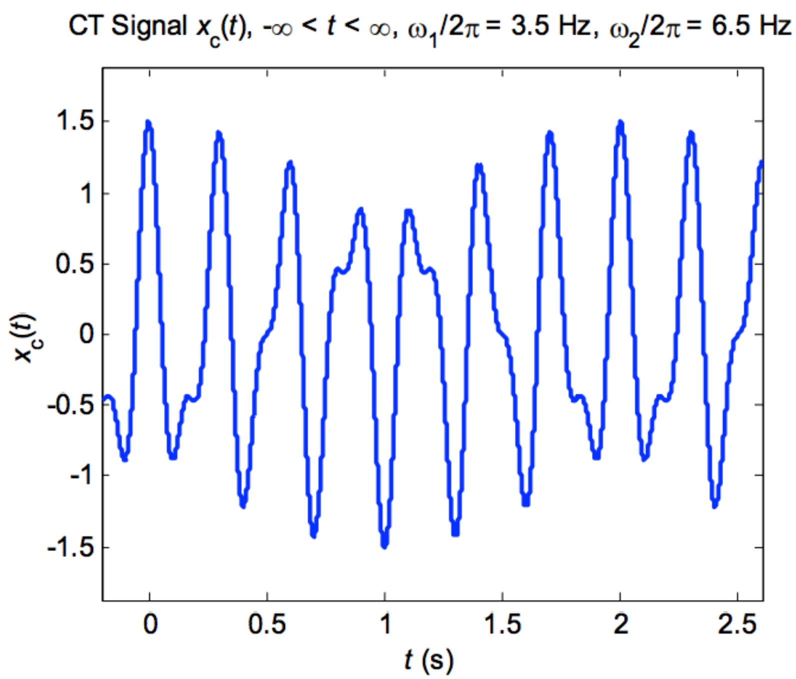
$$x[n] = x_c(t)\big|_{t=nT}, \qquad -\infty < n < \infty$$

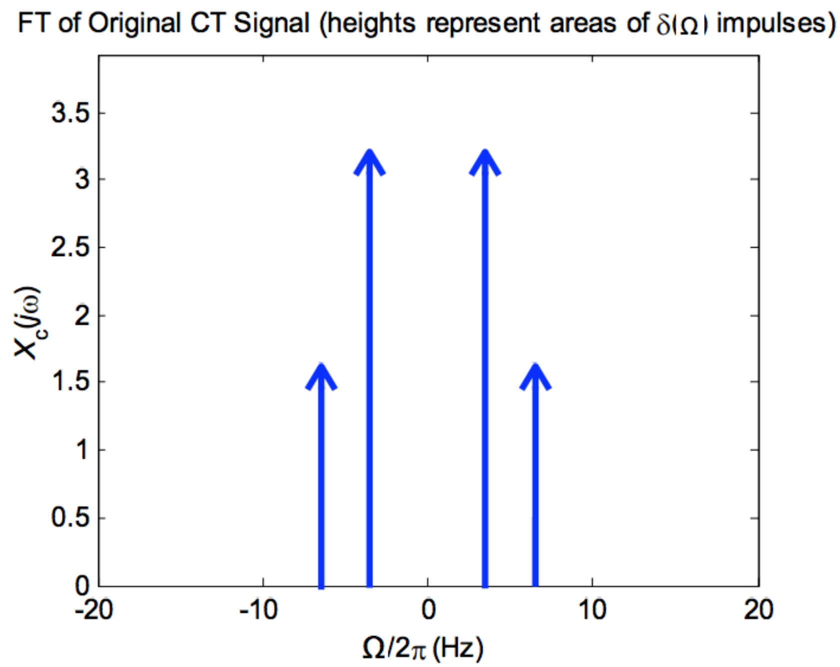
- With the discrete time Fourier transform (DTFT):

$$X(e^{j\Omega T}) = \frac{1}{T}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\Omega - r\frac{2\pi}{T}\right)\right), \qquad -\infty < \Omega < \infty$$

CT Signal Example

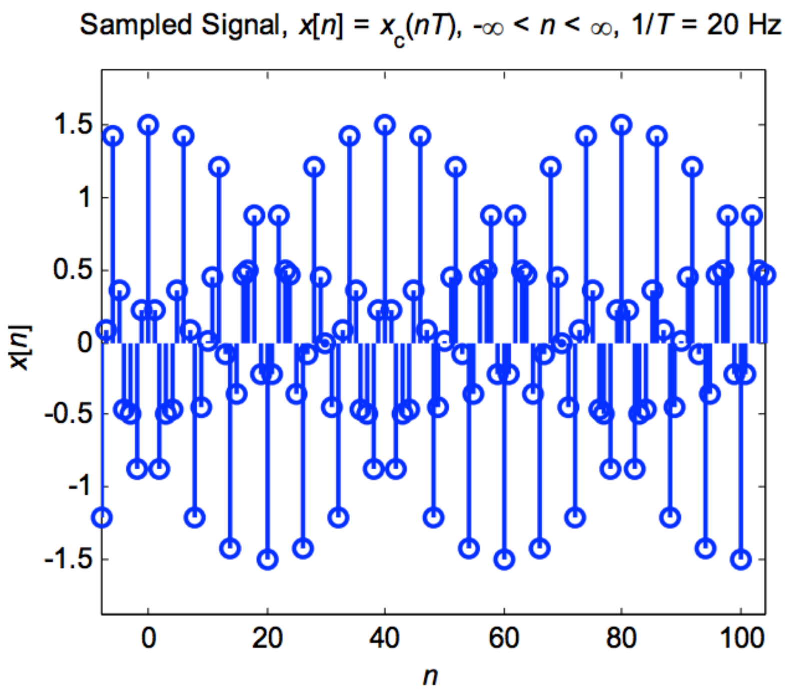
$$x_c(t) = A_1\cos(\omega_1 t) + A_2\cos(\omega_2 t)$$
$$X_c(j\Omega) = A_1\pi\left[\delta(\Omega-\omega_1)+\delta(\Omega+\omega_1)\right] + A_2\pi\left[\delta(\Omega-\omega_2)+\delta(\Omega+\omega_2)\right]$$

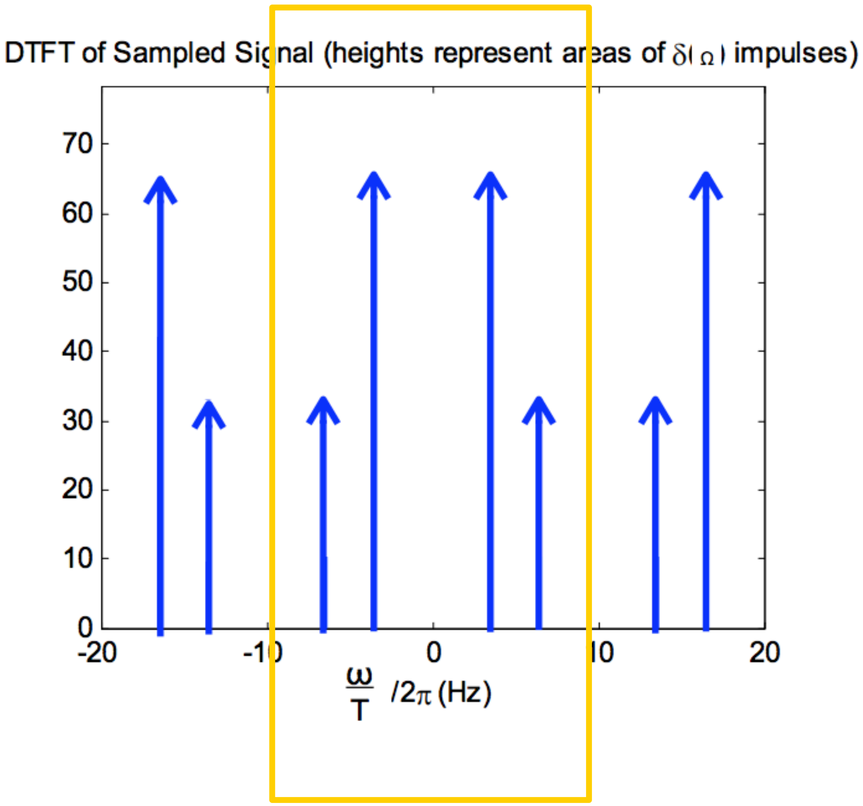




Sampled CT Signal Example

- Sampling with Ωs/2π=1/T=20Hz





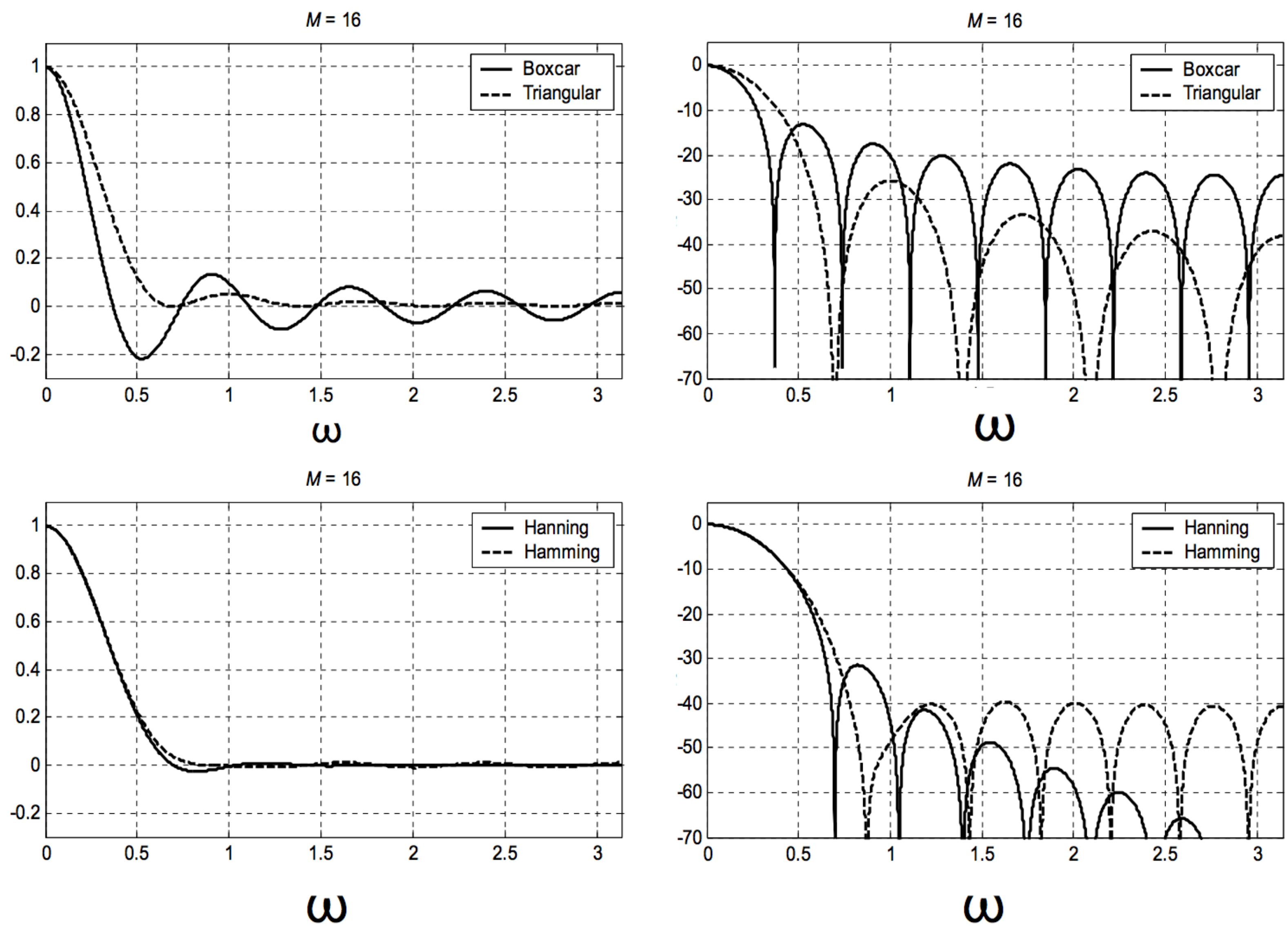
Windowed Sampled CT Signal

- In any real system, we sample only over a finite block of L samples:

$$x[n] = x_c(t)\Big|_{t=nT}, \qquad 0 < n < L-1$$

- This simply corresponds to a rectangular window of duration L
- Recall there are many other window types
  - Hann, Hamming, Blackman, Kaiser, etc.

Windows



M = 16
Boxcar
Triangular
ω
M = 16
Boxcar
Triangular
ω
M = 16
Hanning
Hamming
ω
M = 16
Hanning
Hamming
ω

Windowed Sampled CT Signal

- We take the block of signal samples and multiply by a window of duration L, obtaining:

$$v[n] = x[n] \cdot w[n], \qquad 0 < n < L-1$$

- If the window w[n] has DTFT, $W(e^{j\omega})$, then the windowed block of signal samples has a DTFT given by the periodic convolution between $X(e^{j\omega})$ and $W(e^{j\omega})$:

$$V(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) W(e^{j(\omega-\theta)}) d\theta$$

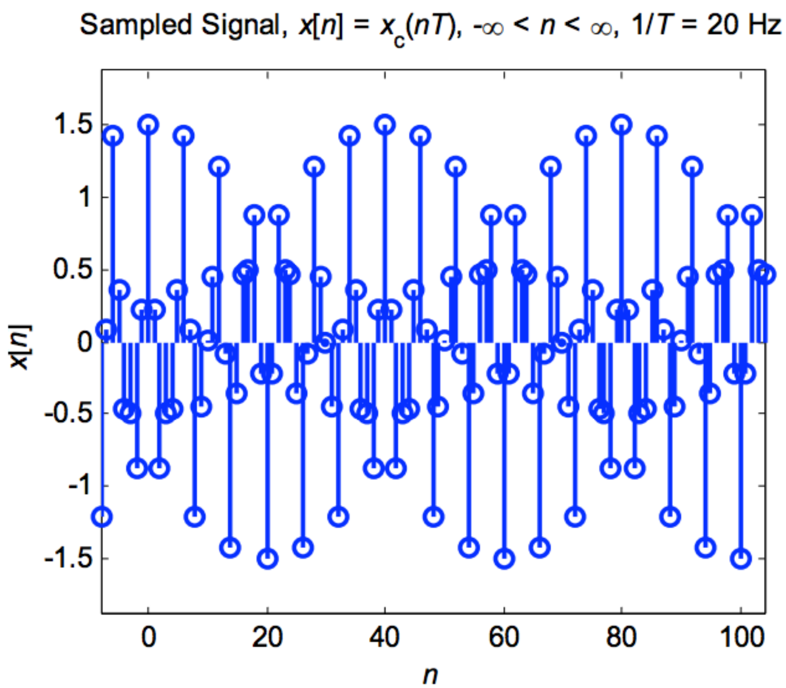
Windowed Sampled CT Signal

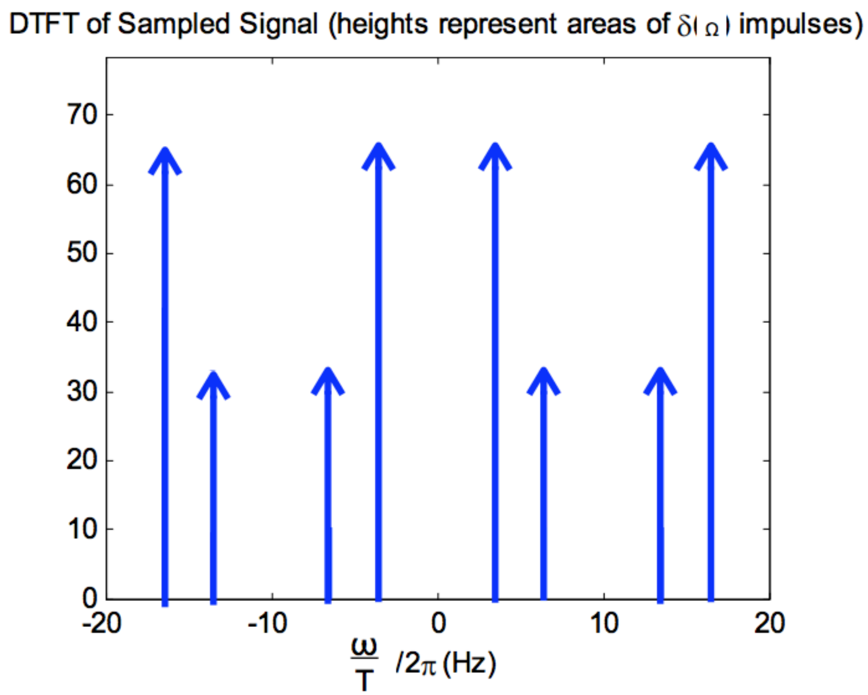
- Convolution with $W(e^{j\omega})$ has two effects in the spectrum:
  - It limits the spectral resolution (spectral spreading)
    - Main lobes of the DTFT of the window
  - The window can produce spectral leakage
    - Side lobes of the DTFT of the window

- These two are always a tradeoff
  - time-frequency uncertainty principle

Sampled CT Signal Example

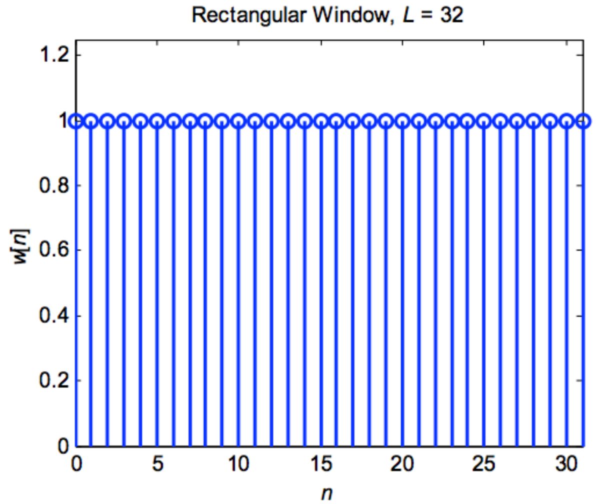
- Sampling with Ωs/2π=1/T=20Hz

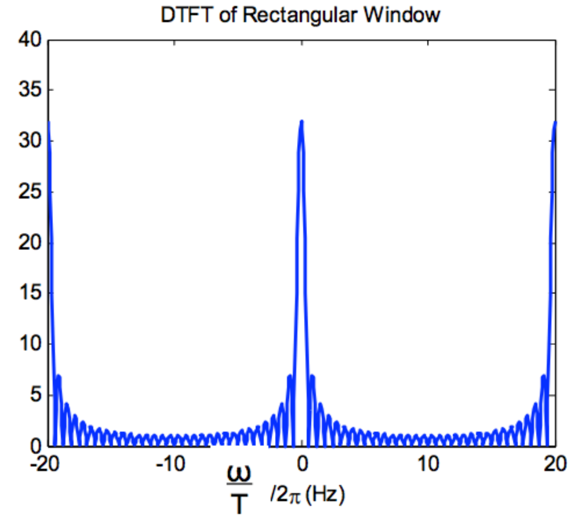


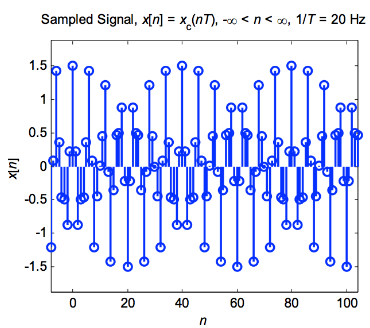


Windowed Sampled CT Signal Example

- As before, the sampling rate is Ωs/2π=1/T=20Hz
- Rectangular Window, L = 32

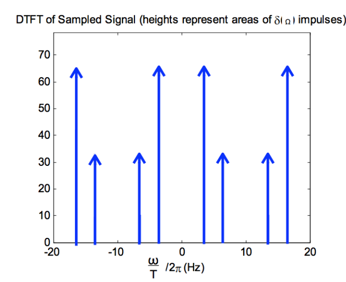






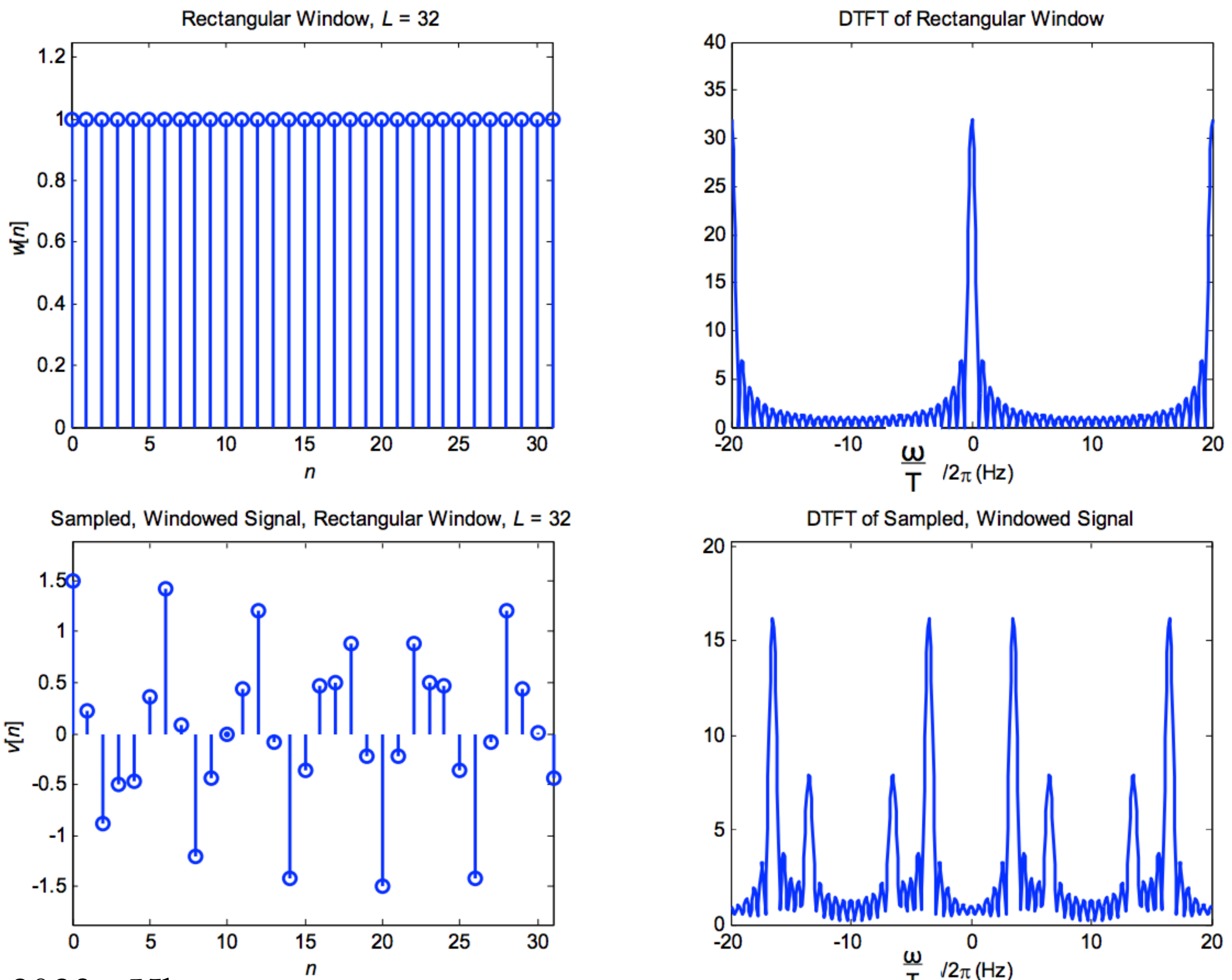
multiply

convolve



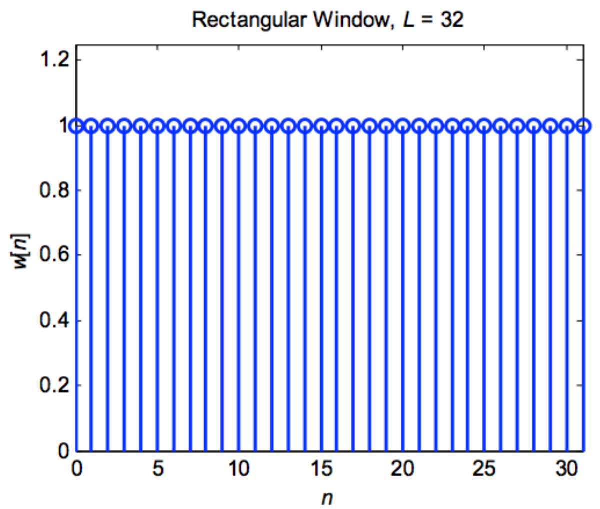
Windowed Sampled CT Signal Example

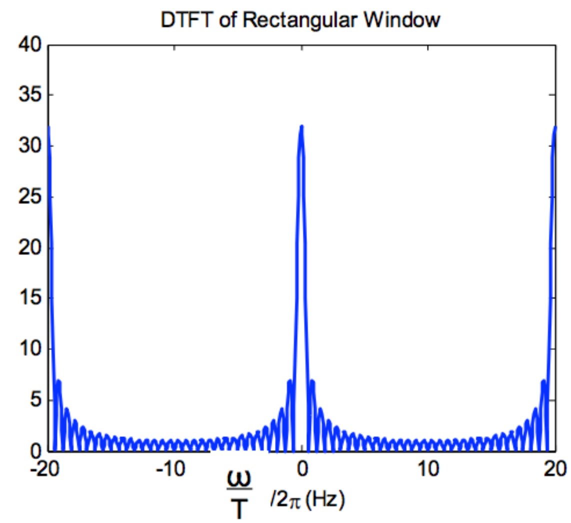
- As before, the sampling rate is $\Omega s/2\pi=1/T=20Hz$
- Rectangular Window, L = 32

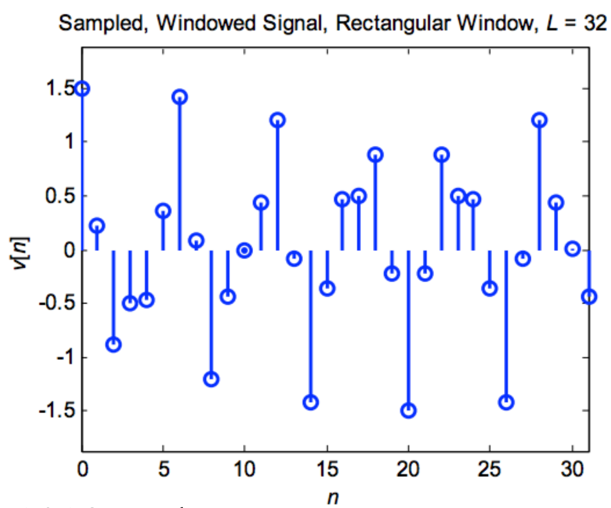


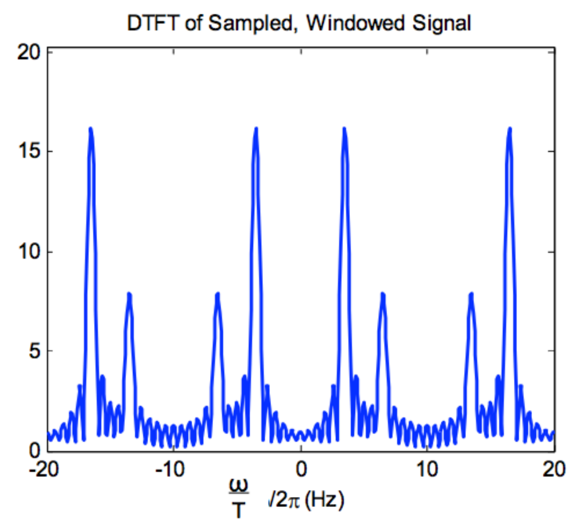
Windowed Sampled CT Signal Example

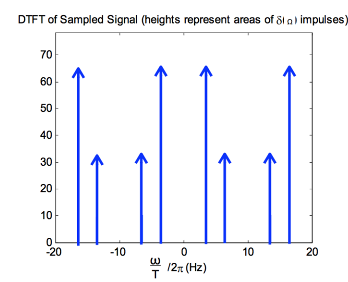
- As before, the sampling rate is $\Omega s/2\pi=1/T=20\text{Hz}$
- Rectangular Window, L = 32





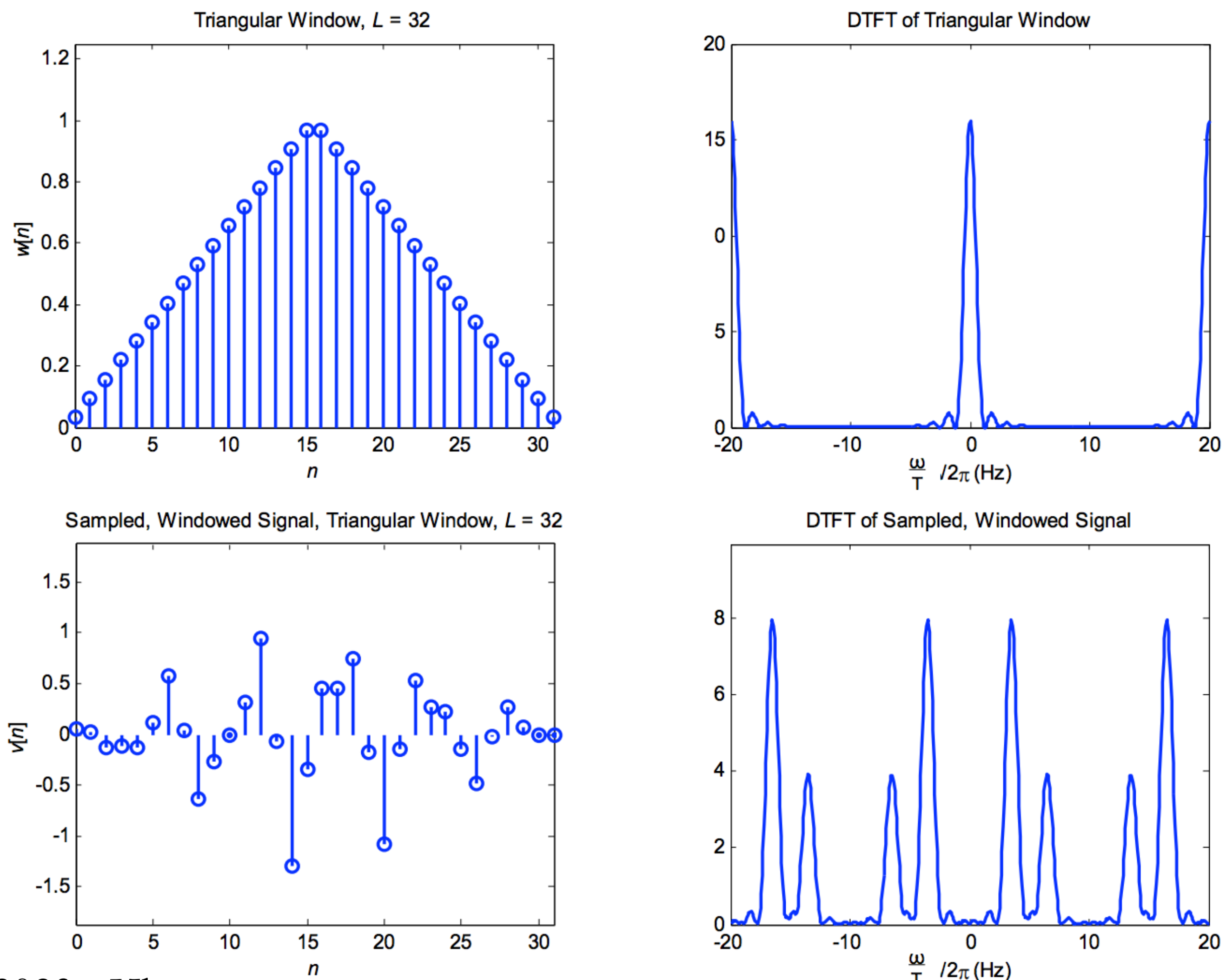






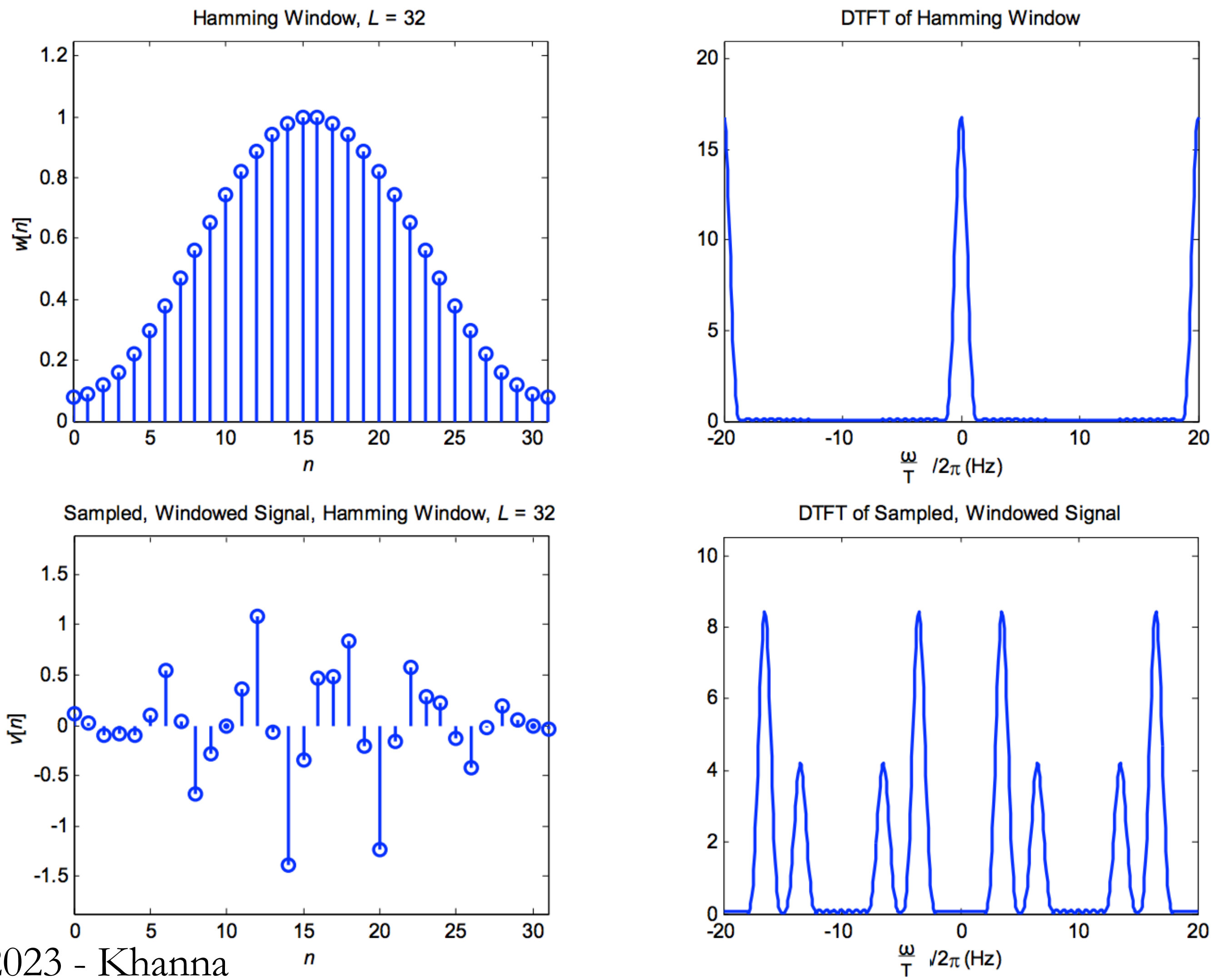
Windowed Sampled CT Signal Example

- As before, the sampling rate is $\Omega s/2\pi = 1/T = 20Hz$
- Triangular Window, L = 32



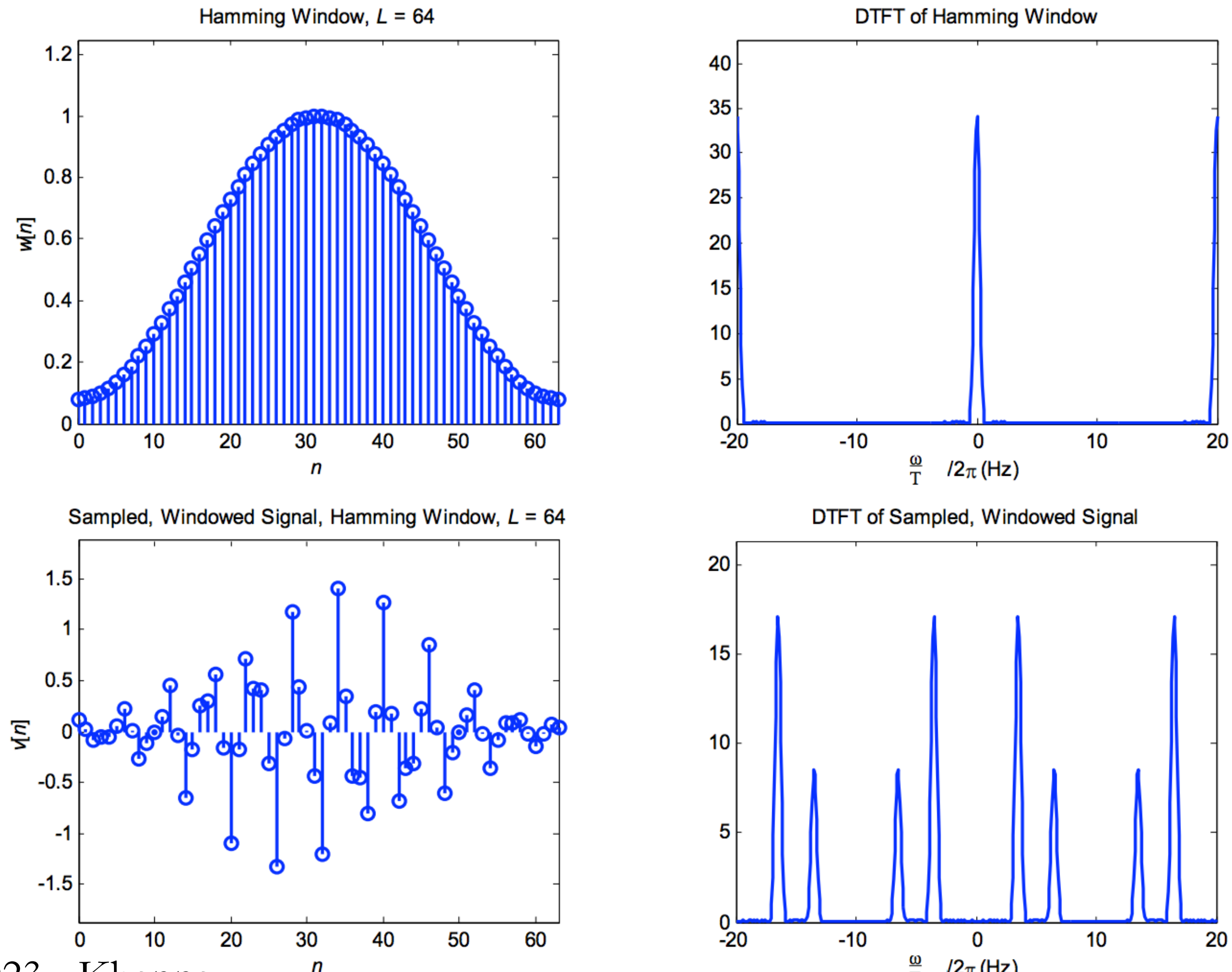
Windowed Sampled CT Signal Example

- As before, the sampling rate is $\Omega s/2\pi = 1/T = 20\text{Hz}$
- Hamming Window, L = 32



Windowed Sampled CT Signal Example

- As before, the sampling rate is $\Omega s/2\pi = 1/T = 20\text{Hz}$
- Hamming Window, L = 64



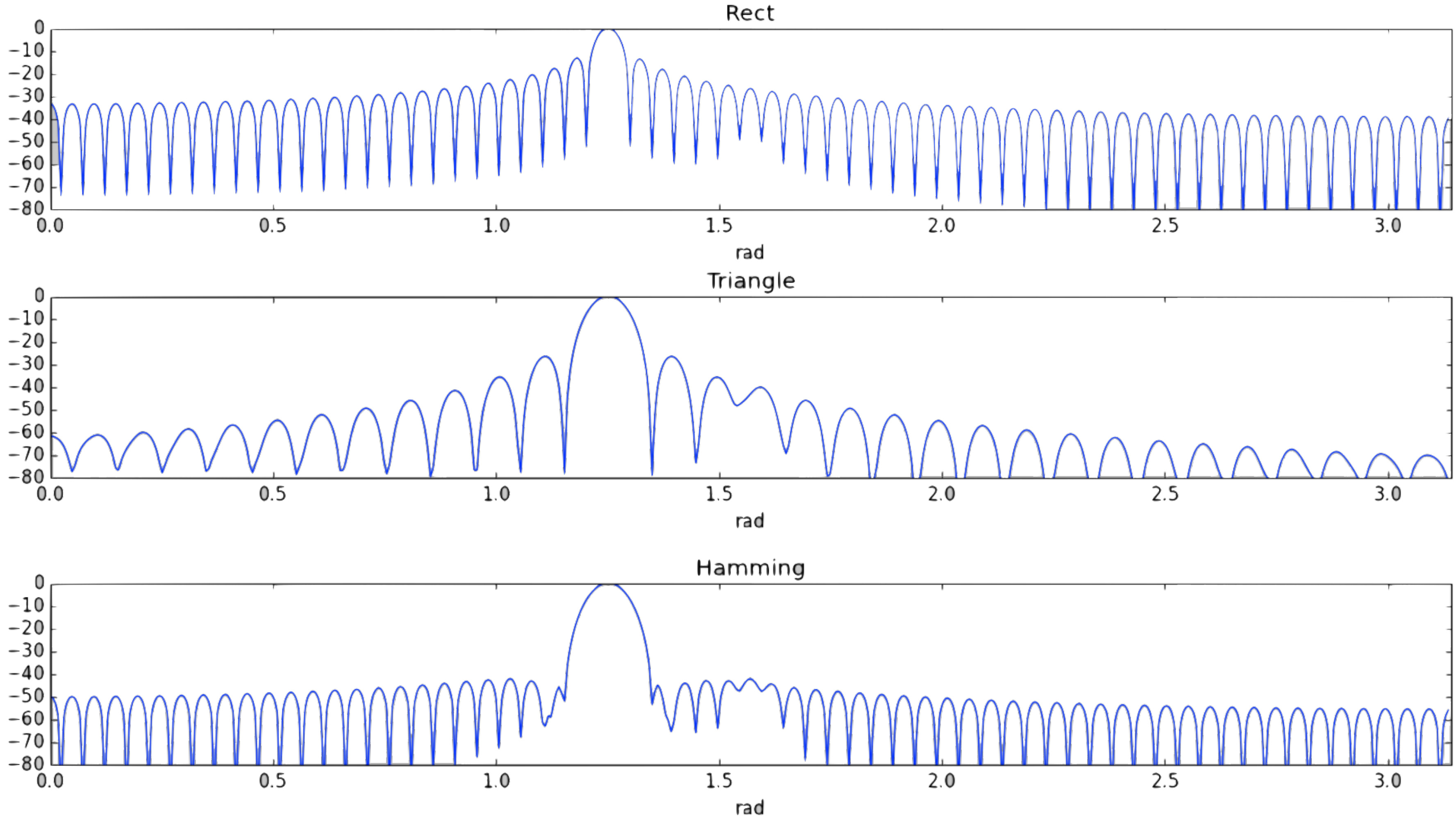
Window Comparison Example

$$y[n] = \underbrace{\sin(2\pi 0.1992 n)}_{1.25} + 0.005\underbrace{\sin(2\pi 0.25 n)}_{1.57} \mid 0 \leq n \leq 128$$

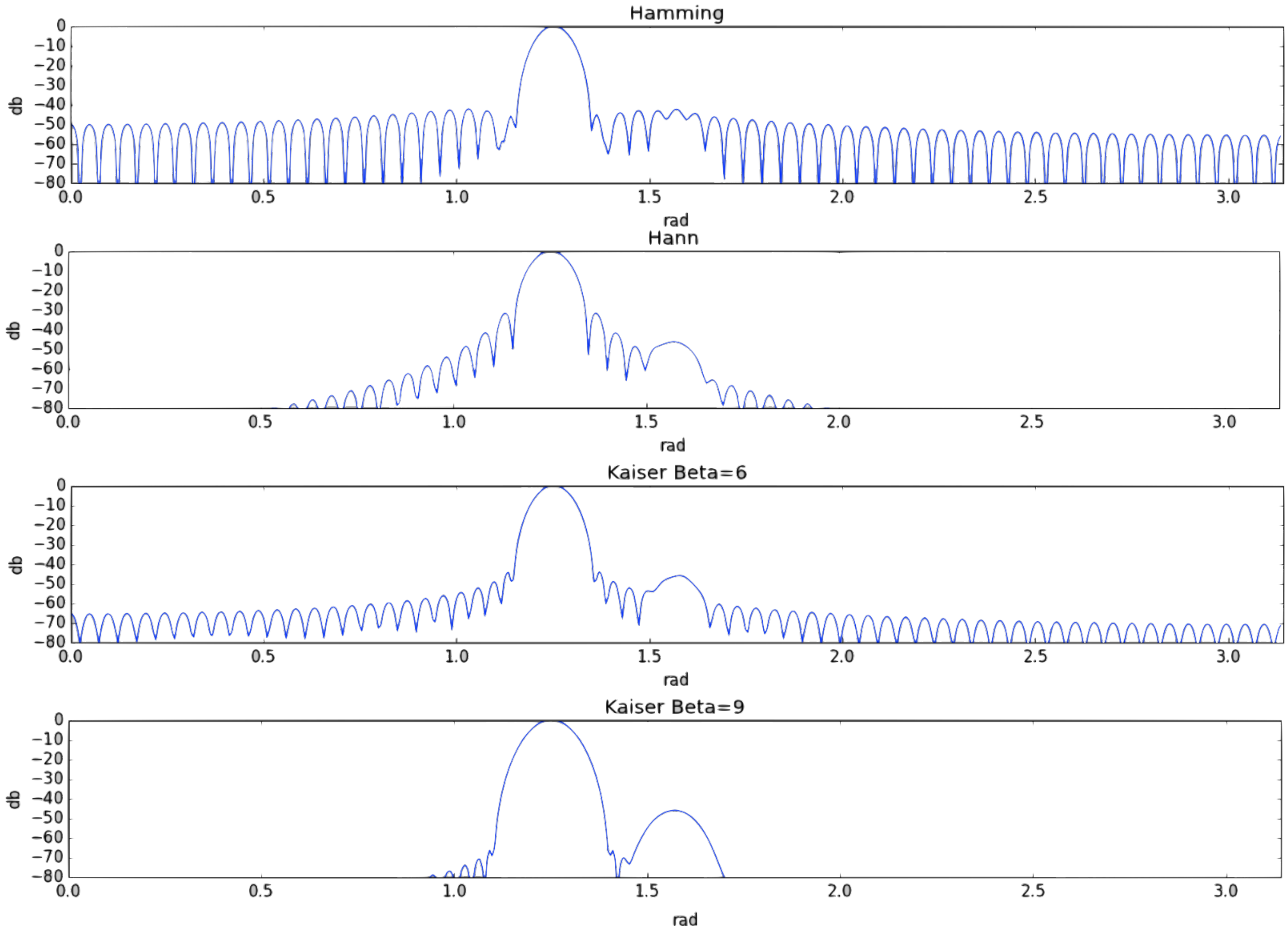
200x smaller → -46dB

Window Comparison Example

$$y[n] = \sin(2\pi 0.1992n) + 0.005\sin(2\pi 0.25n) \,|\, 0 \le n \le 128$$



Window Comparison Example



Zero-Padding

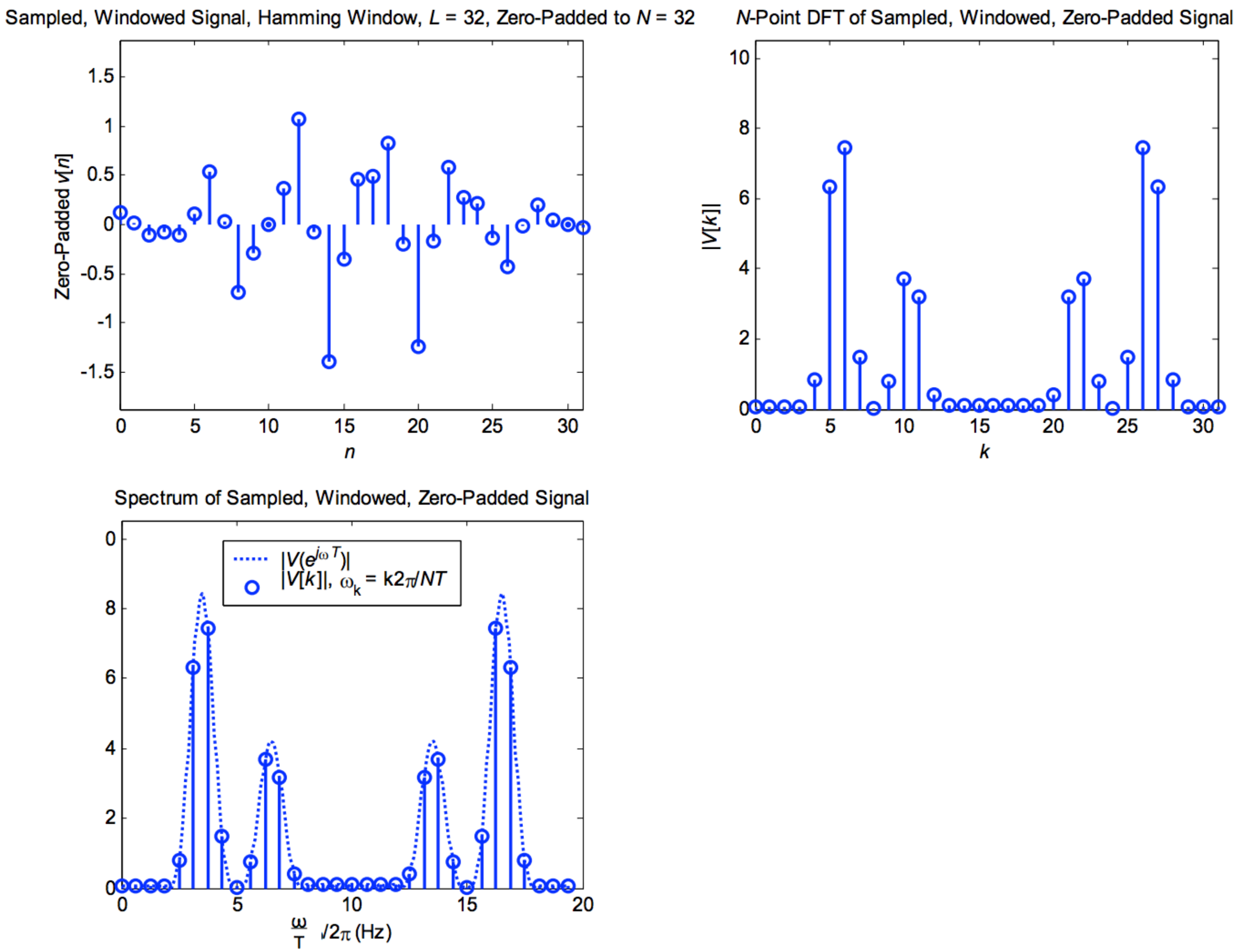
- In preparation for taking an N-point DFT, we may zero-pad the windowed block of signal samples

$$\begin{cases} v[n] & 0 \le n \le L-1 \\ 0 & L \le n \le N-1 \end{cases}$$

- This zero-padding has no effect on the DTFT of v[n], since the DTFT is computed by summing over infinity
- Effect of Zero Padding
  - We take the N-point DFT of the zero-padded v[n], to obtain the block of N spectral samples:

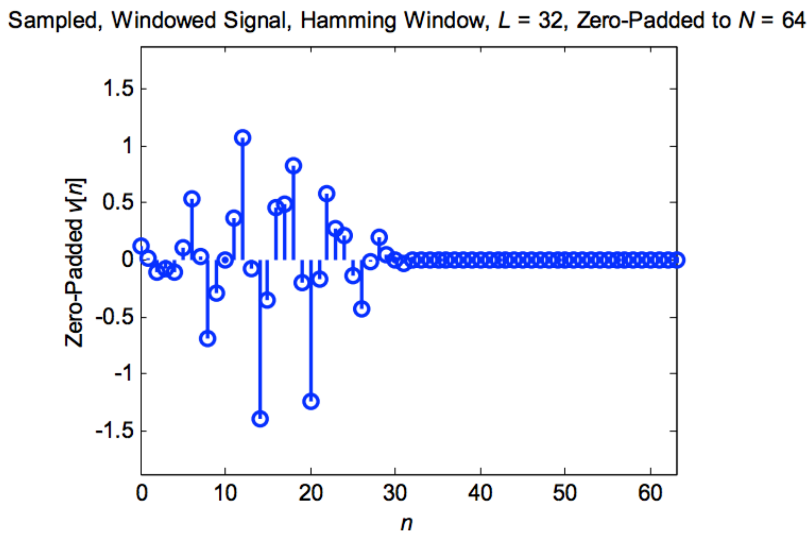
Frequency Analysis with DFT

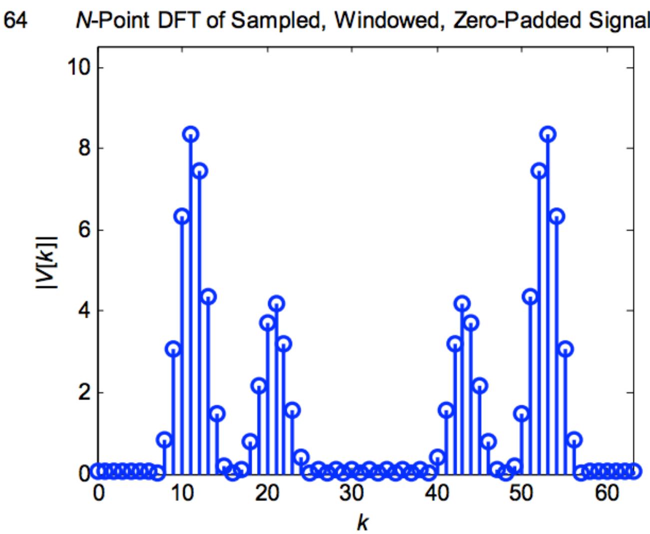
- Hamming window, L = N = 32

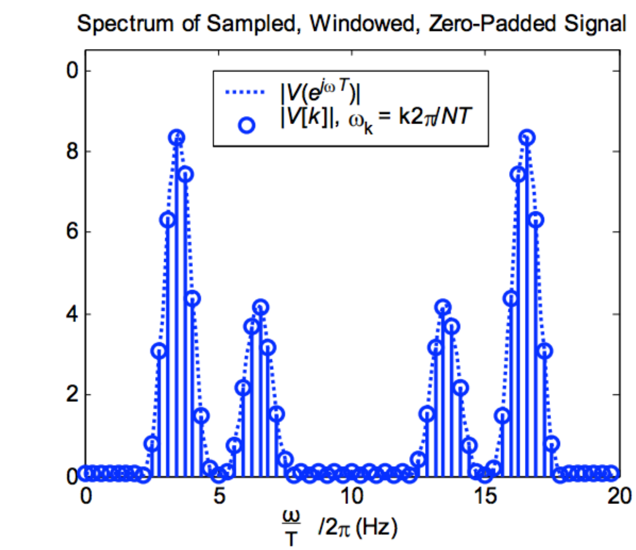


Frequency Analysis with DFT

- Hamming window, L = 32, Zero-padded to N = 64







Frequency Analysis with DFT

- Length of window determines spectral resolution
- Type of window determines side-lobe amplitude/main-lobe width (spectral leakage/spreading)
  - Some windows have better tradeoff between resolution and side-lobe height
- Zero-padding approximates the DTFT better (spectral sampling). Does not introduce new information!

Admin

- Finish Lab 8 by Monday
- Lab 9 on Monday
  - More digital filters