ESE534: Computer Organization

Day 3: January 25, 2010
Arithmetic

Work preclass exercise

Last Time

- Boolean logic ⇒ computing **any** finite function
- Saw gates…and a few properties of logic

Today

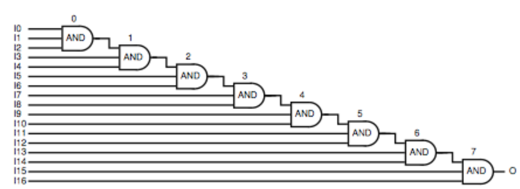
- Addition
  - organization
  - design space
  - parallel prefix

Why?

- Start getting a handle on
  - Complexity
    - Area and time
    - Area-time tradeoffs
  - Parallelism
  - Regularity
- Arithmetic underlies much computation
  - grounds out complexity

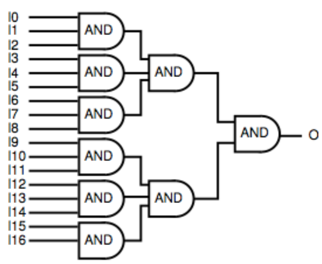
Preclass

Circuit 1

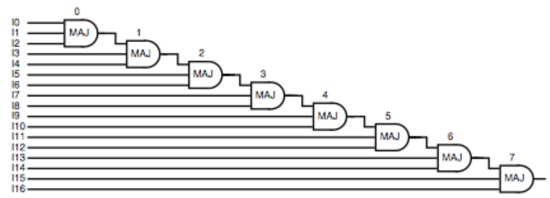


- Can the delay be reduced?
- How?
- To what?

Tree Reduce AND

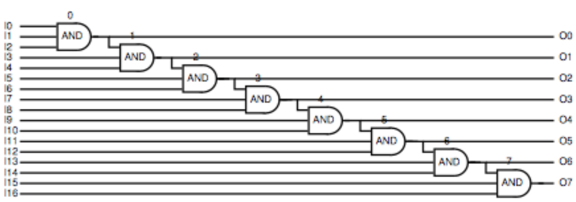


Circuit 2



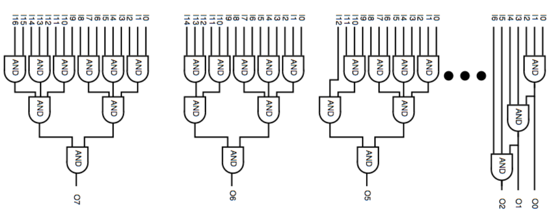
- Can the delay be reduced?

Circuit 3



- Can the delay be reduced?

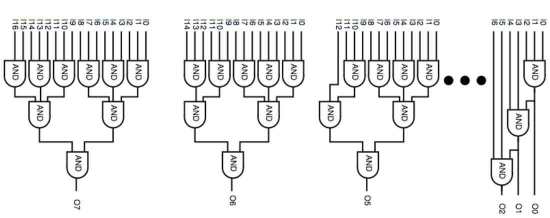
Brute Force Multi-Output AND



- How big?
- ~38 here

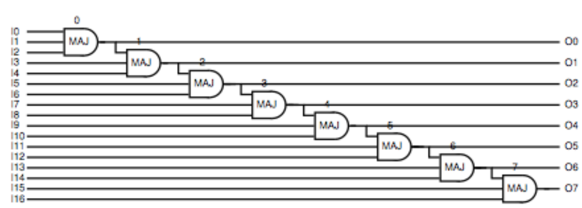
… in general about $N^2/2$

Brute Force Multi-Output AND



- Can we do better?

Circuit 4



- Can the delay be reduced?

Addition

Example: Bit Level Addition

- Addition
  - Base 2 example

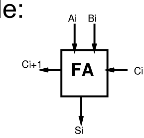
C: 11011010000
A: 01101101010
B: 01100101100
S: 11010010110

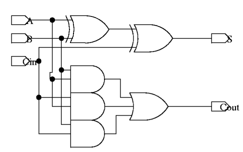
Addition Base 2

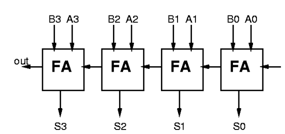
- $A = a_{n-1}*2^{(n-1)}+a_{n-2}*2^{(n-2)}+\ldots a_1*2^1+ a_0*2^0$
  $= \Sigma (a_i*2^i)$
- S=A+B
- What is the function for s_i … $carry_i$?
- s_i= $carry_i$ **xor** a_i **xor** b_i
- $carry_i = (a_{i-1} + b_{i-1} + carry_{i-1}) \geq 2$
  $= a_{i-1}*b_{i-1}+a_{i-1}*carry_{i-1} + b_{i-1}*carry_{i-1}$
  $= MAJ(a_{i-1},b_{i-1},carry_{i-1})$

Ripple Carry Addition

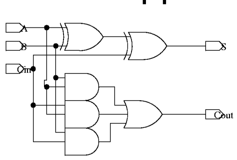
- Shown operation of each bit
- Often convenient to define logic for each bit, then assemble:
  - **bit slice**

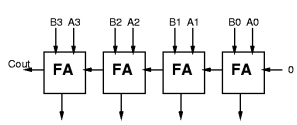






Ripple Carry Analysis



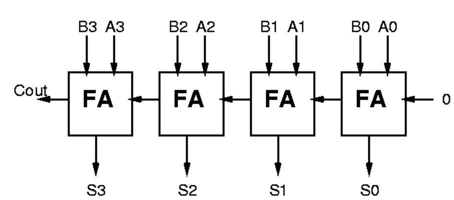


What is area and delay for N-bit RA adder? [unit delay gates]

- Area: O(N) [6n]
- Delay: O(N) [2n]

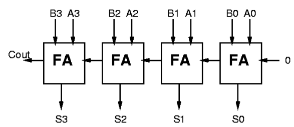
Can we do better?

- Lower delay?



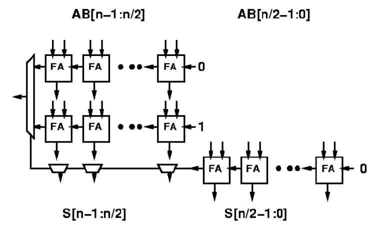
Important Observation

- Do we have to wait for the carry to show up to begin doing useful work?
  - We do have to know the carry to get the right answer.
  - How many values can the carry take on?



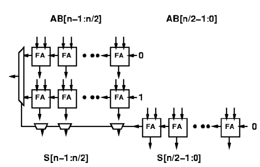
Idea

- Compute both possible values and select correct result when we know the answer



Preliminary Analysis

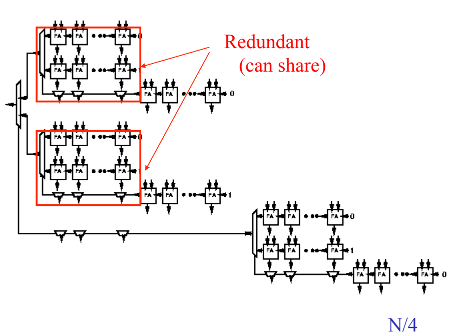
- Delay(RA) --Delay Ripple Adder
- Delay(RA(n)) = k*n [k=2 this example]
- Delay(RA(n)) = 2*(k*n/2)=2*DRA(n/2)
- Delay(P2A) -- Delay Predictive Adder
- Delay(P2A)=DRA(n/2)+D(mux2)
- …almost half the delay!



Recurse

- If something works once, do it again.
- Use the predictive adder to implement the first half of the addition

Recurse



Recurse

- If something works once, do it again.
- Use the predictive adder to implement the first half of the addition
- Delay(P4A(n))=
  Delay(RA(n/4)) + D(mux2) + D(mux2)
- Delay(P4A(n))=Delay(RA(n/4))+2*D(mux2)

Recurse

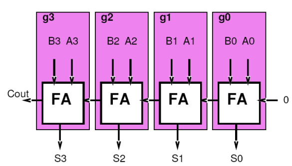
- By know we realize we've been using the wrong recursion
  - should be using the Predictive Adder in the recursion
- Delay(PA(n)) = Delay(PA(n/2)) + D(mux2)
- Every time cut in half…?
- How many times cut in half?
- Delay(PA(n))=$\log_2(n)$*D(mux2)+C
  - C = Delay(PA(1))
    - if use FA for PA(1), then C=2

Another Way

(Parallel Prefix)

CLA

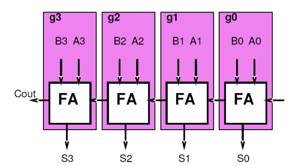
- Think about each adder bit as a computing a function on the carry in
  - C[i]=g(c[i-1])
  - Particular function f will depend on a[i], b[i]
  - g=f(a,b)



Maybe better to show this as a specialization:
g(c) = carry(a=0,b=0,c)
= carry(a=1,b=0,c)
= carry(a=0,b=1,c)
= carry(a=1,b=1,c)

Functions

- What functions can g(c[i-1]) be?
  - g(x)=1
    - a[i]=b[i]=1
  - g(x)=x
    - a[i] xor b[i]=1
  - g(x)=0
    - a[i]=b[i]=0

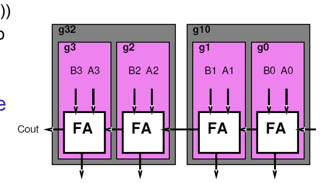


Functions

- What functions can g(c[i-1]) be?
  - g(x)=1 Generate
    - a[i]=b[i]=1
  - g(x)=x Propagate
    - a[i] xor b[i]=1
  - g(x)=0 Squash
    - a[i]=b[i]=0

Combining

- Want to combine functions
  - Compute c[i]=$g_i(g_{i-1}$(c[i-2]))
  - Compute compose of two functions
- What functions will the compose of two of these functions be?
  - Same as before
    - Propagate, generate, squash



Compose Rules
(LSB MSB)

- GG
- GP
- GS
- PG
- PP
- PS
- SG
- SP
- SS

[work on board]

31

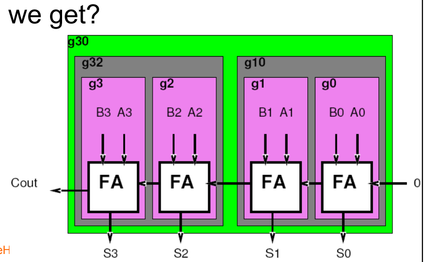
Compose Rules
(LSB MSB)

- GG = G
- GP = G
- GS = S
- PG = G
- PP = P
- PS = S
- SG = G
- SP = S
- SS = S

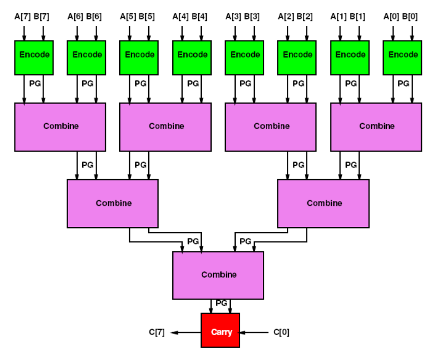
32

Combining

- Do it again…
- Combine g[i-3,i-2] and g[i-1,i]
- What do we get?



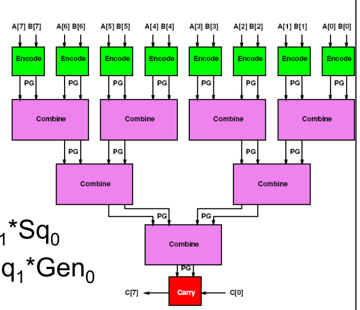
Reduce Tree



34

Reduce Tree

- Sq=/A*/B
- Gen=A*B

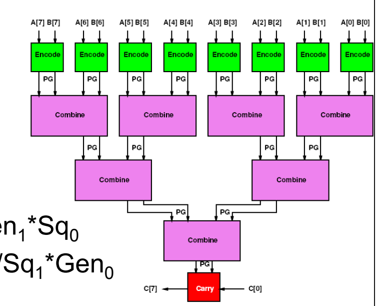


- $Sq_{out}=Sq_1+/Gen_1*Sq_0$
- $Gen_{out}=Gen_1+/Sq_1*Gen_0$

35

Reduce Tree

- Sq=/A*/B
- Gen=A*B



- $Sq_{out}=Sq_1+/Gen_1*Sq_0$
- $Gen_{out}=Gen_1+/Sq_1*Gen_0$

- Delay and Area?

36

Reduce Tree

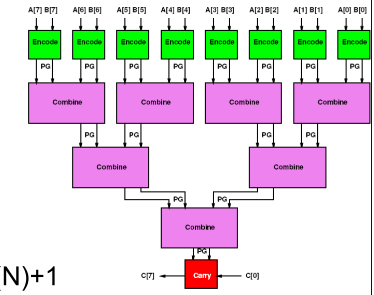
- Sq=/A*/B
- Gen=A*B

- Sq_{out}=Sq_1+/Gen_1*Sq_0
- Gen_{out}=Gen_1+/Sq_1*Gen_0

- A(Encode)=2
- D(Encode)=1
- A(Combine)=4
- D(Combine)=2
- A(Carry)=2
- D(Carry)=1

Reduce Tree: Delay?

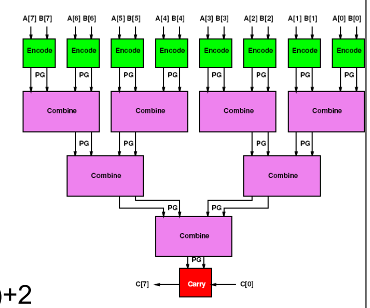
- D(Encode)=1
- D(Combine)=2
- D(Carry)=1



Delay = 1+2$\log_2$(N)+1

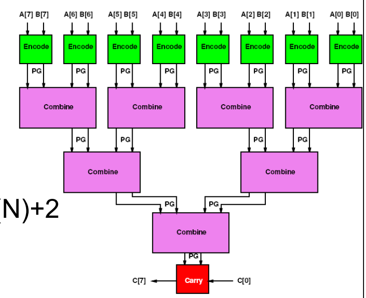
Reduce Tree: Area?

- A(Encode)=2
- A(Combine)=4
- A(Carry)=2



Area= 2N+4(N-1)+2

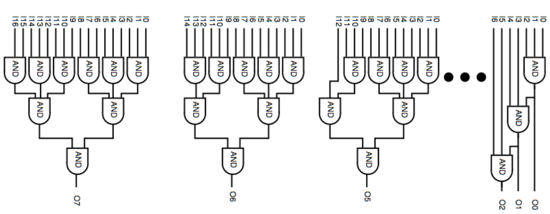
Reduce Tree: Area & Delay



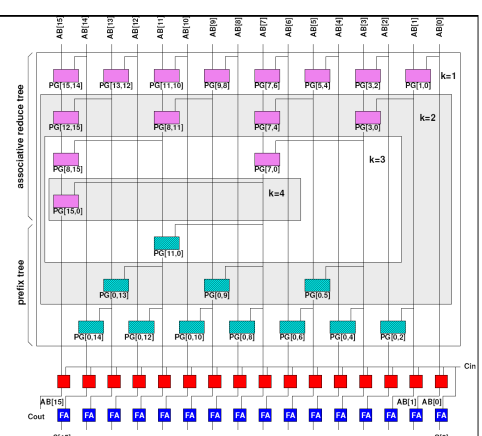
- Area(N) = 6N-2
- Delay(N) = 2$\log_2$(N)+2

Intermediates

- Can we compute intermediates efficiently?

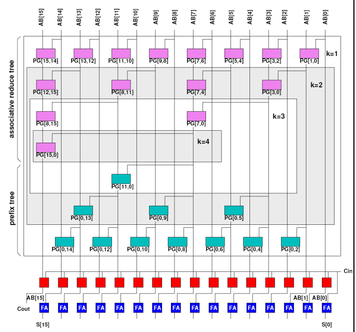


Prefix Tree



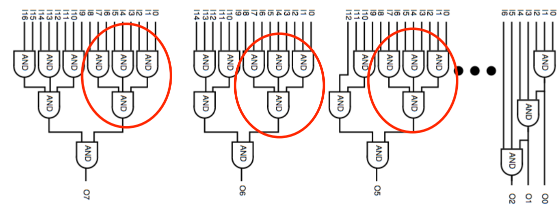
Prefix Tree

- Share terms



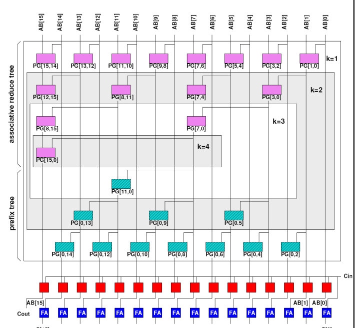
Intermediates

- Share common terms

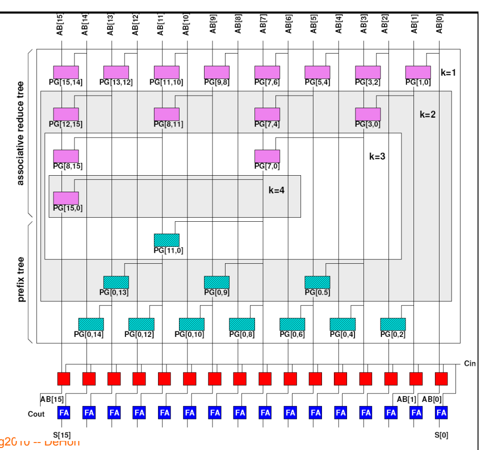


Prefix Tree

- Share terms
- Reduce computes spans
  - 0:3, 4:5
- Reverse tree
  - Combine spans for missing 0:i (i<N)
    - *E.g.* 0:3+4:5→0:5
- Same size as reduce tree

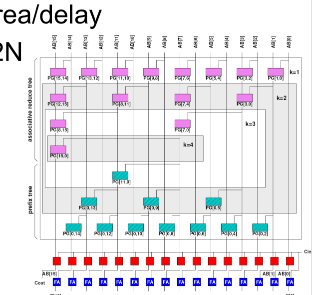


Prefix Tree



Parallel Prefix Area and Delay?

- Roughly twice the area/delay
- Area= 2N+4N+4N+2N
  = 10N
- Delay = $4\log_2(N)+2$



Parallel Prefix

- Important **Pattern**
- Applicable any time operation is *associative*
  - Or can be made assoc. as in MAJ case
- Examples of associative functions?
  - Non-associative?
- Function Composition is always associative

Note: Constants Matter

- Watch the constants
- Asymptotically this Carry-Lookahead Adder (CLA) is great
- For small adders can be smaller with
  - fast ripple carry
  - larger combining than 2-ary tree
  - mix of techniques
- …will depend on the technology primitives and cost functions

Two's Complement

- positive numbers in binary
- negative numbers
  - subtract 1 and invert
  - (or invert and add 1)

Two's Complement

- 2 = 010
- 1 = 001
- 0 = 000
- -1 = 111
- -2 = 110

Addition of Negative Numbers?

- …just works

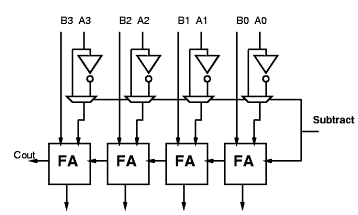
A: 111	A: 110	A: 111	A: 111
B: 001	B: 001	B: 010	B: 110
S: 000	S: 111	S: 001	S: 101

Subtraction

- Negate the subtracted input and use adder
  - which is:
    - invert input and add 1
    - works for both positive and negative input
      - 001 → 110 +1 = 111
      - 111 → 000 +1 = 001
      - 000 → 111 +1 = 000
      - 010 → 101 +1 = 110
      - 110 → 001 +1 = 010

Subtraction (add/sub)

- **Note:** you can use the "unused" carry input at the LSB to perform the "add 1"



Overflow?

A: 111
B: 001
S: 000

A: 110
B: 001
S: 111

A: 111
B: 010
S: 001

A: 111
B: 110
S: 101

A: 001
B: 001
S: 010

A: 011
B: 001
S: 100

A: 111
B: 100
S: 011

- Overflow=(A.s==B.s)*(A.s!=S.s)

Admin

- Office Hours: W2pm

Big Ideas [MSB Ideas]

- Can build arithmetic out of logic

Big Ideas [MSB-1 Ideas]

- Associativity
- Parallel Prefix
- Can perform addition
  - in log time
  - with linear area