OPINION FORMATION IN ISING NETWORKS

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AN ISING MODEL OF INTERACTIONS

A fully-connected network of $n$ vertices:

- **Symmetric interactions** $\{w_{ij}\}$: $w_{ij} = w_{ji}$, $w_{ii} \geq 0$.
- **State (opinions)** $x = (x_1, ..., x_n)$ in $\{-1, +1\}^n$.
- **Update sums**: $S_i(x) = \sum_j w_{ij}x_j$.

Asynchronous state update $x \mapsto x'$:

- **Given**: initial state $x(0)$, an honest asynchronous update schedule $\{i(k), k \geq 1\}$.

Asynchronous dynamics on state space $\{-1, +1\}^n$:

- **Theorem**: Under any honest update schedule, the system converges to a fixed point $x^*$,
  
  $$x_i' = \text{sgn} S_i'(x) = \text{sgn} \sum_j w_{ij}x_j$$
  
  (some $i'$),

  $$x_i' = x_i$$
  
  (for $i \neq i'$).

\[ \begin{align*}
  x(0) & \mapsto x(1) \mapsto x(2) \mapsto \cdots \\
  x_i^* &= \text{sgn} S_i(x^*) \quad (1 \leq i \leq n).
\end{align*} \]
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- **Theorem**: Under any honest update schedule, the system converges to a fixed point $x^*$,

$$x_i' = \text{sgn} S_i'(x) = \text{sgn} \sum_j w_{i'j}x_j \quad \text{(some } i'),$$

$$x_i' = x_i \quad \text{(for } i \neq i').$$

$$x(0) \mapsto x(1) \mapsto x(2) \mapsto \cdots$$

$$x_i^* = \text{sgn} S_i(x^*) \quad (1 \leq i \leq n).$$

**Criterion**: a state $x$ is a fixed point if, and only if, $x_iS_i(x) > 0$ for each $i = 1, \ldots, n$. 

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The classical Ising model: the interactions \( \{w_{ij}, i < j\} \) are i.i.d., \( \mathcal{N}(0,1) \) random variables.

**Theorem:** The expected number of fixed points is exponential in \( n \) (approximately \( 1.05 \times 2^{0.29n} \)).

A new question: what kind of stochastic specification for the interactions permits more control over the fixed point structure?

\[
S_i(x) = \sum_j w_{ij} x_j.
\]

\[
x_i^* = \text{sgn} S_i(x^*) \quad (1 \leq i \leq n).
\]
A two party system: partition the vertices \{1, ..., n\} into two parties \{G_1, G_2\} (generalisation: m parties \{G_1, ..., G_m\}):

- Intra-party interactions exhibit a positive bias.
- Inter-party interactions are neutral or exhibit a negative bias.

\[ \tilde{w}_{ij} = \begin{cases} w_{ij} & \text{if} \ (i, j) \in G_1 \times G_1 \text{ or } G_2 \times G_2, \\ -w_{ij} & \text{if} \ (i, j) \in G_1 \times G_2. \end{cases} \]

- An exchangeable family of random variables with a positive bias: \{\tilde{w}_{ij}, i < j \}.
A SINGLE PARTY ISOMETRY

\[
\begin{align*}
\{w'_{ij}, i \leq j\} & \\
\begin{array}{ll}
w'_{ij} & = \begin{cases} 
  w_{ij} & \text{if } i \neq k \text{ and } j \neq k, \\
  w_{kk} & \text{if } i = k \text{ and } j = k, \\
  -w_{ij} & \text{if either } i = k \text{ or } j = k.
\end{cases}
\end{array} & \\
x' \leftrightarrow x
\end{align*}
\]

\[
x'_i = \begin{cases} 
  x_i & \text{if } i \neq k, \\
  -x_i & \text{if } i = k.
\end{cases}
\]

\[
x'(0) \mapsto x'(1) \mapsto x'(2) \mapsto \cdots & \quad x(0) \mapsto x(1) \mapsto x(2) \mapsto \cdots
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The isomorphism preserves dynamics and probabilities.
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Relabel: we may consider an equivalent single party system characterised by an exchangeable system, \( \{w_{ij}, i < j\} \), of random variables with positive bias.

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A RANDOM INTERACTIONS MODEL

\[ S_i(x) = \sum_j w_{ij} x_j \]

- The interactions \( \{w_{ij}, i < j\} \) are i.i.d., signed Bernoulli variables with a positive bias \( (w_{ii} = 0) \):
  \[ P\{w_{ij} = +1\} = p > 1/2, \]
  \[ P\{w_{ij} = -1\} = 1 - p < 1/2. \]

- **Fixed points**: let \( P_n(x^*) \) denote the probability that a state \( x^* \) is a fixed point:

  \[ P_n(x^*) = P\{x_1^* S_1(x^*) > 0, \ldots, x_n^* S_n(x^*) > 0\}. \]

- **Attraction region**: say that \( x \) is in the region of attraction of \( x^* \) if, starting with initial state \( x \), the system dynamics eventually settles into the fixed point \( x^* \). Write \( P_n(x, x^*) \) for the probability of this event.
Random interactions with positive bias: 
\[ P\{w_{ij} = +1\} = p > 1/2, \quad P\{w_{ij} = -1\} = 1 - p < 1/2. \]

Random walk: 
\[ S_i(x) = \sum_j w_{ij} x_j \]
An intuitive guess at the structure of the fixed points: $x^+ = (+1, +1, \ldots, +1), \quad x^- = -x^+ = (-1, -1, \ldots, -1)$.

Observation: Fixed points arise in pairs as $S_i(-x) = -S_i(x)$.

Random interactions with positive bias: $P\{w_{ij} = +1\} = p > 1/2,\quad P\{w_{ij} = -1\} = 1 - p < 1/2$.

Random walk: $S_i(x) = \sum_j w_{ij}x_j$
Random interactions with positive bias: $P(w_{ij} = +1) = p > 1/2,$ $P(w_{ij} = -1) = 1 - p < 1/2.$

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Observation: Fixed points arise in pairs as $S_i(-x) = -S_i(x)$.

Theorem: Suppose $\delta > 0$ and

$$p = p_n \geq \frac{1}{2} + \sqrt{\frac{\log(n/\delta)}{2(n - 1)}}.$$  

Then $P_n(x^+) \geq 1 - \delta$. If $p$ is bounded away from $1/2$ then $P_n(x^+) \to 1$ as $n \to \infty$.

Observation: $\frac{1}{n-1} S_i(x^+) = \frac{1}{n-1} \sum_{j \neq i} w_{ij}$ is concentrated at $2p - 1$.  

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**Theorem:** Suppose \( 0 < \alpha < \alpha_0 \) where \( \alpha_0 = \alpha_0(p) \) is an absolute constant determined by \( p \). Then, for every \( \epsilon > 0 \), we have \( P_n(x, x^+) > 1 - \epsilon \), eventually, for every \( x \) in the Hamming ball of radius \( \alpha n \) centred at \( x^+ \).

Observation: Consider iterated monotone dynamics.
Relative size of majority opinion

n = 400

n = 1000

n = 2000
High-level idea: each vertex is characterised by a profile of a few bits representing a past history of opinions (on independent issues). Interaction weights are determined by how similar the vertex profiles are.

\[ \pi_i \quad w_{ij} \quad \pi_j \]

- The nitty gritty: each vertex is labelled with a \( \kappa \)-bit profile, \( \pi_i = (\pi_{i1}, \ldots, \pi_{i\kappa}) \), where \( \{\pi_{il}, 1 \leq l \leq \kappa, 1 \leq i \leq n\} \) are i.i.d., signed Bernoulli variables with a positive bias:

\[ P\{\pi_{il} = +1\} = p > 1/2, \quad P\{\pi_{il} = -1\} = 1 - p < 1/2. \]

- Interaction weights are determined by profile inner products:

\[ w_{ij} = \langle \pi_i, \pi_j \rangle = \sum_{l=1}^{\kappa} \pi_{il} \pi_{jl}. \]
EXTREMES OF PROFILE SIZES

If \( \kappa = 1 \), there are exactly two stochastic equilibria (fixed points).

If \( \kappa = 2 \), there are exactly four stochastic equilibria.

If \( \kappa = \kappa_n \) grows sufficiently fast with respect to \( n \), then \( x^+ \) and \( x^- \) are the only two equilibria.

\[
\kappa_n = \frac{4}{(2p - 1)^2} n^2 \log n + o \left( \frac{n}{\log n} \right)
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The equilibria change from stochastic to deterministic as $\kappa$ increases.

What is the behaviour in the non-asymptotic regime for small $\kappa$?
CHARACTERISATION BY PROFILE CLUSTERS

At equilibrium, vertices in a cluster all have the same opinion.

At equilibrium, clusters \( C_\nu \) and \( C_{2^\kappa-1-\nu} \) have opposed opinions.

Cluster state equilibria: A vector of cluster states \( X = (X_0, X_1, ..., X_{2^\kappa-1}) \) is a fixed point if, and only if,

\[
P\{\pi_{i1} = +1\} = p, \quad P\{\pi_{i1} = -1\} = 1 - p, \quad w_{ij} = \langle \pi_i, \pi_j \rangle = \sum_{l=1}^{\kappa} \pi_{il} \pi_{jl}
\]

\[
\begin{array}{c|cccc}
\text{Cluster } C_\nu & \Pi_0 & \Pi_1 & \ldots & \Pi_{2^\kappa-1} \\
\text{Occupancy } c_\nu & c_0 & c_1 & \ldots & c_{2^\kappa-1} \\
\end{array}
\]

\[
X_\nu \sum_{\mu=0}^{2^\kappa-1} \langle \Pi_\nu, \Pi_\mu \rangle c_\mu X_\mu > 0 \quad (0 \leq \nu \leq 2^\kappa - 1).
\]
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$$X_\nu \sum_{\mu=0}^{2^\kappa-1} \langle \Pi_\nu, \Pi_\mu \rangle c_\mu X_\mu > 0 \quad (0 \leq \nu \leq 2^\kappa - 1).$$

Goldilocks theorem: The number of fixed points in the profile-model is bounded above by $2^{2^\kappa-1}$. 

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CONCENTRATION AT MEAN CLUSTER SIZES

\[ P\{\pi_{il} = +1\} = p, \quad P\{\pi_{il} = -1\} = 1 - p, \quad w_{ij} = \langle \pi_i, \pi_j \rangle = \sum_{l=1}^{\kappa} \pi_{il} \pi_{jl} \]

\[ \mu_\nu := p \frac{|\Pi_\nu| + \kappa}{2} (1 - p) \frac{|\Pi_\nu| - \kappa}{2} \quad (0 \leq \nu \leq 2^\kappa - 1) \]

**Theorem:** The fractional cluster sizes \( \frac{1}{n} c_\nu \) (\( 0 \leq \nu \leq 2^\kappa - 1 \)) are all simultaneously concentrated at their respective mean values \( \mu_\nu \).

**Theorem:** Except for a finite set of \( p \) values, with probability tending asymptotically to one, the number of fixed points is the same as that of a deterministic profile-based network with cluster sizes fixed at their expected values.
Expected number of equilibria

\[ \kappa = 3 \]  
\[ \kappa = 4 \]  
\[ \kappa = 5 \]
A FEW CONCLUDING REMARKS

Random interactions model:

- It is plausible that the only equilibria in the random interactions model are the polarised equilibria \( x^+ \) and \( x^- \). But we haven't shown this. One way to handle the technical difficulties would be to craft a central limit theorem where the number of dimensions increases with \( n \).

- The size of the domain of attraction should be expandable to near \( n/2 \). The monotone dynamics invoked in the proof may be too limiting.

Profile-based model:

- We haven't made much progress on improving the estimates for the number of fixed points, especially for \( p \) near 1/2. The number empirically grows nicely as \( \kappa \) increases but the pigeon-hole upper bound is patently too generous. Improvements require a characterisation of the zeros of certain polynomials vaguely connected to the Chebyshev clan. The problem is that the behaviour is (surprisingly) not monotone in \( p \).

- We have made little progress on attraction basin estimates.

- We have scatter-shot results on many party extensions but as yet no systematic theory.