Practical Aspects of the Calculation of DP Bounds for the QAP

- Dual Procedure (DP) provides lower bounds for the branch & bound solution of the Quadratic Assignment Problem (QAP).

- The DP is an iterative procedure for a near optimal dual solution to a continuous relaxation of a linearization of the QAP. Thus,

- The DP produces lower bounds that are as tight as IPLP bounds.

- Furthermore, these bounds are produced efficiently and cheaply.

- The relaxed dual problem while allowing continuous variables requires an integer implementation to avoid error.

- The dual problem variables are scaled upward to approximate continuous variables while maintaining integer accuracy.

- This paper reports on the variation in bound quality and branch and bound runtime with scaling.
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The Quadratic Assignment Problem

Given $N^4$ costs $C_{ijkn} \geq 0$ ($i, j, k, n = 1, 2, \ldots, N$) determine

$$U = [u_{ab}]$$  \hspace{1cm} (1)

called an "assignment", so as to minimize a cost function,

$$R(U) = \sum_{ijkn} C_{ijkn} \cdot u_{ij} \cdot u_{kn}$$  \hspace{1cm} (2)

subject to the following constraints on $U$:

$$u_{ij} = 0,1 \quad (i,j=1,2,\ldots,N), \hspace{1cm} (3)$$

$$\sum_{i=1}^{N} u_{ij} = 1 \quad (j = 1,2,\ldots,N), \hspace{1cm} (4)$$

$$\sum_{j=1}^{N} u_{ij} = 1 \quad (i = 1,2,\ldots,N) \hspace{1cm} (5)$$
Kronecker Product Representation

• Lawler introduced the concept of an $N^2$ by $N^2$ solution (or assignment) matrix $V$ which is a Kronecker product of the $N \times N$ assignment matrix $U$ with itself.

That is,

$$V = U \times U = \begin{pmatrix}
  u_{11}U & u_{12}U & \cdots & u_{1N}U \\
  u_{21}U & u_{22}U & \cdots & u_{2N}U \\
  \vdots & \vdots & \ddots & \vdots \\
  u_{N1}U & u_{N2}U & \cdots & u_{NN}U
\end{pmatrix} = [v_{ijkn}] \quad (6)$$

where $v_{ijkn} = u_{ij} u_{kn} = u_{kn} u_{ij} = v_{knij}$ (i≠k and j≠n) \quad (7)

• (7) says if an element $v_{ijkn}$ is equal to 1 then it has a “complementary element” $v_{knij}$ that is also equal to 1
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Example (N=3):

\[ V = \begin{bmatrix}
\{0\} & 0 & 0 & \{0\} & 0 & 0 & \{1\} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\{1\} & 0 & 0 & \{0\} & 0 & 0 & \{0\} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\{0\} & 0 & 0 & \{1\} & 0 & 0 & \{0\}
\end{bmatrix} \]

\[ U = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix} \]

Certain elements of \( V \) are always zero, specifically

\[ v_{ibkb} = 0 \text{ if } i \neq k \]

\[ v_{bjbn} = 0 \text{ if } j \neq n \]
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**Corresponding Cost Matrix:**

\[
C = \begin{pmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
C_{1111} & * & * & * & C_{1212} & * & * & * & C_{1313} \\
* & C_{1122} & C_{1123} & C_{1221} & * & C_{1223} & C_{1321} & C_{1322} & * \\
* & C_{1132} & C_{1133} & C_{1231} & * & C_{1233} & C_{1331} & C_{1332} & * \\
* & C_{2112} & C_{2113} & C_{2211} & * & C_{2213} & C_{2311} & C_{2312} & * \\
* & C_{2121} & * & * & C_{2222} & * & * & * & C_{2323} \\
* & C_{2132} & C_{2133} & C_{2231} & * & C_{2233} & C_{2331} & C_{2332} & * \\
* & C_{3112} & C_{3113} & C_{3211} & * & C_{3213} & C_{3311} & C_{3312} & * \\
* & C_{3122} & C_{3132} & C_{3221} & * & C_{3223} & C_{3321} & C_{3322} & * \\
C_{3131} & * & * & * & C_{3232} & * & * & * & C_{3333}
\end{pmatrix}
\]
Practical Aspects of the Calculation of DP Bounds for the QAP

- Certain operations may be performed on $C = \{C_{ijkn}\}$ which will change the cost $R(U)$ of assignments $U$ in such a way that all assignment costs are shifted by an identical amount, thus preserving their order with respect to cost. These operations are divided into two classes:

  Class 1: Addition (or subtraction) of a constant to all feasible elements of a submatrix ($C_{ij}$) row or column and the corresponding subtraction (or addition) of this constant from either another row or column of the submatrix or from the submatrix linear cost element.

  Class 2: Addition or subtraction of a constant to all feasible elements of any row or column in matrix $C$. 
Practical Aspects of the Calculation of DP Bounds for the QAP

Dual Procedure similar to Hungarian Method for LAP

• If the operations on $C$ decrease the cost by an amount $R'$ and are performed in a way that keeps the elements of $C$ non-negative, then no assignment cost can become negative and the following relationship holds:

$$R' \leq \min_U R(U)$$

• If furthermore $R'$ can be increased until the equality holds, then the elements of the adjusted cost matrix involved in an optimum assignment would necessarily be zero.
A Reformulation Linearization Technique (RLT) formulation of QAP - called LP (Johnson - 1992, Adams and Johnson - 1994):

LP : Minimize:
\[ \sum_{i,j,k,n \atop i \neq k \atop j \neq n} C_{ijkn} v_{ijkn} + \sum_{i,j} C_{ijij} v_{ijij} \]

Subject to:
\[ \sum_{k \atop k \neq i} v_{ijkn} = v_{ijij} = 0,1 \quad \forall (i, j, n), j \neq n \quad (8) \]
\[ \sum_{n \atop n \neq j} v_{ijkn} = v_{ijij} = 0,1 \quad \forall (i, j, k), i \neq k \quad (9) \]
\[ v_{knij} = v_{ijkn} \quad \forall (i, j, k, n), k < i, j \neq n \quad (10) \]
\[ \sum_{i} v_{ijij} = 1 \quad \forall (j) \quad (11) \]
\[ \sum_{j} v_{ijij} = 1 \quad \forall (i) \quad (12) \]
\[ v_{ijkn} \geq 0 \quad \forall (i, j, k, n), k \neq i, j \neq n \quad (13) \]
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Continuous Relaxation of LP:

CLP1: Minimize:

\[
\sum_{i,j,k,n} \tilde{C}_{ijkn} v_{ijn} + \sum_{i,j} C_{iijj} v_{ijij}
\]

where \( \tilde{C} = C_{ijkn} + C_{knij} \)

Subject to:

\[
0 \leq \sum_{k \leq i} v_{ijn} + \sum_{k > i} v_{knij} = v_{ijij} \leq 1 \quad \forall (i,j,n), j \neq n \tag{14}
\]

\[
0 \leq \sum_{n \neq j} v_{ijn} = v_{ijij} \leq 1 \quad \forall (i,j,k), i > k \tag{15}
\]

\[
0 \leq \sum_{n \neq j} v_{knij} = v_{ijij} \leq 1 \quad \forall (i,j,k), i < k \tag{15a}
\]

\[
\sum_{i} v_{ijij} = 1 \quad \forall (j) \tag{16}
\]

\[
\sum_{j} v_{ijij} = 1 \quad \forall (i) \tag{17}
\]

\[
v_{ijkn} \geq 0 \quad \forall (i,j,k,n), i > k, j \neq n \tag{18}
\]
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Now consider:
A Lagrangian dual, whereby constrains (14), (15) and (15a) are placed into the objective function using multipliers a, b and b' respectively.

LD1: Maximize \( \theta(a, b, b') \) where

\[
\theta(a, b, b') = \min \sum_{i,j,k,n} (\tilde{C}_{ijkn} - a_{ijn} - a_{knj} - b_{ijk} - b'_{kni})v_{ijkn} + \sum_{i,j} (C_{ijij} + a_{ijn} + b_{ijk} + b'_{ijk})v_{ijij}
\]  

(19)

such that \( 0 \leq v_{ijij} \leq 1 \) and subject to (16), (17) and (18).

A further relaxation, whereby constraints (16) and (17) are placed into the objective function using multipliers c and d respectively.

LD2: Maximize \( \theta(a, b, b', c, d) \) where

\[
\theta(a, b, b', c, d) = \min \sum_{i,j,k,n} (\tilde{C}_{ijkn} - a_{ijn} - a_{knj} - b_{ijk} - b'_{kni})v_{ijkn} + \sum_{i,j} (C_{ijij} + a_{ijn} + b_{ijk} + b'_{ijk} - c_i - d_j)v_{ijij}
\]  

(20)

\[+ \sum_i c_i + \sum_j d_j \]

such that \( 0 \leq v_{ijij} \leq 1 \) and subject to (18).
Practical Aspects of the Calculation of DP Bounds for the QAP

Relationship to Other Bounds

- Problem CLP1 is equivalent to problem CLP in Adams and Johnson, “Improved Linear Programming -Based Lower Bounds for the QAP”, DIMACS 1994.

- In fact, the bounds achieved by the DP come within a hair of equaling the linear programming solution of CLP (which is identical to the IPLP bounds of Resende et al. (1995)).

- A&J explain: the following bounds can all be characterized in terms of progressively, more restricted subsets of the dual region encompassed by the CLP:
  1. the reduction scheme of Li et al. (1992)
  2. the row and column reduction strategies of Burkard and Stratmann (1978), Roucairol (1979) and Edwards (1980)
  3. the celebrated Gilmore-Lawler bound

- Thus, the linear programming solution to the CLP (and consequently the DP bounds) must be tighter than the bounds in these three categories.
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DUAL PROCEDURE (RLT) BOUNDS for the QUADRATIC ASSIGNMENT PROBLEM

<table>
<thead>
<tr>
<th></th>
<th>GL BOUND</th>
<th>DP (RLT) BOUND</th>
<th>RLT OPTIMUM</th>
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DP (RTL) Bounds are within at least 99.67% of the RTL Optimal Value
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Effect of Scaling on Dual Procedure Bounds

Effect of Scaling on Nugent15

![Graph showing the effect of scaling on calculated bounds for different mults.]

- Mult by E0
- Mult by E1
- Mult by E2
- Mult by E5
- IPLP

Calculated Bound vs. Iterations x 100
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Dual Procedure in Branch-and-bound (DPB&B)

• The DP attempts to solve the original problem.
• B&B is applied to the revised cost matrix after DP attempt.
• Single facility-location assignment is the first level of partial assignment as well as subsequent levels of partial assignment.
• The DP calculates the bounds as follows.
  • Partial assignment results in a reduced size cost matrix. The DP is applied to this matrix for bounding this partial assignment.
  • This reduced size cost matrix (after DP processing) is stored and utilized as the basis for further partial assignments.
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Scaling Effect on DP Branch-and-Bound Performance

Runtime vs. Scaling

Runtime (secs) - magnified except for Nugent22

Scaling

Instance
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Scaling Effect on DP Branch-and-Bound Performance

Nodes Evaluated vs. Scaling

Nodes Evaluated

Scaling

Instance

<table>
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<tr>
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<th>Nodes Evaluated</th>
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</tr>
<tr>
<td>E-02</td>
<td>1,000,000</td>
</tr>
<tr>
<td>E-03</td>
<td>1,200,000</td>
</tr>
<tr>
<td>E-04</td>
<td>600,000</td>
</tr>
<tr>
<td>E-05</td>
<td>400,000</td>
</tr>
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Nugent15
Nugent20
Nugent22
Hadley16
Hadley18
Practical Aspects of the Calculation of DP Bounds for the QAP

Conclusions

• Scaling is an essential process in the calculation of RLT bounds.

• Achieving near optimum solution of the RLT formulation requires $10^{**5}$ scaling.

• $10^{**2}$ scaling achieves lower bounds almost as good as those achieved by $10^{**5}$ scaling.

• There is a distinct computational advantage to smaller scalings
  1. $10^{**1}$ is good for small problems
  2. $10^{**2}$ is apparently good for the larger problems solved to date.
Locate N facilities among N fixed locations, where we know for each pair of facilities i,k a flow of commodities f(i,k) and for each pair of locations j,n a corresponding distance d(j,n).

Transportation cost between facilities is \( f(i,k) \cdot d(j,n) \), given that i-th facility is assigned to j-th location and k-th facility to n-th location.

The objective is to find an assignment minimizing the sum of all transportation costs.

Find an assignment U to minimize:

\[
R(U) = \sum_{i,j,k,n} f_{ik} \cdot d_{jn} \cdot u_{ij} \cdot u_{kn}
\]
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The Dual Procedure

1. Solve the $N^2$ LAPs (of size N-1 X N-1) in each submatrix $C_{ij}$ of $C$, assuring first that all of the complementary costs are collected in the $C_{ij}$ being solved. The resulting solution value of the LAP is added to the corresponding linear cost $c_{iijj}$.

2. The revised linear costs N x N matrix LAP is solved. The optimum value is accumulated in a reduction constant, $R'$.

3. A test is performed to see if the pattern of solution zeros in the linear cost matrix matches the patterns of zeros in each of the submatrices whose linear costs are zeros, thus constituting a terminating situation.
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4. The new linear cost matrix is investigated for non-zero (positive) elements. If no such elements are found, the procedure terminates with a lower bound, $R'$, albeit no solution. If one or more non-zero linear costs are found, as there usually are, each is distributed among the elements of its submatrix.

5. Each submatrix LAP is re-solved, beginning with those submatrices whose linear costs were zero prior to step 3. Again, prior to its solution, the complementary costs are collected in the matrix being solved.

6. Go to step 2.

Note: While resembling the Gilmore-Lawler and other bounds, the collecting of complementary costs makes the DP unique.
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Example for N=4:

\[
C = \begin{bmatrix}
105 & * & * & * \\
* & 60 & 120 & 30 \\
* & 70 & 140 & 35 \\
* & 20 & 40 & 10 \\
90 & * & * & * \\
* & 50 & 100 & 25 \\
* & 60 & 120 & 30 \\
70 & 140 & 35 & 126 & 63 & 28 & 35 & 42 & * & 56 & 56 & 0 & 105 & * \\
90 & * & 45 & 20 & 25 & 30 & * & 40 & 40 & 0 & 75 & * \\
105 & * & 45 & 20 & 25 & 30 & * & 40 & 40 & 0 & 75 & * \\
105 & * & * & * & 105 & * & * & 105 & * & * & * & 105 \\
105 & * & * & * & 105 & * & * & 105 & * & * & * & 105 \\
105 & * & * & * & 105 & * & * & 105 & * & * & * & 105 \\
105 & * & * & * & 105 & * & * & 105 & * & * & * & 105 \\
105 & * & * & * & 105 & * & * & 105 & * & * & * & 105
\end{bmatrix}
\]

Leaders = \[
\begin{bmatrix}
105 & 105 & 105 & 105 \\
90 & 90 & 90 & 90 \\
105 & 105 & 105 & 105 \\
90 & 90 & 90 & 90
\end{bmatrix}
\]
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Complementary Costs Added and Entered in Uppermost Position:

\[
C = \begin{bmatrix}
105 & * & * & * \\
* & 168 & 150 & 78 \\
* & 196 & 175 & 91 \\
* & 56 & 50 & 26 \\
105 & 0 & 0 & 0 \\
90 & * & * & * \\
* & 140 & 125 & 65 \\
* & 168 & 150 & 78 \\
105 & * & * & * \\
* & 28 & 25 & 13 \\
90 & * & * & * \\
\end{bmatrix}
\]

Leaders = \[
\begin{bmatrix}
105 & 105 & 105 & 105 \\
90 & 90 & 90 & 90 \\
105 & 105 & 105 & 105 \\
90 & 90 & 90 & 90 \\
\end{bmatrix}
\]

Leaders = \[
\begin{bmatrix}
105 & 105 & 105 & 105 \\
90 & 90 & 90 & 90 \\
105 & 105 & 105 & 105 \\
90 & 90 & 90 & 90 \\
\end{bmatrix}
\]
Practical Aspects of the Calculation of DP Bounds for the QAP

Step 1 - Submatrix Elements Reduced

$$C = \begin{bmatrix}
402 & * & * & * & 279 & * & * & 398 & * & * & 257 \\
* & 0 & 0 & 0 & 0 & * & 0 & 45 & 0 & 0 & 0 & 23 & 55 & 0 & * \\
* & 27 & 0 & 0 & 52 & * & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 40 & * \\
* & 0 & 0 & 48 & 0 & * & 0 & 44 & 0 & 32 & * & 0 & 0 & 36 & 0 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & * \\
316 & * & * & * & 201 & * & * & * & * & 295 & * & * & * & 179 \\
* & 0 & 28 & 0 & 38 & * & 0 & 0 & 41 & 0 & * & 0 & 0 & 42 & * \\
* & 15 & 30 & 0 & 68 & * & 0 & 0 & 51 & 0 & * & 8 & 0 & 0 & 58 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
151 & * & * & * & 109 & * & * & * & * & 140 & * & * & * & 109 \\
* & 0 & 0 & 0 & 15 & * & 8 & 0 & 0 & 0 & * & 0 & 0 & 16 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
108 & * & * & * & 90 & * & * & * & * & 104 & * & * & * & 90 \\
\end{bmatrix}$$

Leaders = $$\begin{bmatrix}
402 & 279 & 398 & 257 \\
316 & 201 & 295 & 179 \\
151 & 109 & 140 & 109 \\
108 & 90 & 104 & 90 \\
\end{bmatrix}$$
**Practical Aspects of the Calculation of DP Bounds for the QAP**

**Step 2 - Leader Elements Reduced**

Superleader = 706

\[
\begin{pmatrix}
88 & * & * & * & 0 & * & * & 0 & * & * & 88 & * & * & 0 \\
* & 0 & 0 & 0 & 0 & * & 0 & 45 & 0 & 0 & * & 0 & 23 & 55 & 0 & * \\
* & 27 & 0 & 0 & 52 & * & 0 & 0 & 0 & 0 & * & 8 & 0 & 0 & 40 & * \\
* & 0 & 0 & 48 & 0 & * & 0 & 44 & 0 & 32 & * & 0 & 0 & 36 & 0 & * \\
80 & * & * & * & * & 0 & * & * & * & * & 63 & * & * & * & * & 0 \\
* & 0 & 28 & 0 & 38 & * & 0 & 0 & 41 & 0 & * & 0 & 0 & 42 & * \\
* & 15 & 30 & 0 & 68 & * & 0 & 0 & 51 & 0 & * & 8 & 0 & 0 & 58 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
7 & * & * & * & * & 0 & * & * & * & * & 0 & * & * & * & 22 \\
* & 0 & 0 & 0 & 15 & * & 8 & 0 & 0 & 0 & * & 0 & 0 & 0 & 16 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
\end{pmatrix}
\]

\[
C =
\begin{pmatrix}
88 & 0 & 88 & 0 \\
80 & 0 & 63 & 0 \\
7 & 0 & 0 & 22 \\
0 & 17 & 0 & 39 \\
\end{pmatrix}
\]

Leaders
Practical Aspects of the Calculation of DP Bounds for the QAP

Step 4 - Leader Elements Redistributed

Superleader = 706

\[
\begin{bmatrix}
0 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & 0 \\
* & 30 & 51 & 30 & 27 & * & 21 & 45 & 57 & 30 & * & 30 & 50 & 55 & 21 & * \\
* & 56 & 29 & 37 & 55 & * & 0 & 8 & 32 & 29 & * & 45 & 3 & 0 & 40 & * \\
* & 35 & 29 & 90 & 0 & * & 0 & 57 & 29 & 67 & * & 42 & 0 & 42 & 0 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
0 & * & * & * & * & 0 & * & * & * & * & 0 & * & * & * & * & * & 0 \\
* & 27 & 55 & 34 & 40 & * & 0 & 7 & 64 & 21 & * & 28 & 2 & 0 & 42 & * \\
* & 47 & 56 & 39 & 68 & * & 0 & 13 & 72 & 27 & * & 42 & 0 & 6 & 58 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
0 & * & * & * & * & 0 & * & * & * & * & 0 & * & * & * & * & * & 0 \\
* & 7 & 2 & 15 & 15 & * & 8 & 13 & 0 & 5 & * & 13 & 7 & 12 & 23 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
0 & * & * & * & * & 0 & * & * & * & * & 0 & * & * & * & * & * & 0 \\
\end{bmatrix}
\]

C =

Leaders =

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
Practical Aspects of the Calculation of DP Bounds for the QAP

Step 5 Completed - All Submatrices Now Reduced

Superleader = 706

\[
\begin{bmatrix}
63 & * & * & * & 29 & * & * & 38 & * & * & * & 21 \\
* & 0 & 14 & 0 & 0 & * & 0 & 0 & 13 & 3 & * & 0 \\
* & 7 & 0 & 0 & 29 & * & 0 & 0 & 4 & 0 & * & 19 \\
* & 0 & 0 & 9 & 0 & * & 0 & 11 & 0 & 38 & * & 0 \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 \\
83 & * & * & * & 37 & * & * & * & 73 & * & * & * & 30 \\
* & 0 & 2 & 10 & 12 & * & 0 & 0 & 0 & * & 10 & 0 & 0 & * \\
* & 31 & 0 & 0 & 65 & * & 0 & 0 & 0 & 0 & 0 & 34 & 0 & 0 & 44 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
28 & * & * & * & 35 & * & * & * & 28 & * & * & * & 24 \\
* & 0 & 0 & 12 & * & * & 0 & 0 & 20 & * & 0 & 0 & 0 & 0 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
0 & * & * & * & 6 & * & * & * & 2 & * & * & * & 38 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
63 & 29 & 38 & 21 \\
83 & 37 & 73 & 30 \\
28 & 35 & 28 & 24 \\
0 & 6 & 2 & 38 \\
\end{bmatrix}
\]
Practical Aspects of the Calculation of DP Bounds for the QAP

Second Round - Step 5 (Completed)

Superleader = 792

$$C = \begin{bmatrix}
14 & * & * & * & 1 & * & * & 15 & * & * & * & 1 \\
* & 0 & 10 & 14 & 0 & * & 0 & 0 & 13 & 2 & * & 0 & 1 & 0 & 0 & * \\
* & 21 & 0 & 0 & 30 & * & 0 & 0 & 4 & 0 & * & 5 & 0 & 0 & 11 & * \\
* & 0 & 0 & 22 & 0 & * & 0 & 11 & 0 & 5 & * & 0 & 0 & 16 & 0 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * \\
48 & * & * & * & 13 & * & * & * & 28 & * & * & * & 0 & \end{bmatrix}$$

Leaders = $$\begin{bmatrix}
14 & 1 & 15 & 1 \\
48 & 13 & 28 & 0 \\
0 & 2 & 0 & 13 \\
0 & 33 & 0 & 40 \end{bmatrix}$$
Practical Aspects of the Calculation of DP Bounds for the QAP

Termination of Dual Procedure.

Superleader = 793

$$\begin{pmatrix}
13 & * & * & * & * & \{0\} & * & * & * & 14 & * & * & * & 0 \\
* & 0 & 10 & 14 & 0 & * & 0 & \{0\} & 13 & 2 & * & 0 & 1 & 0 & 0 & * \\
* & 21 & 0 & 0 & 30 & * & \{0\} & 0 & 4 & 0 & 5 & 0 & 0 & 11 & * \\
* & 0 & 0 & 22 & \{0\} & * & 0 & 11 & 0 & 5 & * & 0 & 0 & 16 & 0 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & \{0\} & 0 & * \\
\end{pmatrix}$$

$$C =$$

$$\begin{pmatrix}
48 & * & * & * & 13 & * & * & * & 28 & * & * & * & \{0\} \\
* & 3 & 5 & 0 & 0 & * & 0 & 0 & 0 & 4 & 0 & 0 & 1 & 0 & \{0\} & * \\
* & 38 & 0 & 0 & 52 & * & 0 & 0 & 8 & 0 & 46 & \{0\} & 0 & 44 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & \{0\} & * & 0 & 0 & 0 & * \\
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & \{0\} & 0 & 0 & 0 & * \\
0 & * & * & * & 2 & * & * & * & \{0\} & * & * & * & 13 \\
* & 0 & 0 & 12 & 0 & * & 0 & 0 & \{0\} & 0 & * & 0 & 0 & 6 & 0 & * \\
* & \{0\} & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
* & 0 & 0 & \{0\} & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
* & 0 & \{0\} & 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\
\{0\} & * & * & * & 33 & * & * & * & 0 & * & * & * & 40 \\
\end{pmatrix}$$

$$\text{Leaders} = $$

$$\begin{pmatrix}
13 & \{0\} & 14 & 0 \\
48 & 13 & 28 & \{0\} \\
0 & 2 & \{0\} & 13 \\
\{0\} & 33 & 0 & 40 \\
\end{pmatrix}$$
## Practical Aspects of the Calculation of DP Bounds for the QAP

<table>
<thead>
<tr>
<th>Problem</th>
<th>RR&amp;D IPLP bound</th>
<th>secs</th>
<th>DP 100 Iterations bound</th>
<th>secs</th>
<th>DP 500 Iterations bound</th>
<th>secs</th>
<th>DP 2000 Iterat. bound</th>
<th>secs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nug08</td>
<td>203.5</td>
<td>9.5</td>
<td>201.8</td>
<td>0.8</td>
<td>203.3</td>
<td>4.0</td>
<td>203.3</td>
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<td>754.1</td>
<td>516.2</td>
<td>4.7</td>
<td>521.2</td>
<td>22.2</td>
<td>522.4</td>
<td>82.5</td>
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<td>5203.8</td>
<td>1028.3</td>
<td>17.8</td>
<td>1037.6</td>
<td>85.3</td>
<td>1039.7</td>
<td>325.3</td>
</tr>
<tr>
<td>Nug20</td>
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<td>3611.4</td>
<td>2158.5</td>
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<td>2174.7</td>
<td>382.7</td>
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<tr>
<td>Rou20</td>
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<td>4427.6</td>
<td>636036</td>
<td>62.7</td>
<td>640459</td>
<td>290.2</td>
<td>641663</td>
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<tr>
<td>Kra30a</td>
<td>76002.3</td>
<td>24679.0</td>
<td>74933.6</td>
<td>452.6</td>
<td>75678.1</td>
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<td>75853.4</td>
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<tr>
<td>Kra30b</td>
<td>76751.8</td>
<td>30481.8</td>
<td>75634.6</td>
<td>453.2</td>
<td>76374.6</td>
<td>2136.7</td>
<td>76562.2</td>
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<td>Nug30</td>
<td>4804.4</td>
<td>28819.6</td>
<td>4744.7</td>
<td>411.9</td>
<td>4779.4</td>
<td>1897.1</td>
<td>4788.9</td>
<td>6893.9</td>
</tr>
</tbody>
</table>
Partial assignment (facility 1 to location 2) results in cost matrix:

\[
C = \begin{bmatrix}
\ast \ast \ast \ast & \ast 0 \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast \\
\ast \ast \ast \ast & 0 * 0 45 & \ast \ast \ast \ast & \ast \ast \ast \ast \\
\ast \ast \ast \ast & 52 * 0 0 & \ast \ast \ast \ast & \ast \ast \ast \ast \\
\ast \ast \ast \ast & 0 * 0 44 & \ast \ast \ast \ast & \ast \ast \ast \ast \\
\ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast \\
80 & \ast \ast \ast \ast & \ast \ast \ast \ast & 63 & \ast \ast \ast \ast & 0 \\
\ast \ast \ast \ast & 28 & 0 & \ast \ast \ast \ast & 41 & \ast \ast \ast \ast & 0 & 42 & \ast \\
\ast \ast \ast \ast & 30 & 0 & \ast \ast \ast \ast & 51 & \ast \ast \ast \ast & 0 & 58 & \ast \\
\ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast \\
\ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & 0 & 0 & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast \\
7 & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & 0 & 0 & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & 22 \\
\ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & 0 & 0 & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & 16 \\
\ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast \\
\ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & 0 & 0 & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast \\
\ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast \\
0 & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & \ast \ast \ast \ast & 39
\end{bmatrix}
\]

\[\text{Superleader} = 706\]
Practical Aspects of the Calculation of DP Bounds for the QAP

Reduced size cost matrix for facility 1 to location 2 assignment:

Superleader = 706

\[
\begin{pmatrix}
80 & * & * & | & * & 63 & * & | & * & * & 45 \\
* & 28 & 0 & | & 41 & * & 0 & | & 0 & 42 & *
\\
* & 30 & 0 & | & 51 & * & 8 & | & 0 & 58 & *
\\
59 & * & * & | & * & 0 & * & | & * & * & 22 \\
* & 0 & 0 & | & 0 & * & 0 & | & 0 & 16 & *
\\
0 & * & * & | & * & 0 & * & | & * & * & 83
\end{pmatrix}
\]
Practical Aspects of the Calculation of DP Bounds for the QAP

Bound calculation for facility 1 to location 2 assignment:

Superleader = 706 + 87 = 793

\[
C = \begin{bmatrix}
21 & * & * & 25 & * & * & * & \{0\} \\
* & 0 & 0 & 0 & * & 0 & 0 & \{0\} & * \\
* & 2 & 0 & 2 & * & 0 & \{0\} & 0 & * \\
* & * & * & * & * & * & * & * & *
\end{bmatrix}
\]

\[
59 & * & * & \{0\} & * & * & 22 \\
* & 0 & 0 & \{0\} & * & 0 & 0 & 0 & * \\
* & * & * & * & * & * & * & * & *
\]

\[
\begin{bmatrix}
{0} & * & * & * & 16 & * & * & 83
\end{bmatrix}
\]
Practical Aspects of the Calculation of DP Bounds for the QAP

**Additional Strategies**

- The number of dual iterations at a given partial assignment becomes a dynamic decision based on progress in bounding.
- The search strategy for selecting the next node is a simple depth first strategy.
- Partial assignments are selected from top-left to bottom right.
- When sufficient fathoming permits permanent decisions, these are communicated to the DP by setting costs to a high value.
- In the Nugent test instances we took advantage of the symmetries to eliminate “mirror image” partial assignments.
- The starting upper bound was the best known feasible solution.
Practical Aspects of the Calculation of DP Bounds for the QAP

Improved DPB&B Runtimes

Comparative Runtimes

Size of Nugent Instance

Log of Runtime (Mins)