4. Convex optimization problems

- optimization problem in standard form
- convex optimization problems
- quasiconvex optimization
- linear optimization
- quadratic optimization
- geometric programming
- generalized inequality constraints
- semidefinite programming
- vector optimization

Optimization problem in standard form

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad h_i(x) = 0, \quad i = 1, \ldots, p
\end{align*}
\]

- \( x \in \mathbb{R}^n \) is the optimization variable
- \( f_0 : \mathbb{R}^n \rightarrow \mathbb{R} \) is the objective or cost function
- \( f_i : \mathbb{R}^n \rightarrow \mathbb{R} \), \( i = 1, \ldots, m \), are the inequality constraint functions
- \( h_i : \mathbb{R}^n \rightarrow \mathbb{R} \) are the equality constraint functions

optimal value:

\[
p^* = \inf \{ f_0(x) \mid f_i(x) \leq 0, \ i = 1, \ldots, m, \ h_i(x) = 0, \ i = 1, \ldots, p \}
\]

- \( p^* = \infty \) if problem is infeasible (no \( x \) satisfies the constraints)
- \( p^* = -\infty \) if problem is unbounded below
Optimal and locally optimal points

$x$ is feasible if $x \in \text{dom} f_0$ and it satisfies the constraints

a feasible $x$ is optimal if $f_0(x) = p^*; \ X_{\text{opt}}$ is the set of optimal points

$x$ is locally optimal if there is an $R > 0$ such that $x$ is optimal for

\[
\begin{align*}
\text{minimize} \ (\text{over } z) \quad & f_0(z) \\
\text{subject to} \quad & f_i(z) \leq 0, \ i = 1, \ldots, m, \ h_i(z) = 0, \ i = 1, \ldots, p \\
& \|z - x\|_2 \leq R
\end{align*}
\]

examples (with $n = 1, m = p = 0$)

- $f_0(x) = 1/x, \ \text{dom} f_0 = \mathbb{R}^+: \ p^* = 0$, no optimal point
- $f_0(x) = -\log x, \ \text{dom} f_0 = \mathbb{R}^+: \ p^* = -\infty$
- $f_0(x) = x \log x, \ \text{dom} f_0 = \mathbb{R}^+: \ p^* = -1/e, \ x = 1/e$ is optimal
- $f_0(x) = x^3 - 3x, \ p^* = -\infty$, local optimum at $x = 1$

Implicit constraints

the standard form optimization problem has an implicit constraint

\[ x \in \mathcal{D} = \bigcap_{i=0}^{m} \text{dom} f_i \cap \bigcap_{i=1}^{p} \text{dom} h_i, \]

- we call $\mathcal{D}$ the domain of the problem
- the constraints $f_i(x) \leq 0, \ h_i(x) = 0$ are the explicit constraints
- a problem is unconstrained if it has no explicit constraints ($m = p = 0$)

example:

\[
\begin{align*}
\text{minimize} \quad & f_0(x) = -\sum_{i=1}^{k} \log(b_i - a_i^T x) \\
\end{align*}
\]

is an unconstrained problem with implicit constraints $a_i^T x < b_i$
Feasibility problem

find $x$
subject to $f_i(x) \leq 0$, $i = 1, \ldots, m$
$h_i(x) = 0$, $i = 1, \ldots, p$

can be considered a special case of the general problem with $f_0(x) = 0$:

minimize $0$
subject to $f_i(x) \leq 0$, $i = 1, \ldots, m$
$h_i(x) = 0$, $i = 1, \ldots, p$

- $p^* = 0$ if constraints are feasible; any feasible $x$ is optimal
- $p^* = \infty$ if constraints are infeasible

Convex optimization problem

standard form convex optimization problem

minimize $f_0(x)$
subject to $f_i(x) \leq 0$, $i = 1, \ldots, m$
$a_i^T x = b_i$, $i = 1, \ldots, p$

- $f_0, f_1, \ldots, f_m$ are convex; equality constraints are affine
- problem is quasiconvex if $f_0$ is quasiconvex (and $f_1, \ldots, f_m$ convex)

often written as

minimize $f_0(x)$
subject to $f_i(x) \leq 0$, $i = 1, \ldots, m$
$Ax = b$

important property: feasible set of a convex optimization problem is convex
example

minimize \( f_0(x) = x_1^2 + x_2^2 \)
subject to \( f_1(x) = x_1/(1 + x_2^2) \leq 0 \)
\( h_1(x) = (x_1 + x_2)^2 = 0 \)

- \( f_0 \) is convex; feasible set \( \{(x_1, x_2) \mid x_1 = -x_2 \leq 0\} \) is convex
- not a convex problem (according to our definition): \( f_1 \) is not convex, \( h_1 \) is not affine
- equivalent (but not identical) to the convex problem

minimize \( x_1^2 + x_2^2 \)
subject to \( x_1 \leq 0 \)
\( x_1 + x_2 = 0 \)

Local and global optima

any locally optimal point of a convex problem is (globally) optimal

proof: suppose \( x \) is locally optimal and \( y \) is optimal with \( f_0(y) < f_0(x) \)
\( x \) locally optimal means there is an \( R > 0 \) such that

\( z \) feasible, \( ||z - x||_2 \leq R \implies f_0(z) \geq f_0(x) \)

consider \( z = \theta y + (1 - \theta)x \) with \( \theta = R/(2||y - x||_2) \)

- \( ||y - x||_2 > R \), so \( 0 < \theta < 1/2 \)
- \( z \) is a convex combination of two feasible points, hence also feasible
- \( ||z - x||_2 = R/2 \) and

\( f_0(z) \leq \theta f_0(x) + (1 - \theta)f_0(y) < f_0(x) \)

which contradicts our assumption that \( x \) is locally optimal
Optimality criterion for differentiable $f_0$

$x$ is optimal if and only if it is feasible and

$$\nabla f_0(x)^T (y - x) \geq 0 \quad \text{for all feasible } y$$

if nonzero, $\nabla f_0(x)$ defines a supporting hyperplane to feasible set $X$ at $x$
Equivalent convex problems

two problems are (informally) equivalent if the solution of one is readily obtained from the solution of the other, and vice-versa.

some common transformations that preserve convexity:

• eliminating equality constraints

\[
\begin{align*}
\text{minimize } & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\text{minimize (over } z) & \quad f_0(Fz + x_0) \\
\text{subject to} & \quad f_i(Fz + x_0) \leq 0, \quad i = 1, \ldots, m
\end{align*}
\]

where \( F \) and \( x_0 \) are such that

\[
Ax = b \iff x = Fz + x_0 \text{ for some } z
\]

• introducing equality constraints

\[
\begin{align*}
\text{minimize } & \quad f_0(A_0 x + b_0) \\
\text{subject to} & \quad f_i(A_i x + b_i) \leq 0, \quad i = 1, \ldots, m
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\text{minimize (over } x, y_i) & \quad f_0(y_0) \\
\text{subject to} & \quad f_i(y_i) \leq 0, \quad i = 1, \ldots, m \\
& \quad y_i = A_i x + b_i, \quad i = 0, 1, \ldots, m
\end{align*}
\]

• introducing slack variables for linear inequalities

\[
\begin{align*}
\text{minimize } & \quad f_0(x) \\
\text{subject to} & \quad a_i^T x \leq b_i, \quad i = 1, \ldots, m
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\text{minimize (over } x, s) & \quad f_0(x) \\
\text{subject to} & \quad a_i^T x + s_i = b_i, \quad i = 1, \ldots, m \\
& \quad s_i \geq 0, \quad i = 1, \ldots, m
\end{align*}
\]
• **epigraph form**: standard form convex problem is equivalent to

\[
\begin{align*}
\text{minimize (over } x, t) \quad & t \\
\text{subject to} \quad & f_0(x) - t \leq 0 \\
& f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{align*}
\]

• **minimizing over some variables**

\[
\begin{align*}
\text{minimize} \quad & f_0(x_1, x_2) \\
\text{subject to} \quad & f_i(x_1) \leq 0, \quad i = 1, \ldots, m
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\text{minimize} \quad & \tilde{f}_0(x_1) \\
\text{subject to} \quad & f_i(x_1) \leq 0, \quad i = 1, \ldots, m
\end{align*}
\]

where \( \tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2) \)

---

**Quasiconvex optimization**

\[
\begin{align*}
\text{minimize} \quad & f_0(x) \\
\text{subject to} \quad & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{align*}
\]

with \( f_0 : \mathbb{R}^n \to \mathbb{R} \) quasiconvex, \( f_1, \ldots, f_m \) convex

can have locally optimal points that are not (globally) optimal
convex representation of sublevel sets of $f_0$

if $f_0$ is quasiconvex, there exists a family of functions $\phi_t$ such that:

- $\phi_t(x)$ is convex in $x$ for fixed $t$
- $t$-sublevel set of $f_0$ is 0-sublevel set of $\phi_t$, i.e.,

$$f_0(x) \leq t \iff \phi_t(x) \leq 0$$

does not necessarily require $f_0 \leq 0$.

example

$$f_0(x) = \frac{p(x)}{q(x)}$$

with $p$ convex, $q$ concave, and $p(x) \geq 0$, $q(x) > 0$ on $\text{dom } f_0$

can take $\phi_t(x) = p(x) - tq(x)$:

- for $t \geq 0$, $\phi_t$ convex in $x$
- $p(x)/q(x) \leq t$ if and only if $\phi_t(x) \leq 0$

quasiconvex optimization via convex feasibility problems

$$\phi_t(x) \leq 0, \quad f_i(x) \leq 0, \quad i = 1, \ldots, m, \quad Ax = b \quad (1)$$

- for fixed $t$, a convex feasibility problem in $x$
- if feasible, we can conclude that $t \geq p^*$; if infeasible, $t \leq p^*$

Bisection method for quasiconvex optimization

given $l \leq p^*$, $u \geq p^*$, tolerance $\epsilon > 0$.

repeat

1. $t := (l + u)/2$.
2. Solve the convex feasibility problem (1).
3. if (1) is feasible, $u := t$; else $l := t$.
until $u - l \leq \epsilon$.

requires exactly $\lceil \log_2((u - l)/\epsilon) \rceil$ iterations (where $u$, $l$ are initial values)
Linear program (LP)

minimize \( c^T x + d \)
subject to \( Gx \preceq h \)
\( Ax = b \)

- convex problem with affine objective and constraint functions
- feasible set is a polyhedron

Examples

diet problem: choose quantities \( x_1, \ldots, x_n \) of \( n \) foods
- one unit of food \( j \) costs \( c_j \), contains amount \( a_{ij} \) of nutrient \( i \)
- healthy diet requires nutrient \( i \) in quantity at least \( b_i \)

to find cheapest healthy diet,

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \succeq b, \quad x \succeq 0
\end{align*}
\]

piecewise-linear minimization

\[
\begin{align*}
\text{minimize} & \quad \max_{i=1,\ldots,m}(a_i^T x + b_i) \\
\text{equivalent to an LP}
\end{align*}
\]

\[
\begin{align*}
\text{minimize} & \quad t \\
\text{subject to} & \quad a_i^T x + b_i \leq t, \quad i = 1, \ldots, m
\end{align*}
\]
Chebyshev center of a polyhedron

Chebyshev center of

\[ P = \{ x \mid a_i^T x \leq b_i, \quad i = 1, \ldots, m \} \]

is center of largest inscribed ball

\[ B = \{ x_c + u \mid \|u\|_2 \leq r \} \]

- \( a_i^T x \leq b_i \) for all \( x \in B \) if and only if

\[
\sup \{ a_i^T (x_c + u) \mid \|u\|_2 \leq r \} = a_i^T x_c + r \|a_i\|_2 \leq b_i
\]

- hence, \( x_c, r \) can be determined by solving the LP

\[
\begin{align*}
\text{maximize} & \quad r \\
\text{subject to} & \quad a_i^T x_c + r \|a_i\|_2 \leq b_i, \quad i = 1, \ldots, m
\end{align*}
\]

(Convex optimization problems 4–19)

(Generalized) linear-fractional program

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad Gx \preceq h \\
& \quad Ax = b
\end{align*}
\]

linear-fractional program

\[
f_0(x) = \frac{c^T x + d}{e^T x + f}, \quad \text{dom} f_0(x) = \{ x \mid e^T x + f > 0 \}
\]

- a quasiconvex optimization problem; can be solved by bisection

- also equivalent to the LP (variables \( y, z \))

\[
\begin{align*}
\text{minimize} & \quad c^T y + d z \\
\text{subject to} & \quad Gy \leq h z \\
& \quad Ay = b z \\
& \quad e^T y + f z = 1 \\
& \quad z \geq 0
\end{align*}
\]

(Convex optimization problems 4–20)
generalized linear-fractional program

\[ f_0(x) = \max_{i=1,\ldots,r} \frac{c_i^T x + d_i}{e_i^T x + f_i}, \quad \text{dom } f_0(x) = \{ x \mid e_i^T x + f_i > 0, \ i = 1, \ldots, r \} \]

a quasiconvex optimization problem; can be solved by bisection

**example:** Von Neumann model of a growing economy

\[
\begin{align*}
\text{maximize (over } x, \ x^+) & \quad \min_{i=1,\ldots,n} \frac{x_i^+}{x_i} \\
\text{subject to} & \quad x^+ \geq 0, \ Bx^+ \leq Ax
\end{align*}
\]

- \( x, x^+ \in \mathbb{R}^n \): activity levels of \( n \) sectors, in current and next period
- \((Ax)_i, (Bx^+)_i\): produced, resp. consumed, amounts of good \( i \)
- \( x_i^+/x_i \): growth rate of sector \( i \)

allocate activity to maximize growth rate of slowest growing sector

**Quadratic program (QP)**

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} x^T P x + q^T x + r \\
\text{subject to} & \quad Gx \leq h \\
& \quad Ax = b
\end{align*}
\]

- \( P \in \mathcal{S}_+^n \), so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron
Examples

least-squares

\[
\text{minimize } \| Ax - b \|^2_2
\]

- analytical solution \( x^* = A^\dagger b \) (\( A^\dagger \) is pseudo-inverse)
- can add linear constraints, \( e.g., \ l \leq x \leq u \)

linear program with random cost

\[
\begin{align*}
\text{minimize} & \quad \bar{c}^T x + \gamma x^T \Sigma x = \mathbf{E} c^T x + \gamma \text{var}(c^T x) \\
\text{subject to} & \quad Gx \leq h, \quad Ax = b
\end{align*}
\]

- \( c \) is random vector with mean \( \bar{c} \) and covariance \( \Sigma \)
- hence, \( c^T x \) is random variable with mean \( \bar{c}^T x \) and variance \( x^T \Sigma x \)
- \( \gamma > 0 \) is risk aversion parameter; controls the trade-off between expected cost and variance (risk)

Quadratically constrained quadratic program (QCQP)

\[
\begin{align*}
\text{minimize} & \quad (1/2)x^T P_0 x + q_0^T x + r_0 \\
\text{subject to} & \quad (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

- \( P_i \in \mathbf{S}^n_+ \); objective and constraints are convex quadratic
- if \( P_1, \ldots, P_m \in \mathbf{S}^n_++ \), feasible region is intersection of \( m \) ellipsoids and an affine set
**Second-order cone programming**

\[
\begin{align*}
\text{minimize} & \quad f^T x \\
\text{subject to} & \quad \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m \\
& \quad Fx = g
\end{align*}
\]

\((A_i \in \mathbb{R}^{n_i \times n}, F \in \mathbb{R}^{p \times n})\)

- inequalities are called second-order cone (SOC) constraints:

\[(A_i x + b_i, c_i^T x + d_i) \in \text{second-order cone in } \mathbb{R}^{n_i+1}\]

- for \(n_i = 0\), reduces to an LP; if \(c_i = 0\), reduces to a QCQP

- more general than QCQP and LP

---

**Robust linear programming**

the parameters in optimization problems are often uncertain, *e.g.*, in an LP

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad a_i^T x \leq b_i, \quad i = 1, \ldots, m,
\end{align*}
\]

there can be uncertainty in \(c, a_i, b_i\)

two common approaches to handling uncertainty (in \(a_i\), for simplicity)

- deterministic model: constraints must hold for all \(a_i \in \mathcal{E}_i\)

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad a_i^T x \leq b_i \quad \text{for all } a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m,
\end{align*}
\]

- stochastic model: \(a_i\) is random variable; constraints must hold with probability \(\eta\)

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad \text{prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \ldots, m
\end{align*}
\]
**deterministic approach via SOCP**

- choose an ellipsoid as \( \mathcal{E}_i \):
  \[
  \mathcal{E}_i = \{ \bar{a}_i + P_i u \mid \|u\|_2 \leq 1 \} \quad (\bar{a}_i \in \mathbb{R}^n, \ P_i \in \mathbb{R}^{n \times n})
  \]
  center is \( \bar{a}_i \), semi-axes determined by singular values/vectors of \( P_i \)

- robust LP

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad a_i^T x \leq b_i \quad \forall a_i \in \mathcal{E}_i, \ i = 1, \ldots, m
\end{align*}
\]

is equivalent to the SOCP

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad \bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i, \quad i = 1, \ldots, m
\end{align*}
\]

(follows from \( \sup_{\|u\|_2 \leq 1} (\bar{a}_i + P_i u)^T x = \bar{a}_i^T x + \|P_i^T x\|_2 \))

---

**stochastic approach via SOCP**

- assume \( a_i \) is Gaussian with mean \( \bar{a}_i \), covariance \( \Sigma_i \) (\( a_i \sim \mathcal{N}(\bar{a}_i, \Sigma_i) \))

- \( a_i^T x \) is Gaussian r.v. with mean \( \bar{a}_i^T x \), variance \( x^T \Sigma_i x \); hence

\[
\text{prob}(a_i^T x \leq b_i) = \Phi \left( \frac{b_i - \bar{a}_i^T x}{\|\Sigma_i^{1/2} x\|_2} \right)
\]

where \( \Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-t^2/2} dt \) is CDF of \( \mathcal{N}(0, 1) \)

- robust LP

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad \text{prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \ldots, m,
\end{align*}
\]

with \( \eta \geq 1/2 \), is equivalent to the SOCP

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad \bar{a}_i^T x + \Phi^{-1}(\eta)\|\Sigma_i^{1/2} x\|_2 \leq b_i, \quad i = 1, \ldots, m
\end{align*}
\]
Geometric programming

monomial function
\[ f(x) = cx_1^{a_1}x_2^{a_2} \cdots x_n^{a_n}, \quad \text{dom } f = \mathbb{R}_+^n \]
with \( c > 0 \); exponent \( \alpha_i \) can be any real number

posynomial function: sum of monomials
\[ f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}}x_2^{a_{2k}} \cdots x_n^{a_{nk}}, \quad \text{dom } f = \mathbb{R}_+^n \]

domains of monomials and posynomials can be extended using logarithms

geometric program (GP)

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 1, \quad i = 1, \ldots, m \\
& \quad h_i(x) = 1, \quad i = 1, \ldots, p
\end{align*}
\]
with \( f_i \) posynomial, \( h_i \) monomial

Convex optimization problems

Geometric program in convex form

change variables to \( y_i = \log x_i \), and take logarithm of cost, constraints

- monomial \( f(x) = cx_1^{a_1} \cdots x_n^{a_n} \) transforms to
\[
\log f(e^{y_1}, \ldots, e^{y_n}) = a^T y + b \quad (b = \log c)
\]

- posynomial \( f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}}x_2^{a_{2k}} \cdots x_n^{a_{nk}} \) transforms to
\[
\log f(e^{y_1}, \ldots, e^{y_n}) = \log \left( \sum_{k=1}^{K} e^{a_{1k}y + b_{1k}} \right) \quad (b_k = \log c_k)
\]

- geometric program transforms to convex problem
\[
\begin{align*}
\text{minimize} & \quad \log \left( \sum_{k=1}^{K} \exp(a_{0k}^T y + b_{0k}) \right) \\
\text{subject to} & \quad \log \left( \sum_{k=1}^{K} \exp(a_{ik}^T y + b_{ik}) \right) \leq 0, \quad i = 1, \ldots, m \\
& \quad Gy + d = 0
\end{align*}
\]
Design of cantilever beam

- $N$ segments with unit lengths, rectangular cross-sections of size $w_i \times h_i$
- given vertical force $F$ applied at the right end

**design problem**

minimize total weight

subject to upper & lower bounds on $w_i, h_i$

upper bound & lower bounds on aspect ratios $h_i/w_i$

upper bound on stress in each segment

upper bound on vertical deflection at the end of the beam

variables: $w_i, h_i$ for $i = 1, \ldots, N$

**objective and constraint functions**

- total weight $w_1h_1 + \cdots + w_Nh_N$ is posynomial

- aspect ratio $h_i/w_i$ and inverse aspect ratio $w_i/h_i$ are monomials

- maximum stress in segment $i$ is given by $6iF/(w_i h_i^3)$, a monomial

- the vertical deflection $y_i$ and slope $v_i$ of central axis at the right end of segment $i$ are defined recursively as

$$
v_i = 12(i - 1/2) \frac{F}{Ew_i h_i^3} + v_{i+1}
$$

$$
y_i = 6(i - 1/3) \frac{F}{Ew_i h_i^3} + v_{i+1} + y_{i+1}
$$

for $i = N, N - 1, \ldots, 1$, with $v_{N+1} = y_{N+1} = 0$ ($E$ is Young’s modulus)

$v_i$ and $y_i$ are posynomial functions of $w, h$
formulation as a GP

\[
\begin{align*}
\text{minimize} & \quad w_1 h_1 + \cdots + w_N h_N \\
\text{subject to} & \quad w_{\text{max}}^{-1} w_i \leq 1, \quad w_{\text{min}} w_i^{-1} \leq 1, \quad i = 1, \ldots, N \\
& \quad h_{\text{max}}^{-1} h_i \leq 1, \quad h_{\text{min}} h_i^{-1} \leq 1, \quad i = 1, \ldots, N \\
& \quad S_{\text{max}}^{-1} w_i^{-1} h_i \leq 1, \quad S_{\text{min}} w_i h_i^{-1} \leq 1, \quad i = 1, \ldots, N \\
& \quad 6iF \sigma_{\text{max}}^{-1} w_i^{-1} h_i^{-2} \leq 1, \quad i = 1, \ldots, N \\
& \quad y_{\text{max}} y_1 \leq 1
\end{align*}
\]

note

- we write \( w_{\text{min}} \leq w_i \leq w_{\text{max}} \) and \( h_{\text{min}} \leq h_i \leq h_{\text{max}} \)

\[
\begin{align*}
& w_{\text{min}} / w_i \leq 1, \quad w_i / w_{\text{max}} \leq 1, \quad h_{\text{min}} / h_i \leq 1, \quad h_i / h_{\text{max}} \leq 1
\end{align*}
\]

- we write \( S_{\text{min}} \leq h_i / w_i \leq S_{\text{max}} \) as

\[
\begin{align*}
& S_{\text{min}} w_i / h_i \leq 1, \quad h_i / (w_i S_{\text{max}}) \leq 1
\end{align*}
\]

Convex optimization problems 4-33

Minimizing spectral radius of nonnegative matrix

Perron-Frobenius eigenvalue \( \lambda_{\text{pf}}(A) \)

- exists for (elementwise) positive \( A \in \mathbb{R}^{n \times n} \)
- a real, positive eigenvalue of \( A \), equal to spectral radius \( \max_i |\lambda_i(A)| \)
- determines asymptotic growth (decay) rate of \( A^k \): \( A^k \sim \lambda_{\text{pf}}^k \) as \( k \to \infty \)
- alternative characterization: \( \lambda_{\text{pf}}(A) = \inf\{\lambda \mid Av \preceq \lambda v \text{ for some } v > 0\} \)

minimizing spectral radius of matrix of posynomials

- minimize \( \lambda_{\text{pf}}(A(x)) \), where the elements \( A(x)_{ij} \) are posynomials of \( x \)
- equivalent geometric program:

\[
\begin{align*}
\text{minimize} & \quad \lambda \\
\text{subject to} & \quad \sum_{j=1}^n A(x)_{ij} v_j / (\lambda v_i) \leq 1, \quad i = 1, \ldots, n \\
\end{align*}
\]

variables \( \lambda, v, x \)
**Generalized inequality constraints**

A convex problem with generalized inequality constraints can be formulated as:

\[
\begin{align*}
\text{minimize} \quad & f_0(x) \\
\text{subject to} \quad & f_i(x) \preceq_{K_i} 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{align*}
\]

- \(f_0 : \mathbb{R}^n \to \mathbb{R}\) convex; \(f_i : \mathbb{R}^n \to \mathbb{R}^{k_i}\) \(K_i\)-convex w.r.t. proper cone \(K_i\)
- same properties as standard convex problem (convex feasible set, local optimum is global, etc.)

**Conic form problem**: special case with affine objective and constraints

\[
\begin{align*}
\text{minimize} \quad & c^T x \\
\text{subject to} \quad & Fx + g \preceq_K 0 \\
& Ax = b
\end{align*}
\]

extends linear programming (\(K = \mathbb{R}_+^m\)) to nonpolyhedral cones

---

**Semidefinite program (SDP)**

\[
\begin{align*}
\text{minimize} \quad & c^T x \\
\text{subject to} \quad & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0 \\
& Ax = b
\end{align*}
\]

with \(F_i, G \in \mathbb{S}^k\)

- inequality constraint is called linear matrix inequality (LMI)
- includes problems with multiple LMI constraints: for example,

\[
x_1 \hat{F}_1 + \cdots + x_n \hat{F}_n + \hat{G} \preceq 0, \quad x_1 \tilde{F}_1 + \cdots + x_n \tilde{F}_n + \tilde{G} \preceq 0
\]

is equivalent to single LMI

\[
x_1 \begin{bmatrix} \hat{F}_1 & 0 \\ 0 & \tilde{F}_1 \end{bmatrix} + x_2 \begin{bmatrix} \hat{F}_2 & 0 \\ 0 & \tilde{F}_2 \end{bmatrix} + \cdots + x_n \begin{bmatrix} \hat{F}_n & 0 \\ 0 & \tilde{F}_n \end{bmatrix} + \begin{bmatrix} \hat{G} & 0 \\ 0 & \tilde{G} \end{bmatrix} \preceq 0
\]
LP and SOCP as SDP

LP and equivalent SDP

LP: \[\text{minimize} \quad c^T x\]
subject to \[A x \leq b\]

SDP: \[\text{minimize} \quad c^T x\]
subject to \[\text{diag}(A x - b) \leq 0\]

(note different interpretation of generalized inequality \[\leq\])

SOCP and equivalent SDP

SOCP: \[\text{minimize} \quad f^T x\]
subject to \[\|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m\]

SDP: \[\text{minimize} \quad f^T x\]
subject to \[\begin{bmatrix}(c_i^T x + d_i) I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i\end{bmatrix} \succeq 0, \quad i = 1, \ldots, m\]

Convex optimization problems 4–37

Eigenvalue minimization

\[\text{minimize} \quad \lambda_{\max}(A(x))\]

where \(A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n\) (with given \(A_i \in \mathbb{S}^k\))

equivalent SDP

\[\text{minimize} \quad t\]
subject to \[A(x) \preceq tI\]

- variables \(x \in \mathbb{R}^n, \quad t \in \mathbb{R}\)
- follows from \[\lambda_{\max}(A) \leq t \quad \iff \quad A \preceq tI\]
Matrix norm minimization

minimize $\|A(x)\|_2 = (\lambda_{\text{max}}(A(x)^T A(x)))^{1/2}$

where $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$ (with given $A_i \in S^{p \times q}$)

equivalent SDP

minimize $t$

subject to

$$
\begin{bmatrix}
    tI & A(x) \\
    A(x)^T & tI
\end{bmatrix} \succeq 0
$$

• variables $x \in \mathbb{R}^n$, $t \in \mathbb{R}$
• constraint follows from

$$
\|A\|_2 \leq t \iff A^T A \leq t^2 I, \quad t \geq 0
$$

$$
\iff \begin{bmatrix}
    tI & A \\
    A^T & tI
\end{bmatrix} \succeq 0
$$

Vector optimization

general vector optimization problem

minimize (w.r.t. $K$) $f_0(x)$

subject to $f_i(x) \leq 0, \quad i = 1, \ldots, m$

$h_i(x) \leq 0, \quad i = 1, \ldots, p$

vector objective $f_0 : \mathbb{R}^n \to \mathbb{R}^q$, minimized w.r.t. proper cone $K \in \mathbb{R}^q$

convex vector optimization problem

minimize (w.r.t. $K$) $f_0(x)$

subject to $f_i(x) \leq 0, \quad i = 1, \ldots, m$

$Ax = b$

with $f_0$ $K$-convex, $f_1, \ldots, f_m$ convex
Optimal and Pareto optimal points

set of achievable objective values

\[ \mathcal{O} = \{ f_0(x) \mid x \text{ feasible} \} \]

- feasible \( x \) is **optimal** if \( f_0(x) \) is a minimum value of \( \mathcal{O} \)
- feasible \( x \) is **Pareto optimal** if \( f_0(x) \) is a minimal value of \( \mathcal{O} \)

Multicriterion optimization

vector optimization problem with \( K = \mathbb{R}_+^q \)

\[ f_0(x) = (F_1(x), \ldots, F_q(x)) \]

- \( q \) different objectives \( F_i \); roughly speaking we want all \( F_i \)’s to be small
- feasible \( x^* \) is optimal if

\[
y \text{ feasible } \implies f_0(x^*) \preceq f_0(y)\]

if there exists an optimal point, the objectives are noncompeting
- feasible \( x^{\text{po}} \) is Pareto optimal if

\[
y \text{ feasible}, \quad f_0(y) \preceq f_0(x^{\text{po}}) \implies f_0(x^{\text{po}}) = f_0(y)\]

if there are multiple Pareto optimal values, there is a trade-off between the objectives
Regularized least-squares

multicriterion problem with two objectives

\[ F_1(x) = \|Ax - b\|_2^2, \quad F_2(x) = \|x\|_2^2 \]

- example with \( A \in \mathbb{R}^{100 \times 10} \)
- shaded region is \( \mathcal{O} \)
- heavy line is formed by Pareto optimal points

Risk return trade-off in portfolio optimization

\[
\begin{align*}
\text{minimize (w.r.t. } & R_+^2) \quad (\bar{p}^T x, x^T \Sigma x) \\
\text{subject to} & \quad 1^T x = 1, \quad x \geq 0
\end{align*}
\]

- \( x \in \mathbb{R}^n \) is investment portfolio; \( x_i \) is fraction invested in asset \( i \)
- \( p \in \mathbb{R}^n \) is vector of relative asset price changes; modeled as a random variable with mean \( \bar{p} \), covariance \( \Sigma \)
- \( \bar{p}^T x = E r \) is expected return; \( x^T \Sigma x = \text{var} r \) is return variance

example
Scalarization

to find Pareto optimal points: choose $\lambda \succ_{K^*} 0$ and solve scalar problem

$$\begin{align*}
\text{minimize} & \quad \lambda^T f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad h_i(x) = 0, \quad i = 1, \ldots, p
\end{align*}$$

if $x$ is optimal for scalar problem, then it is Pareto-optimal for vector optimization problem

for convex vector optimization problems, can find (almost) all Pareto optimal points by varying $\lambda \succ_{K^*} 0$

examples

- for multicriterion problem, find Pareto optimal points by minimizing positive weighted sum

  $$\lambda^T f_0(x) = \lambda_1 F_1(x) + \cdots + \lambda_q F_q(x)$$

- regularized least-squares of page 4–43 (with $\lambda = (1, \gamma)$)

  $$\text{minimize} \quad \|Ax - b\|_2^2 + \gamma\|x\|_2^2$$

  for fixed $\gamma > 0$, a least-squares problem

- risk-return trade-off of page 4–44 (with $\lambda = (1, \gamma)$)

  $$\text{minimize} \quad -\bar{p}^T x + \gamma x^T \Sigma x$$

  subject to $\quad 1^T x = 1, \quad x \succeq 0$

  for fixed $\gamma > 0$, a QP