Distributed Receding Horizon Control of Spatially Invariant Systems*

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***Abstract*— We present a rigorous framework for the study of distributed spatially invariant systems with input and state constraints. The proposed approach is based on blending tools from operator theory and Fourier analysis of spatially invariant systems with receding horizon control and Multi Parametric Quadratic Programming (MPQP). Our contributions are two-fold: On one hand, we extend the recent results of Bamieh *et al.* on infinite-horizon optimal control of spatially invariant systems to finite receding horizon control with input and state constraints. On the other hand, our results can be interpreted as an extension of the finite dimensional MPQP-based analysis of receding horizon control to distributed, spatially invariant systems. It is assumed that the dynamics of each subsystem is uncoupled to the others, but the coupling appears through the finite horizon cost function. Specifically, we prove that for spatially invariant systems with constraints, optimal receding horizon controllers are piece-wise affine (represented as a convolution sum plus an offset). Moreover, the kernel of each convolution sum decays exponentially in the spatial domain mirroring the unconstrained infinite-horizon case. Simulation results are provided for a simple example with 5 identical systems coupled in a loop.**

I. INTRODUCTION

Over the past few years, there has been a rapidly growing interest in the systems and control community in the study of coordination and control algorithms for networked dynamic systems. From consensus and agreement problems to formation control, sensing, and coverage, researchers have been interested in control algorithms that are spatially distributed and would achieve a global objective using local interactions [1]–[7]. On the other hand, with advances in real-time optimization-based control, there have been several attempts to develop distributed control algorithms that can handle constraints and can be implemented in real-time. These optimization-based techniques have been used in the context of formation control [8], applications in the manufacturing and process industry where multiple units cooperatively produce a product [9] [10], and large-scale power systems [11]–[15]. In such systems, their dynamics are decoupled, but the coupling occurs through the cost function, which represents some common goal or objective.

In parallel research efforts, there have been some recent developments in the study of infinite dimensional spatially distributed systems with certain symmetries in their structure, such as linear spatially invariant systems. Examples of such systems include a group of identical linear dynamical systems with an interconnection topology of a lattice. Instances of such problems include the optimal control of vehicular platoons, which has been extensively studied (without constraints) in [16]–[18].

In [19], Bamieh *et al.* used spatial Fourier transforms and operator theory to study linear spatially invariant systems with quadratic performance criteria such as the linear quadratic regulator (LQR) (or $\mathcal{H}_2$) and $\mathcal{H}_\infty$. It was shown that such problems can be tackled by solving a parameterized family of finite-dimensional problems in the Fourier domain. Furthermore, it was also shown that the optimal controller has a degree of spatial localization (similar to the underlying system) and can, therefore, be more or less implemented in a distributed fashion. In [20], the authors developed conditions for well-posedness, stability, and performance of spatially interconnected systems whose model is spatially discrete (e.g. over a one-dimensional or two-dimensional lattice) in terms of linear matrix inequalities (LMI). Several other authors, such as [21]–[24], have dealt with the analysis and design of optimal controllers for heterogeneous systems, spatially distributed systems with boundaries, and systems interconnected over an arbitrary graph. Another related work on this subject was reported in [25], where homogeneous interconnected systems are studied using the $\mathcal{Z}$-transform analysis. Furthermore, it is shown that many homogeneous large-scale systems can be reasonably approximated by an infinite number of coupled identical subsystems.

In all of the above results, the underlying system is assumed to be unconstrained. Furthermore, the (often) quadratic performance criteria is evaluated on an infinite time horizon, which makes it possible to characterize the optimal controller in a closed form. To address this void, we build on the recent advances in optimization-based control of constrained systems.

To this end, we consider the receding horizon control (RHC) problem of spatially invariant systems. Similar to [19], the system under study consists of countably many identical units (agents), each being a dynamical system. The coupling of the individual units is assumed to occur in a quadratic cost function of the state and the input variables. Our goal in this paper is to prove that the optimal controller resulting from RHC, which minimizes a quadratic performance criterion subject to discrete-time LTI systems with constraints on inputs and states, has a similar spatial localization property and can, therefore, be implemented in a distributed fashion. The key idea is that the resulting receding horizon controller is piece-wise affine and continuous and can be determined (at least in theory) in an explicit

* This work was supported in parts by ONR YIP-542371, NSF Career ECS-0347285, and ARO MURI W911NF-05-1-0219.

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manner [26]. More importantly, operator-theoretic tools [27] can be used to show that the kernel of the optimal controller decays exponentially in the spatial domain, thereby on one hand extending the results of [19] to constrained systems and on the other hand extending results of [26] to spatially invariant systems.

The paper is organized as follows. We introduce the notation and the basic concepts used throughout the paper in Section II. The class of systems considered in this paper and the RHC formulation is presented in Section III. The main contribution of the paper, i.e. the explicit form of the receding horizon controller and its *spatial locality*, i.e., the exponential decay of the dependence of controller gains on farther systems, is proven in Section IV. Simulation results are included in Section V, in which we demonstrate the exponential decay of spatial coupling in the optimal solution by considering linear systems on the discrete symmetric group of integers modulo 5, i.e., 5 identical linear systems connected in a ring topology. Finally, our concluding remarks are found in Section VI.

II. Fourier Analysis of Translation Invariant Operators

Let $\mathbb{G}$ denote one of the groups $(\mathbb{Z}, +)$, the group of integer numbers, or $(\mathbb{Z}_p\ , \oplus)$, the group of integer numbers modulo p. Denote $\hat{\mathbb{G}}$ to be the dual group corresponding to $\mathbb{G}$. For $\mathbb{G} = \mathbb{Z}$, the dual group is $\hat{\mathbb{G}} = \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$, and $\hat{\mathbb{G}} = \mathbb{Z}_p$ for $\mathbb{G} = \mathbb{Z}_p$. We will refer to $\mathbb{G}$ as the *spatial* domain. Without a loss of generality, all results will be discussed for a one-dimensional spatial domain. The results of this paper can be extended to multi-dimensional spatial domain $\mathbb{G} \times \ldots \times \mathbb{G}$. The Hilbert space ℓ_2 is defined as the set of all maps $u : \mathbb{G} \to \mathbb{R}^n$ for which the quantity $\|u\|_{\ell_2}^2 = \sum_{i\in\mathbb{G}} u_i^* u_i$ is finite. The inner product on ℓ_2 is defined by $\langle u, v\rangle := \sum_{i\in\mathbb{G}} u_i^* v_i$. An operator $\mathcal{P}$ defined from ℓ_2 to ℓ_2 is said to be bounded if

$$\|\mathcal{P}\|_{\ell_2} := \sup_{\substack{u\in\ell_2 \\ u\neq 0}} \frac{\|\mathcal{P}u\|_{\ell_2}}{\|u\|_{\ell_2}} < \infty.$$

An important class of bounded operators on ℓ_2, which is central to our analysis, is the class of translation operators to the left, defined by

$$\mathbf{T}u = \mathbf{T}(\ldots, u_{i-1}, u_i, \ldots) = (\ldots, u_i, u_{i+1}, \ldots) \quad (1)$$

with higher orders defined as $\mathbf{T}^k := \mathbf{T}(\mathbf{T}^{k-1})$ and $\mathbf{T}^0 := \mathrm{I}$ for all $k \in \mathbb{G}$. The identity operator is denoted by I. An operator $\mathcal{P}$ is said to be *translation invariant* if $\mathbf{T}\mathcal{P} = \mathcal{P}\mathbf{T}$. By forming linear combinations of higher order translation operators, we can construct operators of the form

$$\mathcal{P}(\mathbf{T}) = \sum_{k\in\mathbb{G}} P_k \mathbf{T}^k \quad (2)$$

with $P_k \in \mathbb{R}^{n\times n}$. A bounded operator $\mathcal{P}$ is said to be invertible [27] if there exists a bounded operator $\mathcal{Q}$ such that $\mathcal{QP} = \mathcal{PQ} = \mathrm{I}$. Let $\mathcal{P}$ be a bounded linear operator mapping ℓ_2 to itself. Then the adjoint operator of $\mathcal{P}$ is the bounded operator $\mathcal{P}^*$ mapping ℓ_2 into itself [28], such that $\langle \mathcal{P}u, v\rangle = \langle u, \mathcal{P}^* v\rangle$ for all $u, v \in \ell_2$. Operator $\mathcal{P}$ is positive definite if $\langle u, \mathcal{P}u\rangle > 0$ for all nonzero $u \in \ell_2$.

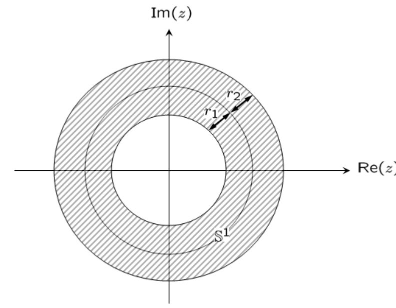

Fig. 1. Analytic continuation to annulus Ω when $\mathbb{G} = \mathbb{Z}$.

In Sections III and IV, we will see that operators in the form (2) will appear in the quadratic performance criteria of our finite horizon cost function as weighting operators as well as kernels in the optimal solution. In the sequel, various properties of operator (2) are discussed. Specifically, we will explore sufficient conditions under which the inverse of (2) decays in spatial domain. Furthermore, we will also assume that all the operators under discussion in this paper can be presented as (2). The key property of a translation invariant operator is that its spatial Fourier transform is a multiplication operator in the Fourier domain, very similar to that of linear time-invariant systems. For $\mathbb{G} = \mathbb{Z}$, the appropriate Fourier transform (bilateral $\mathcal{Z}$-transform) of $u \in \ell_2$ evaluated on $\mathbb{S}^1$ is defined by

$$\hat{u}(\omega) = \sum_{k\in\mathbb{Z}} u_k e^{-ik\omega}. \quad (3)$$

For $\mathbb{G} = \mathbb{Z}_p$, one can define the Fourier transform as

$$\hat{u}_n = \sum_{k\in\mathbb{Z}_p} u_k e^{-i\frac{2\pi nk}{p}}. \quad (4)$$

with $n \in \{0, 1, ..., p-1\}$. For a unique representation of the results, replace $z = e^{-i\omega}$ in (3) and $z = e^{-i\frac{2\pi n}{p}}$ in (4). By a slight abuse of notation, denote $\hat{\mathbb{G}} = \{e^{-i\frac{2\pi n}{p}}\ :\ n \in \mathbb{Z}_p\}$ when $\mathbb{G} = \mathbb{Z}_p$. Therefore, the $\mathcal{Z}$-transform of the operator (2) is

$$\mathrm{P}(z) = \sum_{k\in\mathbb{G}} P_k z^k. \quad (5)$$

evaluated on $\hat{\mathbb{G}}$. We assume that the $\mathcal{Z}$-transform of all operators throughout the paper are continuous functions.

Lemma 1: Operator $\mathcal{P}$ is bounded if and only if

$$\sup_{z\in\hat{\mathbb{G}}} \|\mathrm{P}(z)\|_2 < \infty. \quad (6)$$

Proof: From Plancherel theorem it follows that

$$\|\mathcal{P}\|_{\ell_2} = \sup_{z\in\hat{\mathbb{G}}} \|\mathrm{P}(z)\|_2.$$

has to be bounded. ■

Note that the inverse of a translation invariant operator is also a translation invariant operator.

Corollary 1: Operator $\mathcal{P}$ is invertible if and only if

$$\sup_{z\in\hat{\mathbb{G}}} \|\mathrm{P}(z)^{-1}\|_2 < \infty. \quad (7)$$

Proof: Follows from the definition and lemma 1. ∎

In other words, corollary 1 implies that the operator $\mathcal{P}$ is invertible if and only if $\det(\mathrm{P}(z))$ is nonzero on $\hat{\mathbb{G}}$. The next theorem is the main result of this section.

Theorem 1: Suppose that for operator $\mathcal{P}$ defined by (2) $\det(\mathrm{P}(z)) \neq 0$ on $\hat{\mathbb{G}}$ and $\mathrm{P}(z)$ has analytic continuation to some annulus around $\hat{\mathbb{G}}$

$$\Omega = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2 \ \ , \ \ r_1 < 1 < r_2\}. \tag{8}$$

then operator $\mathcal{P}$ is invertible and the inverse operator $\mathcal{Q}$ can be represented as

$$\mathcal{Q}(\mathbf{T}) = \sum_{k\in\mathbb{G}} Q_k \mathbf{T}^k. \tag{9}$$

Furthermore, for all $k \in \mathbb{G}$

$$\|Q_k\|_2 \leq \ \alpha\ e^{-\beta|k|}. \tag{10}$$

for some $\alpha > 0$ and $0 < \beta < \ln(1+\rho)$ where

$$\rho = \sup\{r : \det(\mathrm{P}(z)) \neq 0 \text{ for all } 1-r \leq |z| \leq 1+r\}.$$

Proof: We refer to [29] for a complete proof. ∎

Remark 1: The decay property of the inverse operator relates closely to the analytic continuation of the operator to some annulus around $\hat{\mathbb{G}}$ (see Figure 1). When $\mathbb{G} = \mathbb{Z}$, this result is similar to that of [30] [19] with the difference that we do not require the additional assumptions on boundedness of $\mathrm{P}(z)$. This is due to the fact that the annulus is a compact set in $\mathbb{C}$, and $\mathrm{P}(z)$ is a continuous function; therefore, the extreme points are attained on the set.

III. Formulation of the Receding Horizon Control Problem

We now formulate the receding horizon control problem for spatially invariant systems such as systems defined on a discrete group $\mathbb{G}$ as depicted in Figures 2 and 3. The development in this section is now a standard procedure in receding horizon control and more or less follows [26]. The model of the system can be written as

$$\begin{aligned} \begin{bmatrix} x_i(t+1) \\ y_i(t) \end{bmatrix} &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_i(t) \\ u_i(t) \end{bmatrix} \\ x_i(0) &= x_0^i. \end{aligned} \tag{11}$$

where index $i \in \mathbb{G}$ is the spatial variable, $x_i \in \mathbb{R}^n$ is the state variables, x_0^i is the initial condition, $u_i \in \mathbb{R}^m$ is the control input, and $y_i \in \mathbb{R}^q$ is the sensor output of agent i, with all matrices having the appropriate dimension.

The objective function is given as a finite horizon quadratic cost. Therefore, the receding horizon control problem can be written as follows

$$\begin{aligned} &\min_{\mathbf{u}^N} \mathfrak{J}(x(0), \mathbf{u}^N; N) && \\ &\text{s.t. Eq. (11)} && ,\ 0 \leq t \leq N \\ &\quad u^i_{\min} \preceq u_i(t) \preceq u^i_{\max} && ,\ 0 \leq t \leq N_c \\ &\quad y^i_{\min} \preceq y_i(t) \preceq y^i_{\max} && ,\ 0 \leq t \leq N_c \\ &\quad u(t) = \mathcal{K}(\mathbf{T})\ x(t) && ,\ N_u \leq t \leq N-1 \\ &\quad i \in \mathbb{G}. && \end{aligned} \tag{12}$$

in which $\mathbf{u}^N = (\ldots, \mathbf{u}^N_{i-1}, \mathbf{u}^N_i, \mathbf{u}^N_{i+1}, \ldots)$ with $\mathbf{u}^N_i = [u_i(0), u_i(1), ..., u_i(N_u-1)]^*$ and the objective function is

$$\begin{aligned} \mathfrak{J}(x(0), \mathbf{u}^N; N) = \langle x(N), \mathcal{P}_N(\mathbf{T})x(N)\rangle + \\ \sum_{t=0}^{N-1} \langle x(t), \mathcal{Q}(\mathbf{T})x(t)\rangle + \langle u(t), \mathcal{R}(\mathbf{T})u(t)\rangle . \end{aligned} \tag{13}$$

It is assumed that $x(0), \mathbf{u}^N, u_{\min}, u_{\max}, y_{\min}, y_{\max} \in \ell_2$ in which for example $x(0) = (\ldots, x_0^{i-1}, x_0^i, x_0^{i+1}, \ldots)$, $u_{\min} = (\ldots, u^{i-1}_{\min}, u^i_{\min}, u^{i+1}_{\min}, \ldots)$. We assume that $\mathcal{Q} = \mathcal{Q}^* \succeq 0$, $\mathcal{P}_N = \mathcal{P}_N^* \succeq 0$, and $\mathcal{R} = \mathcal{R}^* \succ 0$. Also, N_u is the control prediction horizon, N_c the constraint horizon, and N the state prediction horizon. Furthermore, we assume that $N_u \leq N-1$ and $N_c \leq N-1$ (cf. [26]). Note that all operators are in the form of (2). The state feedback kernel $\mathcal{K}$ and the terminal cost $\mathcal{P}_N$ are determined by solving the corresponding infinite horizon LQR problem and the corresponding *Ricatti* equation as in [19]. For a given horizon length, there is a polyhedral set of initial conditions for which feasible trajectories exist, over which the receding horizon controller is stabilizing (cf. [26], [31]–[34] and the references therein). The set of all initial conditions for which an optimal solution of (12) exists is a polyhedral set $\mathbb{X}$. Such a polyhedral set can be characterized as follows

$$\mathbb{X} = \{x \in \ell_2 : \mathcal{A}x \preceq b\}. \tag{14}$$

in which $\mathcal{A}$ is a bounded linear operator. We will assume that $N_u = N_c = N-1$. For $N_u < N-1$, optimization is performed only over N_u control variables, and for the rest of the horizon, we may use the pre-computed feedback gain as in (12) to find control inputs, which in turn will reduce the complexity of the optimization problem. By substituting

$$x_i(t) = A^t x_0^i + \sum_{k=0}^{t-1} A^k B\ u_i(t-k-1)$$

into (11), we have that $[x_i(1), \ldots, x_i(N)]^* = \mathbf{A}\ x_0^i + \mathbf{B}\ \mathbf{u}^N_i$ where the matrix $\mathbf{A}$ and the Toeplitz matrix $\mathbf{B}$ can be determined from system matrices A, B. Hence, (13) can be rewritten as

$$\begin{aligned} \mathfrak{J}(x(0), \mathbf{u}^N; N) = \langle x(0), \mathcal{E}(\mathbf{T})x(0)\rangle \quad + \\ 2\left\langle \mathbf{u}^N, \mathcal{H}(\mathbf{T})x(0)\right\rangle \quad + \quad \left\langle \mathbf{u}^N, \mathcal{P}(\mathbf{T})\mathbf{u}^N\right\rangle . \end{aligned} \tag{15}$$

where

$$\begin{aligned} \mathcal{P}(\mathbf{T}) &= \mathrm{diag}(\mathcal{R}(\mathbf{T}), ..., \mathcal{R}(\mathbf{T})) + \\ &\quad \mathbf{B}^*\ \mathrm{diag}(\mathcal{Q}(\mathbf{T}), ..., \mathcal{Q}(\mathbf{T}), \mathcal{P}_N(\mathbf{T}))\ \mathbf{B} \\ \mathcal{H}(\mathbf{T}) &= \mathbf{B}^*\ \mathrm{diag}(\mathcal{Q}(\mathbf{T}), ..., \mathcal{Q}(\mathbf{T}), \mathcal{P}_N(\mathbf{T}))\ \mathbf{A} \\ \mathcal{E}(\mathbf{T}) &= \mathbf{A}^*\ \mathrm{diag}(\mathcal{Q}(\mathbf{T}), ..., \mathcal{Q}(\mathbf{T}), \mathcal{P}_N(\mathbf{T}))\ \mathbf{A}. \end{aligned} \tag{16}$$

It is clear that $\mathcal{P} = \mathcal{P}^\star \succ 0$. After all substitutions, it can be shown that problem (12) can be written in the following equivalent compact form

$$\begin{aligned} &\min_{\mathbf{u}^N} \frac{1}{2}\left\langle \mathbf{u}^N, \mathcal{P}(\mathbf{T})\mathbf{u}^N\right\rangle + \left\langle \mathbf{u}^N, \mathcal{H}(\mathbf{T})x(0)\right\rangle \\ &\text{s.t.} \quad G\mathbf{u}^N_i \preceq E_i + Fx_0^i \\ &\qquad\ i \in \mathbb{G}. \end{aligned} \tag{17}$$

in which G and H are completely determined from C and

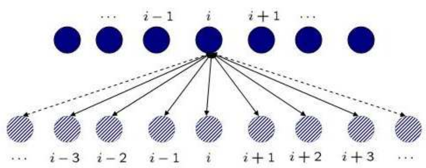

Fig. 2. Topology of the system on group $\mathbb{G} = \mathbb{Z}$. Agents are physically decoupled (lattice in solid color). Coupling between two agents i and k through the cost function is shown by a bidirectional edge from agent i in the top lattice (solid) to the image of agent k in the lower lattice (shaded).

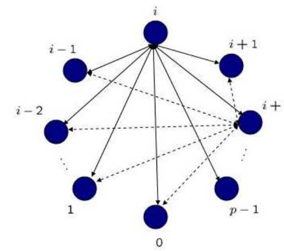

Fig. 3. Topology of the system on group $\mathbb{G} = \mathbb{Z}_p$. Coupling between two agents through cost function is shown by a bidirectional edge between them.

D. The formulation in (17) gives a clear picture of the relationship between the control input variables and initial conditions x_0^i. Problem (17) is a multi-parametric optimization problem on group $\mathbb{G}$ in which $x(0)$ is treated as a vector of parameters. For $\mathbb{G} = \mathbb{Z}$, (17) is an infinite dimensional optimization problem, and in the case of $\mathbb{G} = \mathbb{Z}_p$, it is a finite-dimensional optimization problem. The spatially invariance property of the system induces a symmetry on the formulation (17), which will help to break down the original infinite-dimensional problem into countably many tractable subproblems. In order to apply the results of Section II, we make the following assumption about the $\mathcal{Z}$-transform of operators $\mathcal{R}$ and $\mathcal{Q}$:

Assumption (AC): (Analytic Continuity) $\mathrm{R}(z)$ and $\mathrm{Q}(z)$ have analytic continuation to some annulus $\Omega_0 = \{z \in \mathbb{C} : 1 - r_0 \leq |z| \leq 1 + r_0 \;\;,\;\; r_0 > 0\}$.

The following corollary shows that positive definiteness is sufficient for satisfaction of the first condition of theorem 1:

Corollary 2: Suppose that $\mathcal{P} \succ 0$ as in (16) and assumption (AC) holds. Then operator $\mathcal{P}$ is invertible with bounded inverse $\mathcal{P}^{-1}$ satisfying (9) and (10).

Proof: We only need to prove that $\mathcal{P}$ is invertible. Then corollary 1 and theorem 1 can be applied directly. Assume that $\mathcal{P}$ is not invertible, then the null space of $\mathcal{P}$ is nonempty. Choose $u_0 \neq 0$ to be in the null space of $\mathcal{P}$, and it follows that $\mathcal{P}u_0 = 0$ and $\langle u_0, \mathcal{P}u_0\rangle = \langle u_0, 0\rangle = 0$ which contradicts the fact that $\mathcal{P}$ is a positive definite operator. ■

Remark 2: The key technique for proving the spatial decay of the size of the convolution kernel in [19] is to demonstrate that the *Riccati* solution is an algebraic function, and satisfies the boundedness condition of [30]. In our scenario, this condition for spatial decay reduces to checking whether the operator is positive definite.

IV. Analysis of Receding Horizon Control for Spatially Invariant Systems

As mentioned before, (17) can be solved as a multi-parametric optimization problem, in which parameters are the initial states x_0^i. It can be shown that the set of admissible states $\mathbb{X}$ can be partitioned into countably many partitions [26] over each of which the optimal control law is an affine function of the initial states.

We now show that a similar explicit representation exists for the optimal receding horizon controller in the case of spatially invariant systems. In this scenario, however, the affine representation is in the form of a convolution sum with the controller gains appearing as the kernel. We will show that convolution kernel corresponding to the optimal controllers on each partition, have decay exponentially over the spatial dimension. Before we state the theorem, we recall that [35] if $\mathcal{P} \succ 0$, for every $x(0) \in \mathbb{X}$, there exist a unique optimal solution for (17). The following theorem is the main result of our paper and extends the main result of [26] to the infinite-dimensional case and, more importantly, gives us the spatial decay in the coupling, similar to [19].

Theorem 2: (Main Result) Let $\mathcal{P}$ satisfy the assumptions of corollary 2. For any $i \in \mathbb{G}$, assume that some combination of constraints in (17) are active and the corresponding rows to these active constraints from matrices G, E_i, and F form the full-row rank matrices $\bar{G}_i, \bar{E}_i$, and $\bar{F}_i$. Let $\Pi \subseteq \ell_2$ be the set of all $x(0) \in \ell_2$ with x_0^i as its i^{th} component so that such a combination is active at the optimal solution. Then the optimal solution $\mathbf{u}_i^N$ of (17), as well as the corresponding Lagrange multipliers to index i, are

(a) *affine* maps of $x(0)$ over Π, especially

$$\mathbf{u}_i^N = \sum_{k \in \mathbb{G}} [\Delta]_{ik} \; x_0^k + c_i. \tag{18}$$

(b) *spatially distributed*, in the sense that the coupling decays exponentially in the spatial domain, i.e.

$$\| \, [\Delta]_{ik} \, \|_2 \leq \; \alpha \; e^{-\beta |i-k|}.$$

for some $\alpha > 0$ and $0 < \beta < \hat{\beta}$ in which $\hat{\beta} = \min(\beta_1, \ln(1+\rho), \ln(1+r_0))$ (β_1 is determined explicitly in the proof) and

$\rho = \sup\{r : \det(\mathrm{P}(z)) \neq 0 \text{ for all } 1-r \leq |z| \leq 1+r\}$.

Proof: The Lagrange function for (17) is given by

$$\begin{aligned}\mathfrak{L}(\mathbf{u}^N, \lambda) \;\; &= \;\; \frac{1}{2}\left\langle \mathbf{u}^N, \mathcal{P}(\mathbf{T})\mathbf{u}^N \right\rangle + \left\langle \mathbf{u}^N, \mathcal{H}(\mathbf{T})x(0) \right\rangle \\ &+ \;\; \sum_{i \in \mathbb{G}} \lambda_i^* \; (G\mathbf{u}_i^N - E_i - Fx_0^i).\end{aligned}$$

KKT conditions are given by

$$\mathcal{P}(\mathbf{T})\mathbf{u}^N + \mathcal{H}(\mathbf{T})x(0) + \mathcal{G}^*\lambda = 0 \tag{19}$$

$$\lambda_i^j \; (G\mathbf{u}_i^N - E_i - Fx_0^i)^j = 0 \tag{20}$$

$$j = 1, ..., 2N(m+q)$$

$$\lambda_i \succeq 0 \tag{21}$$

$$G\mathbf{u}_i^N \preceq E_i + Fx_0^i \tag{22}$$

$$i \in \mathbb{G}.$$

in which $\mathcal{G} = \operatorname{diag}_{i\in\mathbb{G}} G$ and λ_i^j or $(.)^j$ represents the j^{th} row. From corollary 2, $\mathcal{P}$ is invertible, therefore from (19) we get

$$\mathbf{u}^N = -\mathcal{P}(\mathbf{T})^{-1}\mathcal{H}(\mathbf{T})x(0) - \mathcal{P}(\mathbf{T})^{-1}\mathcal{G}^*\lambda. \quad (23)$$

According to equation (20) and (21), all Lagrange multipliers corresponding to inactive constraints are zero, and the Lagrange multipliers corresponding to active constraints, stacked in a column vector $\bar{\lambda}_i$, are non-negative numbers. From these assumptions, active constraints result in the following set of equations which allow us to solve them along with (23) for $\bar{\lambda}_i$. By assuming $x(0) \in \Pi$, using (23) it follows that

$$\bar{\mathcal{G}}\mathbf{u}^N - \bar{E} - \bar{\mathcal{F}}x(0) = 0 \quad (24)$$

and that

$$\bar{\mathcal{G}}\mathcal{P}(\mathbf{T})^{-1}\bar{\mathcal{G}}^*\bar{\lambda} + \bar{E} + \left(\mathcal{P}(\mathbf{T})^{-1}\mathcal{H}(\mathbf{T}) + \bar{\mathcal{F}}\right)x(0) = 0. \quad (25)$$

in which $\bar{E}$ and $\bar{\lambda}$ are the concatenation of column vectors $\bar{E}_i$ and $\bar{\lambda}_i$, and $\bar{\mathcal{G}} = \operatorname{diag}_{i\in\mathbb{G}} \bar{G}_i$ and $\bar{\mathcal{F}} = \operatorname{diag}_{i\in\mathbb{G}} \bar{F}_i$. Note that operator $\bar{\mathcal{G}}\mathcal{P}(\mathbf{T})^{-1}\bar{\mathcal{G}}^*$ is not translation invariant. It is invertible, because $\bar{\mathcal{G}}$ is surjective. Let $\mathcal{F}(\mathbf{T}) = \mathcal{P}(\mathbf{T})^{-1}\mathcal{H}(\mathbf{T}) + \bar{\mathcal{F}}$; we have

$$\bar{\lambda} = -\left(\bar{\mathcal{G}}\mathcal{P}(\mathbf{T})^{-1}\bar{\mathcal{G}}^*\right)^{-1}\bar{E} - \left(\bar{\mathcal{G}}\mathcal{P}(\mathbf{T})^{-1}\bar{\mathcal{G}}^*\right)^{-1}\mathcal{F}(\mathbf{T})x(0)$$

and that

$$\mathbf{u}^N = \Delta\ x(0)\ +\ \Gamma\ \bar{E} \quad (26)$$

in which

$$\Delta = \mathcal{P}(\mathbf{T})^{-1}\bar{\mathcal{G}}^*\left(\bar{\mathcal{G}}\mathcal{P}(\mathbf{T})^{-1}\bar{\mathcal{G}}^*\right)^{-1}\mathcal{F}(\mathbf{T}) - \mathcal{P}(\mathbf{T})^{-1}\mathcal{H}(\mathbf{T})$$
$$\Gamma = \mathcal{P}(\mathbf{T})^{-1}\bar{\mathcal{G}}^*\left(\bar{\mathcal{G}}\mathcal{P}(\mathbf{T})^{-1}\bar{\mathcal{G}}^*\right)^{-1}$$

From (26), it appears that the optimal solution $\mathbf{u}^N$ is affine map of $x(0)$. Specifically, if we denote the block elements of Δ by $[\Delta]_{ik}$ for all $i, k \in \mathbb{G}$, (26) can be rewritten as

$$\mathbf{u}_i^N = \sum_{k\in\mathbb{G}} [\Delta]_{ik}\ x_0^k\ +\ c_i \quad (27)$$

for all $i \in \mathbb{G}$. It can be shown that the block elements of the operator $\left(\bar{\mathcal{G}}\mathcal{P}(\mathbf{T})^{-1}\bar{\mathcal{G}}^*\right)^{-1}$ decays exponentially in the spatial domain [29],

$$\|[\left(\bar{\mathcal{G}}\mathcal{P}(\mathbf{T})^{-1}\bar{\mathcal{G}}^*\right)^{-1}]_{ik}\|_2\ \leq\ \alpha_0\ e^{-\beta_0|i-k|} \quad (28)$$

for some $\alpha_0 > 0$ and $0 < \beta_0 < \beta_1$. Also, note that $\mathcal{P}(\mathbf{T})^{-1}$ is analytic on annulus $\{z \in \mathbb{C}\ :\ 1-\rho < |z| < 1-\rho\}$, $\mathcal{H}(\mathbf{T})$ on annulus Ω_0, and $\mathcal{F}(\mathbf{T})$ on annulus $\{z \in \mathbb{C}\ :\ 1 - \min(\rho, r_0) < |z| < 1 + \min(\rho, r_0)\}$. Therefore,

$$\|[\Delta]_{ik}\|\ \leq\ \alpha\ e^{-\beta|i-k|}.$$

for some $\alpha > 0$ and $0 < \beta < \hat{\beta}$ in which $\hat{\beta} = \min(\beta_1, \ln(1+\rho), \ln(1+r_0))$. This completes the proof. ∎

The result of theorem 2 suggests that the coupling between agents diminishes exponentially in the optimal control law. As a result, it would be possible to only take a few neighbors for each agent. In other words, the global optimal solution has an inherently decentralized structure. This suggest that perhaps one can solve the optimization problem (17) locally.

Remark 3: Components of E_i are obtained from $u^i_{\min}, u^i_{\max}, y^i_{\min}$, and $y^i_{\max}$. Equation (27) clearly indicates that not only the optimal solution of agent i depends on the initial state of neighboring agents, also it depends on the upper and lower bounds on the input and output variables of neighboring agents.

V. Simulation Results

In this section, we consider a group of five identical second order systems connected in a loop topology. The objective for these vehicles is to move as a formation with a constant speed and stay in a distance d of each other. We assume that the dynamic of each vehicle can be modeled as a double integrator as follows

$$\begin{bmatrix} x_i(t+1) \\ v_i(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_i(t) \\ v_i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u_i(t)$$

with sampling time $T_s = 1(s)$ and $i \in \mathbb{Z}_5$. Define the following error variables (see [18] and references therein)

$$\begin{aligned} p_i(t) &= x_i(t) - v_d t + id && \text{(Absolute position error)} \\ w_i(t) &= v_i(t) - v_d && \text{(Velocity error)} \\ e_i(t) &= x_i(t) - x_{i-1}(t) + d && \text{(Relative position error)} \end{aligned}$$

By changing variables, we get

$$\begin{aligned} \begin{bmatrix} p_i(t+1) \\ w_i(t+1) \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} p_i(t) \\ w_i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u_i(t) \\ &= A\phi_i(t) + Bu_i(t) && (29) \\ y_i(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix}\phi_i(t) && (30) \end{aligned}$$

At any time step t, the formation objective can be formulated as minimizing the following cost function subject to (29)-(30)

$$\mathfrak{J}(\phi(t), \mathbf{u}^N; N) = \langle\phi(t+N), \mathcal{P}_N(\mathbf{T})\phi(t+N)\rangle + \sum_{k=t}^{t+N-1}\langle\phi(k), \mathcal{Q}(\mathbf{T})\phi(k)\rangle + \langle u(k), \mathcal{R}(\mathbf{T})u(k)\rangle \quad (31)$$

in which weighting operators are defined as

$$\mathcal{Q}(\mathbf{T}) = \sum_{k\in\mathbb{Z}_5} Q_k\mathbf{T}^k \quad , \quad Q_k = \begin{bmatrix} q_p^k & 0 \\ 0 & q_w^k \end{bmatrix}\ ,\ \mathcal{R}(\mathbf{T}) = r$$

$q_p^k > 0$ is weighting on the absolute position error and $q_w^k > 0$ is weighting on the velocity error. Note that all arithmetics are done over $\mathbb{Z}_5$. Terminal costs $\mathcal{P}_N$ is obtained from solving the corresponding LQR problem for cost (31). We also impose some constraints on the outputs as

$$y_{\min} \leq y_i(k) \leq y_{\max}\ \ ,\ \ 0 \leq k \leq N_c$$

Note that (A, B) is controllable and $(\mathrm{Q}(z), A)$ detectable. Multi-Parametric Toolbox (MPT) [36] is used to run simulations for this problem. Simulation parameters are chosen to be $q_p^k = q_w^k = (0.3)^{|k|}$ for $k \in \mathbb{Z}_5$, $r = 1$, $N = N_c = 5$, $y_{\max} = -y_{\min} = 10$. MPT returned 436 partitions for the admissible region and corresponding affine controllers. After careful evaluations, it turns out that number of different optimal control laws are much less than the number of partitions returned by MPT. Figure 4 shows the dependency of the optimal control law of agent number 0 to agents

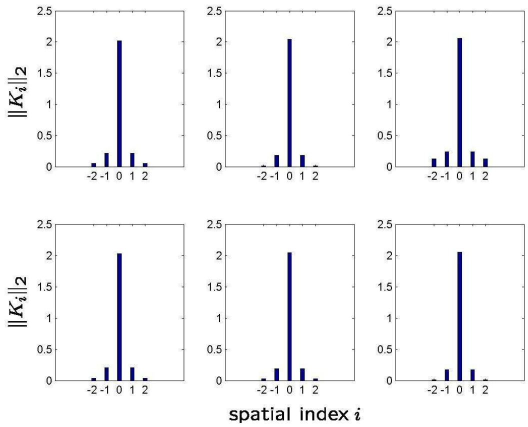

Fig. 4. Decay of feedback gain by spatial index $i \in \mathbb{Z}_5 = \{-2, -1, 0, 1, 2\}$ for six different regions

-1,-2,1, and 2. Six different affine controllers are chosen randomly and norm of the feedback gain is depicted versus the spatial index in figure 4. Although all the agents are coupled together through the cost function, the coupling in the optimal controller gains decays exponentially with the spatial index. As we can see in figure 4, the dependency of the optimal control of agent 0 to agent number -2 and 2 is negligible, and agent 0 needs only to communicate with agents -1 and 1.

VI. Conclusion

We developed a formal study of receding horizon control (RHC) of spatially invariant systems that are defined on discrete groups $\mathbb{Z}$ and $\mathbb{Z}_p$. The systems are assumed to be physically decoupled. However, the coupling between agents occurs through a finite horizon quadratic performance index. We proved that for spatially invariant systems with constraints, optimal receding horizon controllers are piecewise affine functions of the initial condition (the control appears as a convolution sum). Furthermore, using tools from operator theory, we showed that the kernel of each convolution sum decays exponentially in the spatial domain. Thereby extending the previous analysis for infinite horizon optimal controllers to finite horizon constrained receding horizon control. Future work will be focused on analysis and synthesis of receding horizon control of systems defined on arbitrary graphs. We suspect that similar analysis can be extended to any graph, not necessarily lattices or rings.

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