Notes on averaging over acyclic digraphs and discrete coverage control

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***Abstract*— In this paper, we show the relationship between two algorithms and optimization problems that are the subject of recent attention in the networking and control literature. First, we obtain some results on averaging algorithms over acyclic digraphs with fixed and controlled-switching topology. Second, we discuss continuous and discrete coverage control laws. Further, we show how discrete coverage control laws can be cast as averaging algorithms over discrete Voronoi graphs.**

I. INTRODUCTION

Consensus and coverage control are two distinct problems within the recent literature on multiagent coordination and cooperative robotics. Roughly speaking, the objective of the consensus problem is to analyze and design distributed control laws to drive the groups of agents to agree upon certain quantities of interest. On the other hand, the objective of the coverage control problem is to deploy the agents to get optimal sensing performance of an environment of interest.

In the literature, many researchers have used averaging algorithms to solve consensus problems. The spirit of averaging algorithms is to let the state of each agent evolve according to the (weighted) average of the state of its neighbors. Averaging algorithms has been studied both in continuous time [1], [2], [3], [4] and in discrete time [4], [5], [6], [7], [8]. In [1], averaging algorithms are investigated via graph Laplacians under a variety of assumptions, including fixed and switching communication topologies, time delays, and directed and undirected information flow. In [2], a series of consensus protocols are presented, based on the regular averaging algorithms, to drive the agents to agree upon the value of the power mean, see also [3]. A theoretical explanation for the consensus behavior of the Vicsek model [9] is provided in [6], see also the early work in [5], while [4] extends the results of [6] to the case of directed topology. The work [7] adopts a set-valued Lyapunov approach to analyze the convergence properties of averaging algorithms. The works [10], [11] survey the results available for consensus problems using averaging algorithms. In the scenario of coverage control, [12] proposes gradient descent algorithms for optimal coverage, and [13] presents coverage control algorithms for groups of mobile sensors with limited-range interactions. Also, we want to point out that a special kind of directed graphs, namely acyclic digraphs, are presented in the literature to describe the interactions of agents in leader-following formation problems, e.g., [14], [15].

The contributions of this paper are (i) the investigation of the properties of averaging algorithms over acyclic digraphs with fixed and controlled-switching topologies, and (ii) the establishment of the connection between discrete coverage problems and averaging algorithms over acyclic digraphs. Regarding (i), our first contribution is a novel matrix representation of the disagreement function associated with a directed graph. Secondly, we prove that averaging over an fixed acyclic graph drives the agents to an equilibrium determined by the so-called "sinks" of the graph. Finally, we show that averaging over controlled-switching acyclic digraphs also makes the agents converge to the set of equilibria under suitable state-dependent switching signals. Regarding (ii), we present multicenter locational optimization functions in continuous and discrete settings, and discuss distributed coverage control algorithms that optimize them. Finally, we show how discrete coverage control laws over discrete Voronoi graphs can be casted and analyzed as averaging algorithms over a set of controlled-switching acyclic digraphs. In the technical report [16], we provide the proofs for all statements in this paper.

The paper is organized as follows. Section II introduces our novel matrix representation of the disagreement function, and then reviews the current results on consensus problems. We also present convergence results of averaging algorithms over acyclic digraphs with both fixed and controlled-switching topologies. Section III presents locational optimization functions in both continuous and discrete settings, and then discusses appropriate coverage control laws. The main result of the paper shows the relationship between averaging over switching acyclic digraphs and discrete coverage. Finally, we gather our conclusions in Section IV.

II. AVERAGING ALGORITHMS OVER DIGRAPHS

We begin with some basic notation. We let $\mathbb{N}$ and $\mathbb{R}_{\geq 0}$ denote, respectively, the set of natural numbers and the set of non-negative reals. The quadratic form associated with a symmetric matrix $B \in \mathbb{R}^{n\times n}$ is the function defined by $x \mapsto x^T Bx$.

A. Digraphs and disagreement functions

A *weighted directed graph*, in short *digraph*, $\mathcal{G} = (\mathcal{U}, \mathcal{E}, \mathcal{A})$ of order n consists of a *vertex set* $\mathcal{U}$ with n elements, an *edge set* $\mathcal{E} \in 2^{\mathcal{U}\times\mathcal{U}}$ (recall that $2^{\mathcal{U}}$ is the collection of subsets of $\mathcal{U}$), and a *weighted adjacency matrix* $\mathcal{A}$ with nonnegative entries a_{ij}, $i, j \in \{1, \dots, n\}$. For

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simplicity, we take $\mathcal{U} = \{1, \dots, n\}$. For $i, j \in \{1, \dots, n\}$, the entry a_{ij} is positive if and only if the pair (i, j) is an edge of $\mathcal{G}$, i.e., $a_{ij} > 0 \iff (i, j) \in \mathcal{E}$. We also assume $a_{ii} = 0$ for all $i \in \{1, \dots, n\}$ and $a_{ij} = 0$ if $(i, j) \notin \mathcal{E}$, for all $i, j \in \{1, \dots, n\}$ and $i \neq j$. When convenient, we will refer to the adjacency matrix of $\mathcal{G}$ by $\mathcal{A}(\mathcal{G})$.

Let us now review some basic connectivity notions for digraphs. A *directed path* in a digraph is an ordered sequence of vertices such that any two consecutive vertices in the sequence are an edge of the digraph. A *cycle* is a non-trivial directed path that starts and ends at the same vertex. A digraph is *acyclic* if it contains no directed cycles. A node of a digraph is *globally reachable* if it can be reached from any other node by traversing a directed path. A digraph is *strongly connected* if every node is globally reachable.

The *out-degree* and the *in-degree* of node i are defined by $d_{\text{out}}(i) = \sum_{j=1}^{n} a_{ij}$ and $d_{\text{in}}(i) = \sum_{j=1}^{n} a_{ji}$, respectively. The out-degree matrix $D_{\text{out}}(\mathcal{G})$ and the in-degree matrix $D_{\text{in}}(\mathcal{G})$ are the diagonal matrices defined by $(D_{\text{out}}(\mathcal{G}))_{ii} = d_{\text{out}}(i)$ and $(D_{\text{in}}(\mathcal{G}))_{ii} = d_{\text{in}}(i)$, respectively. The digraph $\mathcal{G}$ is *balanced* if $D_{\text{out}}(\mathcal{G}) = D_{\text{in}}(\mathcal{G})$. The *graph Laplacian* of the digraph $\mathcal{G}$ is

$$L(\mathcal{G}) = D_{\text{out}}(\mathcal{G}) - \mathcal{A}(\mathcal{G}).$$

Next, we define reverse and mirror digraphs. Let $\tilde{\mathcal{E}}$ be the set of reverse edges of $\mathcal{G}$ obtained by reversing the order of all pairs in $\mathcal{E}$. The *reverse digraph* $\tilde{\mathcal{G}}$ of $\mathcal{G}$ is $(\mathcal{U}, \tilde{\mathcal{E}}, \tilde{\mathcal{A}})$, where $\tilde{\mathcal{A}} = \mathcal{A}^T$. The *mirror digraph* $\widehat{\mathcal{G}}$ of $\mathcal{G}$ is $(\mathcal{U}, \widehat{\mathcal{E}}, \widehat{\mathcal{A}})$, where $\widehat{\mathcal{E}} = \mathcal{E} \cup \tilde{\mathcal{E}}$ and $\widehat{\mathcal{A}} = (\mathcal{A} + \mathcal{A}^T)/2$. Note that $L(\tilde{\mathcal{G}}) = D_{\text{out}}(\tilde{\mathcal{G}}) - \mathcal{A}(\tilde{\mathcal{G}}) = D_{\text{in}}(\mathcal{G}) - \mathcal{A}(\mathcal{G})^T$.

Given a digraph $\mathcal{G}$ of order n, the *disagreement function* $\Phi_{\mathcal{G}} : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\Phi_{\mathcal{G}}(x) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x_j - x_i)^2. \tag{1}$$

The following characterization of $\Phi_{\mathcal{G}}$ is novel.

Proposition 2.1 (Matrix form of disagreement): Given a digraph $\mathcal{G}$ of order n, the disagreement function $\Phi_{\mathcal{G}} : \mathbb{R}^n \to \mathbb{R}$ is the quadratic form associated with the symmetric positive-semidefinite matrix

$$P(\mathcal{G}) = \frac{1}{2}(D_{\text{out}}(\mathcal{G}) + D_{\text{in}}(\mathcal{G}) - \mathcal{A}(\mathcal{G}) - \mathcal{A}(\mathcal{G})^T).$$

Moreover, $P(\mathcal{G})$ is the graph Laplacian of the mirror graph $\widehat{\mathcal{G}}$, that is, $P(\mathcal{G}) = L(\widehat{\mathcal{G}}) = \frac{1}{2}\big(L(\mathcal{G}) + L(\tilde{\mathcal{G}})\big)$.

Remark 2.2: In general, $P(\mathcal{G}) \neq L(\mathcal{G})$ and, therefore, $\Phi_{\mathcal{G}}(x) \neq x^T L(\mathcal{G}) x$. However, if the digraph $\mathcal{G}$ is balanced, then $D_{\text{out}}(\mathcal{G}) = D_{\text{in}}(\mathcal{G})$ and, in turn, $\Phi_{\mathcal{G}}(x) = x^T L(\mathcal{G}) x$. This is the usual result for undirected graphs, e.g., [1]. •

B. Averaging plus connectivity achieves consensus

To each node $i \in \mathcal{U}$ of a digraph $\mathcal{G}$, we associate a state $x_i \in \mathbb{R}$, that obeys a first-order dynamics of the form $\dot{x}_i = u_i,\ i \in \{1, \dots, n\}$. We say that the nodes of a network have reached a *consensus* if $x_i = x_j$ for all $i, j \in \{1, \dots, n\}$. Our objective is to design control laws u that guarantee that consensus is achieved starting from any initial condition, while u_i depends only on the state of the node i and of its neighbors in $\mathcal{G}$, for $i \in \{1, \dots, n\}$. In other words, the closed-loop system asymptotically achieves consensus if, for any $x_0 \in \mathbb{R}^n$, one has that $x(t) \to \{\alpha(1, \dots, 1) \mid \alpha \in \mathbb{R}\}$ when $t \to +\infty$. If the value α is the average of the initial state of the n nodes, then we say the nodes have reached *average-consensus*.

We refer to the following linear control law, often used in the literature on consensus (e.g., see [6], [8], [10]), as the *averaging protocol*:

$$u_i = \sum_{j=1}^{n} a_{ij}(x_j - x_i). \tag{2}$$

With this control law, the closed-loop system is

$$\dot{x}(t) = -L(\mathcal{G})x(t). \tag{3}$$

Next, we consider a family of digraphs $\{\mathcal{G}_1, \dots, \mathcal{G}_m\}$ with the same vertex set $\{1, \dots, n\}$. A *switching signal* is a map $\sigma : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \{1, \dots, m\}$. Given these objects, we can define the following switched dynamical system

$$\begin{aligned} \dot{x}(t) &= -L(\mathcal{G}_k)x(t), \\ k &= \sigma(t, x(t)). \end{aligned} \tag{4}$$

Note that the notion of solution for this system might not be well-defined for arbitrary switching signals. The properties of the linear system (3) and the system (4) under time-dependent switching signals have been investigated in [1], [4], [7], [17]. Here, we review some of these properties in the following two statements.

Theorem 2.3 (Averaging over digraphs): Let $\mathcal{G}$ be a digraph. The following statements hold:

(i) System (3) asymptotically achieves consensus if and only if $\mathcal{G}$ has a globally reachable node;
(ii) If $\mathcal{G}$ is strongly connected, then system (3) asymptotically achieves average-consensus if and only if $\mathcal{G}$ is balanced.

Next, let $\{\mathcal{G}_1, \dots, \mathcal{G}_m\}$ be digraphs with the same vertex set $\{1, \dots, n\}$, let $\sigma : \mathbb{R}_{\geq 0} \to \{1, \dots, m\}$ be a piecewise constant function. The following statements hold:

(iii) System (4) asymptotically achieves consensus if there exist infinitely many consecutive uniformly bounded time intervals such that the union of the switching graphs across each interval has a globally reachable node;
(iv) If each $\mathcal{G}_i$, $i \in \{1, \dots, m\}$, is strongly connected and balanced, then for any arbitrary piecewise constant function σ, the system (4) globally asymptotically solves the *average*-consensus problem.

C. Averaging protocol over a fixed acyclic digraph

Here we characterize the convergence properties of the averaging protocol (3) under different connectivity properties than the ones stated in Theorem 2.3(i) and (ii), namely assuming that the given digraph is acyclic.

We start by reviewing some basic properties of acyclic digraphs. Given an acyclic digraph $\mathcal{G}$, every vertex of in-degree 0 is named *source*, and every vertex of out-degree 0 is named *sink*. Every acyclic digraph has at least one source and at least one sink. Given an acyclic digraph $\mathcal{G}$, we associate a nonnegative number to each vertex, called *depth*, in the following way. First, we define the depth of the sinks of $\mathcal{G}$ to be 0. Next, we consider the acyclic digraph that results from erasing the 0-depth vertices from $\mathcal{G}$ and the in-edges towards them; the depth of the sinks of this new acyclic digraph are defined to be 1. The higher depth vertices are defined recursively. This process is well-posed as any acyclic digraph has at least one sink. The depth of the digraph is the maximum depth of its vertices. For $n, d \in \mathbb{N}$, $\mathcal{S}_{n,d}$ is the set of acyclic digraphs with vertex set $\{1, \dots, n\}$ and depth d.

Next, it is convenient to relabel the n vertices of the acyclic digraph $\mathcal{G}$ with depth d in the following way: (1) label the sinks from 1 to n_0, where n_0 is the number of sinks; (2) label the vertices of depth k from $\sum_{j=0}^{k-1} n_j + 1$ to $\sum_{j=0}^{k-1} n_j + n_k$, where n_k is the number of vertices of depth k, for $k \in \{1, \dots, d\}$. Note that vertices with the same depth may be labeled in arbitrary order. With this labeling, the adjacency matrix $\mathcal{A}(\mathcal{G})$ is lower-diagonal with vanishing diagonal entries, and the Laplacian $L(\mathcal{G})$ takes the form

$$L(\mathcal{G}) = \begin{bmatrix} 0 & 0 & \dots & 0 \\ -a_{21} & \sum_{j=1}^{1} a_{2j} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ -a_{n1} & -a_{n2} & \dots & \sum_{j=1}^{n-1} a_{nj} \end{bmatrix}.$$

Clearly, all eigenvalues of L are non-negative and the zero eigenvalues are simple.

Proposition 2.4 (Averaging over an acyclic digraph): Let $\mathcal{G}$ be an acyclic digraph of order n with n_0 sinks, assume its vertices are labeled according to their depth, and consider the dynamical system $\dot{x}(t) = -L(\mathcal{G})x(t)$ defined in (3). The following statements hold:

(i) The equilibrium set of (3) is the vector subspace
$$\ker L(\mathcal{G}) = \{(x_{\mathrm{s}}, x_{\mathrm{e}}) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n-n_0} \mid x_{\mathrm{e}} = -L_{22}^{-1} L_{21} x_{\mathrm{s}}\}.$$

(ii) Each trajectory $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ of (3) exponentially converges to the equilibrium x^* defined by
$$x_i^* = \begin{cases} x_i(0), & i \in \{1, \dots, n_0\}, \\ \dfrac{\sum_{j=1}^{i-1} a_{ij} x_j^*}{\sum_{j=1}^{i-1} a_{ij}}, & i \in \{n_0 + 1, \dots, n\}. \end{cases}$$

(iii) If $\mathcal{G}$ has unit depth, then $\Phi_{\mathcal{G}}$ is monotonically decreasing along any trajectory of (3).

Remark 2.5: If the digraph has a single sink, then the convergence statement in part (ii) of Proposition 2.4 is equivalent to part (i) of Theorem 2.3. Note also that statement (iii) is not true for digraphs with depth larger than 1. The digraph in Figure 1 is a counterexample. •

D. Averaging protocol over switching acyclic digraphs

Given a family of digraphs $\Gamma = \{\mathcal{G}_1, \dots, \mathcal{G}_m\}$ with vertex set $\{1, \dots, n\}$, the *minimal disagreement function* $\Phi_\Gamma : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\Phi_\Gamma(x) = \min_{k \in \{1, \dots, m\}} \Phi_{\mathcal{G}_k}(x). \tag{5}$$

Let $I(x) = \operatorname{argmin}\{\Phi_{\mathcal{G}_k}(x) \mid k \in \{1, \dots, m\}\}$. We consider state-dependent switching signals $\sigma : \mathbb{R}^n \to \{1, \dots, m\}$ with

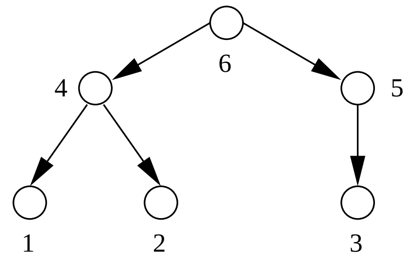

Fig. 1. For this digraph of depth 2, the Lie derivative of the disagreement (1) along the protocol (3) is indefinite.

the property that $\sigma(x) \in I(x)$, that is, at each $x \in \mathbb{R}^n$, $\sigma(x)$ corresponds to the index of a graph with minimal disagreement. Clearly, for any such σ, one has $\Phi_\Gamma(x) = \Phi_{\mathcal{G}_{\sigma(x)}}(x)$.

Proposition 2.6 (Averaging over acyclic digraphs): Let $\Gamma = \{\mathcal{G}_1, \dots, \mathcal{G}_m\}$ be a set of acyclic digraphs with vertices $\{1, \dots, n\}$ and depth 1, i.e., $\Gamma \subset \mathcal{S}_{n,1}$. Assume that $\cup_{k \in \{1,\dots,m\}} \mathcal{G}_k \in \mathcal{S}_{n,1}$ and that $\sigma : \mathbb{R}^n \to \{1, \dots, m\}$ satisfies $\sigma(x) \in I(x)$. Consider the discontinuous system

$$\dot{x}(t) = Y(x(t)) = -L(\mathcal{G}_k)x(t), \quad \text{for} \quad k = \sigma(x(t)), \tag{6}$$

whose solutions are understood in the Filippov sense. The following statements hold:

(i) The point $x^* \in \mathbb{R}^n$ is an equilibrium for (6) if and only if for each $i \in I(x^*)$, there exists scalars $\lambda_i \geq 0$ and $\sum_{i \in I(x^*)} \lambda_i = 1$, such that
$$x^* \in \ker \Big(\sum_{i \in I(x^*)} \lambda_i L(\mathcal{G}_i) \Big). \tag{7}$$

(ii) Each trajectory $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ of (6) converges to the set of equilibria.

(iii) The minimum disagreement function Φ_Γ is monotonically non-increasing along any trajectory of (6).

Remarks 2.7:
- Statement (ii) in this theorem is weaker than statement (ii) in previous one in three ways: first, we are not able to characterize the limit point as a function of the initial state. Second, we require the depth 1 assumption, which is sufficient to ensure convergence, but possibly not necessary. Third, we establish only convergence to a set, rather than an individual point. It remains an open question to obtain necessary and sufficient conditions for convergence to a point.
- Although the statement (ii) is obtained only for digraphs of unit depth, this class of graphs is of interest in the forthcoming sections. •

III. DISCRETE COVERAGE CONTROL

We now consider motion coordination problems for a group of robots described by first order integrators. In other words, we assume that n robotic agents are placed at locations $p_1, \dots, p_n \in \mathbb{R}^2$ and that they move according to $\dot{p}_i = u_i, \ i \in \{1, \dots, n\}$. We denote by P the vector $(p_1, \dots, p_n) \in (\mathbb{R}^2)^n$. Additionally, we define

$$\mathcal{S}_{\text{coinc}} = \{(p_1, \dots, p_n) \in (\mathbb{R}^2)^n \mid p_i = p_j \text{ for some } i \neq j\},$$

and, for $P \notin \mathcal{S}_{\text{coinc}}$, we let $\{V_i(P)\}_{i \in \{1,\dots,n\}}$ denote the Voronoi partition generated by P, e.g., see Figure 2.

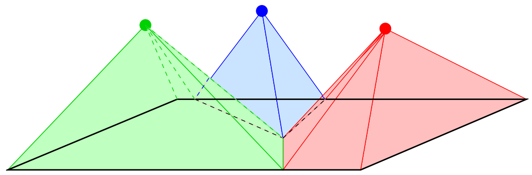

Fig. 2. Voronoi partition of a rectangle. The generators $p_1,\dots,p_n$ are elevated from the plane for intuition's sake.

A. Continuous and discrete multi-center functions

Let Q be a convex polygon in $\mathbb{R}^2$ including its interior and let $\phi:\mathbb{R}^2\to\overline{\mathbb{R}}_+$ be a bounded and measurable function whose support is Q. Analogously, let $\{q_1,\dots,q_N\}\subset\mathbb{R}^2$ be a pointset and $\{\phi_1,\dots,\phi_N\}$ be nonnegative weights associated to them. Given a non-increasing performance function $f:\overline{\mathbb{R}}_+\to\mathbb{R}$, we consider the *continuous* and *discrete multi-center functions* $\mathcal{H}:(\mathbb{R}^2)^n\to\mathbb{R}$ and $\mathcal{H}_{\text{dscrt}}:(\mathbb{R}^2)^n\to\mathbb{R}$ defined by

$$\mathcal{H}(P)=\int_Q \max_{i\in\{1,\dots,n\}} f(\|q-p_i\|)\phi(q)dq,$$
$$\mathcal{H}_{\text{dscrt}}(P)=\sum_{j=1}^{N}\max_{i\in\{1,\dots,n\}}\phi_j f(\|q_j-p_i\|).$$

Let $d(q)=\min_{j\in\{1,\dots,n\}}\|q-p_j\|$, we define

$$\mathcal{S}_{\text{equid}}=\{(p_1,\dots,p_n)\in(\mathbb{R}^2)^n \mid \|q-p_i\|=\|q-p_k\|=d(q) \text{ for some } q\in\{q_1,\dots,q_N\} \text{ and for some } i\neq k\},$$

In other words, if $P\notin\mathcal{S}_{\text{equid}}$, then no point q_j is equidistant to two or more nearest robots. Note that $\mathcal{S}_{\text{equid}}$ is a set of measure zero because it is the union of the solutions of a finite number of algebraic equations. Using Voronoi partitions, for $P\notin\mathcal{S}_{\text{coinc}}$, we may write

$$\mathcal{H}(P)=\sum_{i=1}^{n}\int_{V_i(P)} f(\|q-p_i\|)\phi(q)dq,$$
$$\mathcal{H}_{\text{dscrt}}(P)=\sum_{i=1}^{n}\sum_{q_j\in V_i(P)}\frac{\phi_j}{\operatorname{card}(q_j,P)}f(\|q_j-p_i\|),$$

where $\operatorname{card}:\mathbb{R}^2\times(\mathbb{R}^2)^n\to\{1,\dots,n\}$ denotes the number of indices k for which $\|q_j-p_k\|=\min_{i\in\{1,\dots,n\}}\|q_j-p_i\|$. If q_j is a point in the interior of $V_i(P)$ for some i, then $\operatorname{card}(q_j,P)=1$. The following result is discussed in [13] for the continuous multi-center function; the result for the discrete function is novel.

Proposition 3.1 (Derivatives of $\mathcal{H}$ and $\mathcal{H}_{\text{dscrt}}$): If f is locally Lipschitz, then $\mathcal{H}$ and $\mathcal{H}_{\text{dscrt}}$ are locally Lipschitz on Q^n. If f is differentiable, then

(i) $\mathcal{H}$ is differentiable on $Q^n\setminus\mathcal{S}_{\text{coinc}}$ with

$$\frac{\partial\mathcal{H}}{\partial p_i}(P)=\int_{V_i(P)}\frac{\partial}{\partial p_i}f(\|q-p_i\|)\phi(q)dq,$$

(ii) $\mathcal{H}_{\text{dscrt}}$ is regular on Q^n with generalized gradient

$$\partial\mathcal{H}_{\text{dscrt}}(P)=\sum_{j=1}^{N}\phi_j\operatorname{co}\Big\{\frac{\partial}{\partial P}f(\|q_j-p_k\|)\mid k\in I(q_j,P)\Big\},$$

where $I(q_j,P)$ is the set of indices k for which $f(\|q_j-p_k\|)=\max_{i\in\{1,\dots,n\}}f(\|q_j-p_i\|)$. Additionally, if $P\notin\mathcal{S}_{\text{coinc}}\cup\mathcal{S}_{\text{equid}}$, then $\mathcal{H}_{\text{dscrt}}$ is differentiable at P, and for each $i\in\{1,\dots,n\}$

$$\frac{\partial\mathcal{H}_{\text{dscrt}}}{\partial p_i}(P)=\sum_{q_j\in V_i(P)}\phi_j\frac{\partial}{\partial p_i}f(\|q_j-p_i\|).$$

For particular choices of f, the multi-center functions and their partial derivatives may simplify. For example, if $f(x)=-x^2$, the partial derivative of the multi-center function $\mathcal{H}$ reads, for $P\notin\mathcal{S}_{\text{coinc}}$,

$$\frac{\partial\mathcal{H}}{\partial p_i}(P)=2M_{V_i(P)}(C_{V_i(P)}-p_i),$$

where mass and the centroid of $W\subset Q$ are

$$M_W=\int_W\phi(q)\,dq,\qquad C_W=\frac{1}{M_W}\int_W q\,\phi(q)\,dq.$$

Additionally, the critical points P^* of $\mathcal{H}$ have the property that $p_i^*=C_{V_i(P^*)}$, for $i\in\{1,\dots,n\}$; accordingly, they are called [12] *centroidal Voronoi configurations*. Analogously, if $f(x)=-x^2$, the discrete function reads

$$\mathcal{H}_{\text{dscrt}}(P)=-\sum_{j=1}^{N}\max_{i\in\{1,\dots,n\}}\phi_j\|q_j-p_i\|^2,$$

and its generalized gradient is

$$\partial\mathcal{H}_{\text{dscrt}}(P)=\sum_{j=1}^{N}\phi_j\operatorname{co}\Big\{2(q_j-p_k)\frac{\partial p_k}{\partial P}\mid k\in I(q_j,P)\Big\}.$$

For each $j\in\{1,\dots,N\}$, assume the scalars λ_{ij}, $i\in I(q_j,P)$, satisfy

$$\lambda_{ij}\geq 0,\qquad \sum_{i\in I(q_j,P)}\lambda_{ij}=1, \tag{8}$$

and define $(M_{\text{dscrt}})_{V_i(P)}$ and $(C_{\text{dscrt}})_{V_i(P)}$ by

$$(M_{\text{dscrt}})_{V_i(P)}=\sum_{q_j\in V_i(P)}\lambda_{ij}\phi_j,$$
$$(C_{\text{dscrt}})_{V_i(P)}=\begin{cases}p_i, & \text{if } (M_{\text{dscrt}})_{V_i(P)}=0,\\[2mm] \dfrac{\sum_{q_j\in V_i(P)}\lambda_{ij}\phi_j q_j}{(M_{\text{dscrt}})_{V_i(P)}}, & \text{otherwise.}\end{cases}$$

With this notation, P^* is a critical point of $\partial H_{\text{dscrt}}$, that is, $0\in\partial H_{\text{dscrt}}(P^*)$ if, for any $j\in\{1,\dots,N\}$, there exist λ_{ij} as in equations (8) such that $p_i^*=(C_{\text{dscrt}})_{V_i(P^*)}$, for each $i\in\{1,\dots,n\}$. We call such points P^* *discrete centroidal Voronoi configurations*.

B. Continuous and discrete coverage control

Based on the expressions obtained in the previous subsection, it is possible to design motion coordination algorithms for the robots $p_1, \ldots, p_n$. We call *continuous* and *discrete coverage optimization* the problems of maximizing the multi-center function $\mathcal{H}$ and $\mathcal{H}_{\text{dscrt}}$, respectively. The continuous problem is studied in [12]. We simply impose that the locations $p_1, \ldots, p_n$ follow a gradient ascent law defined over the set $Q^n \setminus \mathcal{S}_{\text{coinc}}$. The (continuous) *coverage control law* is

$$u_i = k_{\text{prop}} \frac{\partial \mathcal{H}}{\partial p_i}(P), \quad i \in \{1, \ldots, n\}, \tag{9}$$

where k_{prop} is a positive gain. Analogously, the *discrete coverage control law* is

$$u_i = k_{\text{prop}}\, X_i(P), \quad i \in \{1, \ldots, n\}, \tag{10}$$

where $X_i : Q^n \to \mathbb{R}^2$ is defined by

$$X_i(P) = \sum_{q_j \in V_i(P)} \frac{\phi_j}{\text{card}(q_j, P)} \frac{\partial}{\partial p_i} f(\|q_j - p_i\|).$$

Note that X_i is discontinuous on Q^n, continuous on $Q^n \setminus \mathcal{S}_{\text{coinc}} \cup \mathcal{S}_{\text{equid}}$, and satisfies

$$X_i(P) = \frac{\partial \mathcal{H}_{\text{dscrt}}}{\partial p_i}(P).$$

Note that both laws are distributed in the sense that each robot only needs information about its Voronoi cell in order to compute its control.

To handle the discontinuity of the discrete coverage control law (10), we define the vector field $X = [X_1, X_2, \ldots, X_n]^T$ and write

$$\dot{P} = k_{\text{prop}}\, X(P). \tag{11}$$

We understand the solution of this equation in the Filippov sense. We then investigate the properties of the solution and analysis the convergence of (9) and (10).

Proposition 3.2: (Continuous and discrete coverage control) For the closed-loop systems induced by equation (9) and by equation (10) starting at $P_0 \in Q^n \setminus \mathcal{S}_{\text{coinc}}$, the agents location converges asymptotically to the set of critical points of $\mathcal{H}$ and of $\mathcal{H}_{\text{dscrt}}$, respectively.

C. The relationship between discrete coverage and averaging over switching acyclic digraphs

Let Q be a convex polygon, let $\{p_1, \ldots, p_n\} \subset Q$ be the position of n robots, let $\{q_1, \ldots, q_N\} \subset Q$ be N fixed points in Q with corresponding nonnegative weights $\{\phi_1, \ldots, \phi_N\}$, and let $I(q_j, P)$ be the set of indices k for which $\|q_j - p_k\| = \min_{i \in \{1,\ldots,n\}} \|q_j - p_i\|$. We begin by defining some useful digraphs.

A *discrete Voronoi graph* $\mathcal{G}_{\text{dscrt-Vor}}$ is a digraph with $(n+N)$ vertices $\{p_1, \ldots, p_n, q_1, \ldots, q_N\}$, with N directed edges $\{(p_i, q_j)|$ for each $j \in \{1, \ldots, N\}$, pick one and only one $i \in I(q_j, P)\}$ and with corresponding edge weights ϕ_j, for all $j \in \{1, \ldots, N\}$. We illustrate one such graph in Figure 3. With our definition, it is possible for one vertex set to generate multiple discrete Voronoi graphs. We will denote

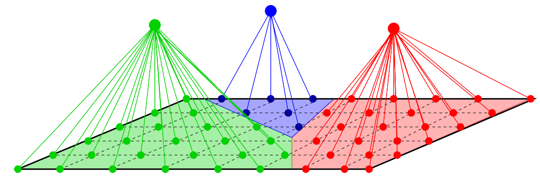

Fig. 3. The discrete Voronoi graph over 3 robots and 6×9 grid points in the rectangle of Figure 2. The edges have top/down direction.

the nodes of $\mathcal{G}_{\text{dscrt-Vor}}$ by $Z = (z_1, \ldots, z_{n+N}) \in (\mathbb{R}^2)^{n+N}$, the weights by $a_{\alpha\beta}$, for $\alpha, \beta \in \{1, \ldots, n+N\}$, with the understanding that:

$$z_\alpha = \begin{cases} p_\alpha, & \text{if } \alpha \in \{1, \ldots, n\}, \\ q_{\alpha-n}, & \text{otherwise}, \end{cases}$$

and that the only non-vanishing weights are $a_{\alpha\beta} = \phi_j$ when $\beta = n + j$, for $j \in \{1, \ldots, N\}$, and when $\alpha \in \{1, \ldots, n\}$ corresponds to the robot p_α closest to q_j and (p_α, q_j) is a directed edge of the graph $\mathcal{G}_{\text{dscrt-Vor}}$. Note that $\mathcal{G}_{\text{dscrt-Vor}}$ depends upon Z. Since $\{q_1, \ldots, q_N\} \subset Q$ are fixed, when we need to emphasize this dependence, we will simply denote it as $\mathcal{G}_{\text{dscrt-Vor}}(P)$.

Let us now define a set of digraphs of which the discrete Voronoi graphs are examples. Let $F(N, n)$ be the set of functions from $\{1, \ldots, N\}$ to $\{1, \ldots, n\}$. Roughly speaking, a function in $F(N, n)$ assigns to each point q_j, $j \in \{1, \ldots, N\}$, a robot p_i, $i \in \{1, \ldots, n\}$. Given $h \in F(N, n)$, let $\mathcal{G}_h$ be the digraph with $(n + N)$ vertices $\{p_1, \ldots, p_n, q_1, \ldots, q_N\}$, with N directed edges $\{(p_{h(j)}, q_j)\}_{j \in \{1,\ldots,N\}}$, and corresponding edge weights ϕ_j, $j \in \{1, \ldots, N\}$. With these notations, it holds that $\mathcal{G}_{\text{dscrt-Vor}}(P) = \mathcal{G}_{h^*(\cdot,P)}$ with any function $h^* : \{1, \ldots, N\} \times Q^n \to \{1, \ldots, n\}$ which satisfies $h^*(j, P) \in \text{argmin}\{\|q_j - p_i\| \mid i \in \{1, \ldots, n\}\}$. Let us state a useful observation about these digraphs.

Lemma 3.3: The set of digraphs $\mathcal{G}_h$, $h \in F(N, n)$, is a set of acyclic digraphs with unit depth, i.e., it is a subset of $\mathcal{S}_{n+N,1}$ (see definition in Subsection II-C). Moreover, $\cup_{h \in F(N,n)} \mathcal{G}_h$ is an acyclic digraph with unit depth, i.e., $\cup_{h \in F(N,n)} \mathcal{G}_h \in \mathcal{S}_{n+N,1}$.

For $h \in F(N, n)$, let us study appropriate disagreement functions for the digraph $\mathcal{G}_h$. We define the function $\Phi_{\mathcal{G}_h} : (\mathbb{R}^2)^{n+N} \to \mathbb{R}$ by

$$\begin{aligned} \Phi_{\mathcal{G}_h}(Z)|_{Z=(p_1,\ldots,p_n,q_1,\ldots,q_N)} &= \frac{1}{2} \sum_{\alpha,\beta=1}^{n+N} a_{\alpha\beta} \|z_\alpha - z_\beta\|^2 \\ &= \frac{1}{2} \sum_{j=1}^{N} \phi_j \|q_j - p_{h(j)}\|^2, \end{aligned}$$

because the weights $a_{\alpha\beta}$, $\alpha, \beta \in \{1, \ldots, n + N\}$ of the $\mathcal{G}_h$ all vanish except for $a_{h(j),j} = \phi_j$, $j \in \{1, \ldots, N\}$. We now state the main result of this section.

Theorem 3.4: (Correspondence between discrete coverage control laws and averaging protocols over acyclic graphs) The following statements hold:

(i) The discrete multi-center function $\mathcal{H}_{\text{dscrt}}$ with $f(x) = -x^2$, and the minimum disagreement function over the set of digraphs $\mathcal{G}_h$, $h \in F(N, n)$, satisfy

$$
\begin{aligned}
-\frac{1}{2}\mathcal{H}_{\text{dscrt}}(P) &= \frac{1}{2}\sum_{j=1}^{N} \min_{i\in\{1,\dots,n\}} \phi_j \|q_j - p_i\|^2 \\
&= \frac{1}{2}\sum_{j=1}^{N} \phi_j \|q_j - p_{h^*(j)}\|^2 \\
&= \Phi_{\mathcal{G}_{\text{dscrt-Vor}}}(p_1, \dots, p_n, q_1, \dots, q_N) \\
&= \min_{h\in F(N,n)} \Phi_{\mathcal{G}_h}(p_1, \dots, p_n, q_1, \dots, q_N).
\end{aligned}
$$

(ii) For $P \notin \mathcal{S}_{\text{coinc}} \cup \mathcal{S}_{\text{equid}}$, the discrete coverage control law for $f(x) = -x^2$ and the averaging protocol over the discrete Voronoi digraph together satisfy, for $i \in \{1, \dots, n\}$,

$$
\frac{1}{2}\frac{\partial \mathcal{H}_{\text{dscrt}}}{\partial p_i}(P) = \sum_{q_j\in V_i(P)} \phi_j (q_j - p_i) = \sum_{\beta=1}^{n+N} a_{\alpha\beta}(z_\beta - z_\alpha),
$$

where z_α and $a_{\alpha\beta}$, $\alpha, \beta \in \{1, \dots, n+N\}$, are nodes and weights of $\mathcal{G}_{\text{dscrt-Vor}}$. Accordingly, the discontinuous coverage control system (11), for $f(x) = -x^2$, and the averaging system (6) over the set of digraphs $\mathcal{G}_h$, $h \in F(N, n)$, together satisfy, for $i \in \{1, \dots, n\}$,

$$
\frac{1}{2}K[X_i](P) = K[Y_i](Z),
$$

with $Z = (p_1, \dots, p_n, q_1, \dots, q_N)$. X_i, Y_i are the i^{th} 2-dimensional block component of X, Y, respectively.

(iii) $P^* \in Q^n$ is an equilibrium of the discrete coverage control system with $f(x) = -x^2$ if and only if $Z^* = (p_1^*, \dots, p_n^*, q_1, \dots, q_N)$ is an equilibrium of system (6) over the set of digraphs $\mathcal{G}_h$, $h \in F(N, n)$, that is:

$$
\begin{gathered}
\forall j \in \{1, \dots, N\},\ \exists \lambda_{ij} \text{ as in (8), such that} \\
p_i^* = (C_{\text{dscrt}})_{V_i(P^*)},\ \forall i \in \{1, \dots, n\}, \\
\Updownarrow \\
\exists \mu_k \geq 0 \text{ and } \sum_k \mu_k = 1, \text{ such that} \\
Z^* \in \ker\Big(\sum_k \mu_k L(\mathcal{G}^k_{\text{dscrt-Vor}}(Z^*))\Big),
\end{gathered}
$$

where $\{\mathcal{G}^k_{\text{dscrt-Vor}}(Z^*)\}_k$ are all possible discrete Voronoi graphs generated by Z^*.

(iv) Given any initial position of robots $P_0 \in Q^n$, the evolution of the discrete coverage control system (11) and the evolution of the averaging system (6) under the switching signal $\sigma : Q^n \to \{\mathcal{G}_h \mid h \in F(N, n)\}$ defined by $\sigma(P) = \mathcal{G}_{\text{dscrt-Vor}}(Z)$ are identical in the Filippov sense and, therefore, the two systems will converge to the same set of equilibrium placement of robots, as described in (iii).

IV. CONCLUSIONS

We have studied averaging protocols over fixed and controlled-switching acyclic digraphs, and characterized their asymptotic convergence properties. We have also discussed continuous and discrete multi-center locational optimization functions, and distributed control laws that optimize them. The main result of the paper shows how these two sets of problems are intimately related: discrete coverage control laws are indeed averaging protocols over acyclic digraphs. As a consequence of our analysis, one can argue that coverage control consensus problems are special cases of a general class of distributed optimization problems.

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