

Distributed Quadratic Programming over Arbitrary Graphs

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Abstract—In this paper, the locality features of infinite-dimensional quadratic programming (QP) optimization problems are studied. Our approach is based on tools from operator theory and ideas from Multi Parametric Quadratic Programming (MPQP). The key idea is to use the spatially decaying operators (SD), which has been recently developed to study spatially distributed systems in [1], to capture couplings between optimization variables in the quadratic cost functional and linear constraints. As an application, it is shown that the problem of receding horizon control of spatially distributed systems with heterogeneous subsystems, input and state constraints, and arbitrary interconnection topologies can be modeled as an infinite-dimensional QP problem. Furthermore, we prove that for a convex infinite-dimensional QP in which the couplings are through SD operators, optimal solution is piece-wise affine— represented as convolution sums. More importantly, we prove that the kernel of each convolution sum decays in the spatial domain at a rate proportional to the inverse of the corresponding coupling function of the optimization problem, thereby providing evidence that even centralized solutions to the infinite-dimensional QP has inherent spatial locality.

I. INTRODUCTION

The problem of performing distributed computations over a network to implicitly solve a global optimization problem has been an active area of research over the past few years. There are many important problems that have been cast in the form of a large-scale finite-dimensional or an infinite-dimensional constraint optimization problem. Substantial progress has been made to understand the fundamental issues regarding this class of problems, for example see [2], [3] and references therein. One of the fundamental problems in this area is to study the locality features of spatially distributed optimization problems which can be advantageous in the development of fast and well-conditioned distributed algorithms.

On the other hand, there has been a rapidly growing interest in systems and control community in the study of coordination and control algorithms for networked dynamic systems. From consensus and agreement problems to formation control, sensing, and coverage, researchers have been interested in algorithms that are spatially distributed and would achieve a global objective using local interactions [4]–[12]. The subject of this paper is mainly motivated by the problem of receding horizon control of spatially distributed systems over infinite graphs. Spatially distributed systems consist of a large, possibly infinite, number of subsystems coupled either through their dynamics or through a single cost function, which represents

some common goal or objective. It is shown that this problem can be cast as an infinite-dimensional quadratic programming problem [13], [14]. There have been a considerable progress in the study of receding horizon control of distributed systems which consist of finitely many subsystems. Previous works include [15]–[20].

With advances in real-time optimization-based control, there have been several attempts to develop distributed control algorithms that can handle constraints and can be implemented in real-time. Receding horizon control is a form of control in which the control action is obtained by solving online a finite horizon open-loop optimal control problem. Applications of receding horizon control range from formation control [16], [21], [22] to applications in manufacturing and process industry where multiple units cooperatively produce a product [23], [24], and large scale power systems [17], [20], [25]–[28].

In [15], the authors proposed a distributed receding horizon control algorithm for systems which consist of subsystems whose dynamics and constraints are uncoupled, and couplings are imposed through a single performance cost function. Stability analysis is based on the fact that the optimal state trajectory of each subsystem satisfies a compatibility constraints condition, and that the receding horizon updates happen sufficiently fast. In [16], a decentralized receding horizon control scheme for systems whose coupling is through cost function and constraints, is proposed. Each subsystems uses only local information of itself and every neighbor to compute the optimal trajectory. Stability and feasibility issues regarding this distributed algorithm is also discussed and compared to those of others being proposed earlier in the literature. Another related work on this subject was reported in [18] where the authors solve a min-max problem for each subsystem. In this work, coupling comes from dynamics and the stability of the proposed algorithm is ensured by imposing a contractive constraint, called stability constraint. In [17], [19], [20], unconstrained coupled subsystems are addressed with a separable quadratic cost function. The primary objective of these papers is to develop decomposition algorithms, with stability and feasibility guarantees, to solve the centralized receding horizon control problem in a distributed fashion.

In this paper, our objective is to study the spatial locality properties of infinite-dimensional linear programming (LP) and quadratic programming (QP) problems. We address this problem by employing the operator theoretic tools developed in [1] to study spatially distributed systems. A new class of linear operators called *spatially decaying* (SD) is introduced in [1] where it is shown that such operators exhibit a localized behavior in spatial domain, i.e., the norm of blocks in the matrix representation of the operator decay as a function of an appropriate measure of distance between subsystems. It is shown that the space of SD operators is a normed vector

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space with respect to a specific operator-norm which is not induced and is denoted by $\|\cdot\|, \|\cdot\|_r^*$. Furthermore, such operators equipped with the norm form a *Banach algebra*. Using this result, we prove certain closure properties of this space with respect to inversion. This will then enable us to study the solution of linear operator equations. Using duality theory and complimentary slackness, it is shown that the optimal solution of infinite-dimensional LP or QP is a piece-wise affine map of parameters, and can be represented as convolution sums, similar to the finite dimensional case [29]. Most importantly, we prove that the kernel of each convolution sum decays, e.g. exponentially or polynomially, in the spatial domain. In other words, change of parameters in one node mainly affects the optimal solution of those nodes which are located in the immediate vicinity of that node. It is important to stress that such spatial locality in the solution of multi-parametric quadratic programs provides a justification for spatial truncation and removing the influence of farther-away subsystems, suggesting the possibility of distributed solutions whose cost is “close” to the centralized one. This issue is highlighted further when we consider multi-parametric quadratic programs in the context of Model Predictive Control (MPC) problems. The structural locality of the optimal state feedback solutions suggest the potential for ignoring the influence of farther away nodes without sacrificing too much performance. Of course, as it becomes clear later on in the paper, the rate of decay is intimately related to how tightly coupled the dynamics and the cost functions of individual subsystems are.

This paper is organized as follows. We introduce the notation and the basic concepts used throughout the paper in Section II. The infinite-dimensional quadratic programming problem is presented in Section III. The concept of spatially decaying operators is introduced in Section IV. The receding horizon control problem for spatially distributed systems is discussed in Section V. Results of Section IV are utilized in Section VI to show that the kernel of each convolution sum inherits spatial locality.

II. PRELIMINARIES

The notation used in this paper is fairly standard. \mathbb{R} denotes the set of real numbers, \mathbb{R}^+ the set of nonnegative real numbers, \mathbb{Z} the set of integer numbers, \mathbb{C} the set of complex numbers, and \mathbb{S}^1 the unit circle in \mathbb{C} . Let $\langle \cdot, \cdot \rangle_E$ and $|\cdot|$ denote the inner product and Euclidean 2-norm on \mathbb{R}^{n_i} for $i \in \mathbb{G}$ where \mathbb{G} is an index set. We refer to \mathbb{G} as the spatial domain. Examples of typical spatial domains include \mathbb{Z}^n and \mathbb{R}^n . Whenever it is clear from the context, all induced norms of linear maps between two Euclidean spaces are simply denoted by $\|\cdot\|$. The Banach space $\ell_p(\mathbb{G})$ for $1 \leq p < \infty$ is defined to be the set of all sequences $x = (x_i)_{i \in \mathbb{G}}$ in which $x_i \in \mathbb{R}^{n_i}$ satisfying

$$\sum_{i \in \mathbb{G}} |x_i|^p < \infty$$

endowed with the norm

$$\|x\|_p := \left(\sum_{i \in \mathbb{G}} |x_i|^p \right)^{\frac{1}{p}}.$$

The Banach space $\ell_\infty(\mathbb{G})$ denotes the set of all bounded sequences endowed with the norm

$$\|x\|_\infty := \sup_{i \in \mathbb{G}} |x_i|.$$

Throughout the paper, we will use the shorthand notation ℓ_p for $\ell_p(\mathbb{G})$. A linear functional F on the space ℓ_p is a linear mapping from ℓ_p to \mathbb{R} . We will use the notation $\langle x, F \rangle$ to denote $F(x)$. An operator $\mathcal{Q} : \ell_p \rightarrow \ell_q$ for $1 \leq p, q \leq \infty$ is bounded if it has a finite induced norm, i.e., the following quantity

$$\|\mathcal{Q}\|_{p/q} := \sup_{\|x\|_p=1} \|\mathcal{Q}x\|_q \quad (1)$$

is bounded. The identity operator is denoted by \mathcal{I} . The set of all bounded linear operators of ℓ_p into ℓ_p for some $1 \leq p \leq \infty$ is denoted by $\mathcal{L}(\ell_p)$. The space $\mathcal{L}(\ell_p)$ equipped with norm (1) is a Banach space (cf. [30]). The dual space of a Banach space X , denoted by X^* , is the space of all bounded linear functionals on X . Since we are interested in Banach space ℓ_p in this paper, we have $(\ell_p)^* = \ell_q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

An operator $\mathcal{Q} \in \mathcal{L}(\ell_p)$ has an *algebraic inverse* [30] if it has an inverse \mathcal{Q}^{-1} in $\mathcal{L}(\ell_p)$:

$$\mathcal{Q}\mathcal{Q}^{-1} = \mathcal{Q}^{-1}\mathcal{Q} = \mathcal{I}.$$

Definition 1: Let $\mathcal{Q} : \ell_p \rightarrow \ell_q$ be a bounded linear operator. The adjoint operator $\mathcal{Q}^* : (\ell_q)^* \rightarrow (\ell_p)^*$ is defined by the following equation

$$\langle x, \mathcal{Q}^*y^* \rangle = \langle \mathcal{Q}x, y^* \rangle$$

for all $x \in \ell_p$ and $y^* \in (\ell_q)^*$.

The space ℓ_2 is a Hilbert space with inner product

$$\langle x, y \rangle = \sum_{i \in \mathbb{G}} \langle x_i, y_i \rangle_E$$

for all $x, y \in \ell_2$. An operator $\mathcal{Q} \in \mathcal{L}(\ell_2)$ is self-adjoint if $\mathcal{Q} = \mathcal{Q}^*$.

Definition 2: An operator $\mathcal{Q} \in \mathcal{L}(\ell_2)$ is *positive definite*, shown as $\mathcal{Q} \succ 0$, if there exists a number $\alpha > 0$ such that

$$\langle x, \mathcal{Q}x \rangle > \alpha \|x\|_2^2$$

for all nonzero $x \in \ell_2$.

Definition 3: A subset P of a linear vector space is called a cone if for every $x \in P$ and $\theta \in \mathbb{R}^+$ it satisfies $\theta x \in P$. A subset P of a real vector space is a convex cone if it is convex and a cone, which means that for any $x_1, x_2 \in P$ and $\theta_1, \theta_2 \in \mathbb{R}^+$, it satisfies

$$\theta_1 x_1 + \theta_2 x_2 \in P.$$

Definition 4: Let P be a convex cone in a vector space V . For $x, y \in V$, we write $x \succeq y$ (with respect to P) if $x - y \in P$. The cone P defining this relation is called the positive cone in V .

In most situations, the choice of P will arise naturally. For example, the set of all continuous functions from $D \subseteq \mathbb{R}$ into \mathbb{R} is a vector space \mathcal{F} over \mathbb{R} . The positive cone in \mathcal{F} is the set of all continuous functions in the space that are nonnegative

everywhere on D . Therefore, for $f, g \in \mathcal{F}$, the notation $f \preceq g$ means the pointwise inequality $f(t) \leq g(t)$ for all $t \in D$.

For a positive cone P in Banach space X , the corresponding positive convex cone P^\oplus in the dual space X^* is defined by

$$P^\oplus = \{x^* \in X^* : \langle x, x^* \rangle \geq 0 \text{ for all } x \in P\}$$

A family of seminorms on \mathcal{F} is defined as $\{\|\cdot\|_T \mid T \in \mathbb{R}^+\}$ in which

$$\|f\|_T := \sup_{s \leq T} |f(s)|$$

for all $f \in \mathcal{F}$. The topology generated by all open $\|\cdot\|_T$ -balls is called the topology generated by the family of seminorms and is denoted by $\|\cdot\|_T$ -topology. Continuity of a function in this topology is equivalent to continuity in every seminorm in the family (cf. [31]).

In this paper we are interested in linear operators $\mathcal{Q} : \ell_p \rightarrow \ell_p$ which have matrix representations

$$\mathcal{Q} \mapsto \begin{bmatrix} \ddots & & & \\ & \ddots & & \\ & & [\mathcal{Q}]_{ki} & \\ & & & \ddots \end{bmatrix}$$

where the block element $[\mathcal{Q}]_{ki}$ is a matrix in $\mathbb{R}^{n_k \times n_i}$.

Given linear operators $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} : \ell_p \rightarrow \ell_p$, whenever dimensions are compatible, the row and column block composition of these operators are defined as follows

$$\begin{bmatrix} \mathcal{A} \\ \mathcal{B} \end{bmatrix}_{ki} = \begin{bmatrix} [\mathcal{A}]_{ki} \\ [\mathcal{B}]_{ki} \end{bmatrix}, \quad \begin{bmatrix} \mathcal{C} & \mathcal{D} \end{bmatrix}_{ki} = \begin{bmatrix} [\mathcal{C}]_{ki} & [\mathcal{D}]_{ki} \end{bmatrix}$$

The other complex block compositions can be defined in terms of these elementary operations. Similarly, we can define the row and column block compositions of elements in ℓ_p .

The concatenation of a sequence of vectors $x_i \in \mathbb{R}^{n_i}$ is defined as $\text{cat}_{i \in \mathbb{G}} x_i$. For a given sequence of matrices $A_i \in \mathbb{R}^{n_i \times n_i}$ with $i \in \mathbb{G}$, the diagonal operator $\text{diag} A_i$ is defined to be an operator that maps $x = \text{cat}_{i \in \mathbb{G}} x_i$ to $y = \text{cat}_{i \in \mathbb{G}} y_i$ such that $y_i = A_i x_i$ for all $i \in \mathbb{G}$.

III. PROBLEM SETUP

In this paper, we consider the infinite-dimensional quadratic programming problem:

$$\begin{aligned} \inf_{x \in \ell_2} & \frac{1}{2} \langle x, \mathcal{P}x \rangle + \langle c, x \rangle \\ \text{subject to:} & \quad \mathcal{G}x \preceq b \end{aligned} \quad (2)$$

where $\mathcal{P}, \mathcal{G} \in \mathcal{L}(\ell_2)$, \mathcal{P} is positive definite, and $c, b \in \ell_2$. We can associate an undirected weighted graph to problem (2) (see Fig. 1) to represent coupling in the cost functions of subsystems. Let denote \mathbb{G} as the set of nodes of the graph. For each $x = (x_k)_{k \in \mathbb{G}}$ in ℓ_2 , element x_k represents the vector of variables corresponding to node k . For a given pair of nodes (k, i) , the block elements $[\mathcal{P}]_{ki}$ and $[\mathcal{G}]_{ki}$ can be thought of as coupling weights on the edge connecting these two nodes. In distributed control applications, very often the underlying system is spatially distributed over an arbitrary graph. Each node corresponds to an individual dynamical subsystem which

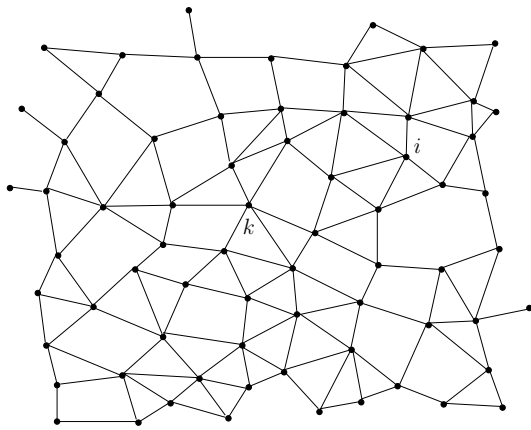


Fig. 1. Topology of the system on an arbitrary connected graph. Coupling between two agents is shown by an undirected edge between them.

is coupled to the other subsystems in the network through their dynamics and collective performance objective function. There are numerous problems concerning this class of systems that can be posed as problem (2). For example, the problem of receding horizon control (or model predictive control) of spatially distributed systems with discrete-time linear models (see Section V for details) and least square optimization problems with linear constraints arisen in different applications in sensor networks can be formulated in the form (2).

Problem (2) is an infinite-dimensional convex optimization problem. In general, it is a tedious task to find numerical solutions for this class of problems except for the special cases where operators \mathcal{P} and \mathcal{G} have some kind of spatial symmetries such as the case where they are Toeplitz operators. Therefore, we focus our study on the structural properties of problems of the form (2). Specifically, we show that the optimal solution of (2) has, to some extent, localized features.

One of the fundamental tools in deriving methods to compute exact or approximate solutions to the infinite-dimensional problems is duality theory [2]. Contrary to the finite-dimensional case, the relationship between primal and dual problems in the infinite-dimensional problems may not be so simple. Problem (2) is a convex optimization problem and, therefore, under suitable constraint qualification conditions strong duality holds. The strong duality (no duality gap) involves cases where the optimal solutions of both primal and dual problems exist and both have the same optimal cost. The strong duality relationship in the primal-dual pair provides a clear insight into the structure of the problem and its optimal solution.

It is worth mentioning that for the case where $\mathcal{P} \equiv 0$, problem (2) reduces to an infinite-dimensional linear programming problem. One example of such problems is receding horizon control of spatially distributed systems with discrete-time linear models and mixed $1/\infty$ -norm performance functions (cf. [32] for finite-dimensional single system case). In what follows, only the infinite-dimensional QP is discussed. A similar analysis is applicable for the infinite-dimensional LP case.

In the following section, we review some of the definitions

and properties of spatially decaying operators [1]. Later in section VI, we will apply results of section IV to problem (2) and prove that the kernel of the optimal solution decays in space, and as a result the infinite-dimensional optimization problem (2) has localized features around each node.

IV. SPATIALLY DECAYING OPERATORS

Spatially decaying (SD) operators have been recently developed to study the structural properties of infinite-horizon optimal control of spatially distributed systems [1]. In the following, first we review briefly the necessary concepts and definitions which will be useful throughout the paper, next, we will show that the set of such operators is closed under inversion (when the inverses exist). Specifically, it is shown that similar to translation invariant operators, if an SD operator in $\mathcal{L}(\ell_2)$ has an algebraic inverse, then the inverse operator is SD as well.

A. Definitions

In the sequel, by a distance function on \mathbb{G} we mean a single-valued function $\text{dis} : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{R}^+$ which has the following properties [33]:

- 1) $\text{dis}(k, i) = 0$ iff $k = i$.
- 2) $\text{dis}(k, i) = \text{dis}(i, k)$.
- 3) $\text{dis}(k, i) \leq \text{dis}(k, j) + \text{dis}(j, i)$.

for all $k, i, j \in \mathbb{G}$.

Definition 5: A nondecreasing continuous function $\chi : \mathbb{R}^+ \rightarrow [1, \infty)$ is called a coupling characteristic function if $\chi(0) = 1$ and $\chi(s+t) \leq \chi(s)\chi(t)$ for all $s, t \in \mathbb{R}^+$.

The constant coupling characteristic function with unit value everywhere is denoted by $\mathbf{1}$.

Definition 6: A one-parameter family of coupling characteristic functions \mathcal{C} is defined to be the set of all characteristic functions χ_α for $\alpha \in \mathbb{R}^+$ such that

- (i) $\chi_0 = \mathbf{1}$.
- (ii) For all $\chi_\alpha, \chi_\beta \in \mathcal{C}$ with $\alpha < \beta$, relation $\chi_\alpha \prec \chi_\beta$ holds (with respect to cone of positive functions).
- (iii) χ_α is a continuous function of α in $\|\cdot\|_T$ -topology.

The definition of a family of coupling characteristic functions enables us to measure the decay rate of the coupling strength between nodes in a coupled network of subsystems as distance increases.

Assumption 1: We assume that for a given family of coupling characteristic functions the following condition satisfies

$$\sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \chi_\alpha(\text{dis}(k, i))^{-1} < \infty$$

for all $0 \leq \alpha < \tau$ and some $\tau > 0$.

The following definition characterizes the class of spatially decaying linear operators on Banach space ℓ_p .

Definition 7: Suppose that a distance function $\text{dis}(\cdot, \cdot)$ and a one-parameter family of parameterized coupling characteristic functions \mathcal{C} are given. A linear operator $\mathcal{Q} \in \mathcal{L}(\ell_p)$ is SD

with respect to \mathcal{C} if there exists a number $\tau > 0$ such that the auxiliary operator $\tilde{\mathcal{Q}}(\alpha)$, defined block-wise as

$$[\tilde{\mathcal{Q}}(\alpha)]_{ki} = [\mathcal{Q}]_{ki} \chi_\alpha(\text{dis}(k, i))$$

is bounded on ℓ_p for all $0 \leq \alpha < \tau$. The number τ is referred to as the *decay margin*.

A simple sufficient condition for an operator to be SD on all ℓ_p spaces is given by

$$\sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \|[\mathcal{Q}]_{ki}\| \chi_\alpha(\text{dis}(k, i)) < \infty \quad (3)$$

for all $0 \leq \alpha < \tau$ (cf. Lemma 1 in [1]).

The interesting result is that the set of all SD operators with decay margin at least $\tau > 0$ forms a *Banach Algebra* with respect to the following operator norm

$$\|\mathcal{Q}\|_\tau^* = \sup_{\alpha \in [0, \tau)} \sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \|[\mathcal{Q}]_{ki}\| \chi_\alpha(\text{dis}(k, i)).$$

This Banach algebra is denoted by

$$\mathcal{S}_\tau(\mathcal{C}) = \{\mathcal{Q} : \|\mathcal{Q}\|_\tau^* < \infty\}$$

The operator norm satisfies the usual conditions, i.e., for all $\mathcal{Q}, \mathcal{P} \in \mathcal{S}_\tau(\mathcal{C})$ and $c \in \mathbb{C}$,

- 1) $\|\mathcal{Q}\|_\tau^* \geq 0$ and $\|\mathcal{Q}\|_\tau^* = 0$ iff $\mathcal{Q} \equiv 0$,
- 2) $\|c \mathcal{Q}\|_\tau^* = |c| \|\mathcal{Q}\|_\tau^*$,
- 3) $\|\mathcal{Q} + \mathcal{P}\|_\tau^* \leq \|\mathcal{Q}\|_\tau^* + \|\mathcal{P}\|_\tau^*$.

Furthermore, it is submultiplicative,

$$4) \quad \|\mathcal{Q}\mathcal{P}\|_\tau^* \leq \|\mathcal{Q}\|_\tau^* \|\mathcal{P}\|_\tau^*.$$

In [1], some of the closure properties of Banach algebra $\mathcal{S}_\tau(\mathcal{C})$ are shown such as closure under addition, multiplication, and limit. These properties utilized to show the structural properties of optimal control of spatially distributed systems. In the sequel, it is shown that under a reasonable assumption, namely invertibility on ℓ_2 , Banach algebra $\mathcal{S}_\tau(\mathcal{C})$ is closed under inversion as well. As we will see in Section VI, the closure under inversion property plays a central role in proving the spatial locality features of the optimal solution of problem (2).

B. Closure under Inversion

The studying of optimal solutions of problem (2) involves solving linear equations of the following form

$$\mathcal{Q}x = y \quad (4)$$

where \mathcal{Q} is an invertible SD operator, $x \in \ell_2$ is the unknown variable, and $y \in \ell_2$ is given.

In the following, it is shown that if $\mathcal{Q}^{-1} \in \mathcal{L}(\ell_2)$, then $\mathcal{Q}^{-1} \in \mathcal{S}_\tau(\mathcal{C})$. Before stating the main results of this section, we recall a motivating result about translation invariant operators [34]. As shown in [1], every translation invariant operator on $\ell_2(\mathbb{Z})$, which its Fourier transform is analytic within an annulus of radius $\tau > 0$ around the unit circle, belongs to $\mathcal{S}_\tau(\mathcal{C})$. Note that \mathcal{C} is the one-parameter family of

parameterized exponential functions $\chi_\alpha(s) = e^{\alpha s}$. We begin by introducing the unit translation operator to the right with respect to the group operation '+' as follows

$$\mathbf{T}u = \mathbf{T}(\dots, |u_i, u_{i+1}, \dots) = (\dots, |u_{i-1}, u_i, \dots).$$

For a translation invariant operator $\mathcal{Q} \in \mathcal{L}(\ell_2(\mathbb{Z}))$ which is defined as

$$\mathcal{Q}(\mathbf{T}) = \sum_{k \in \mathbb{Z}} Q_k \mathbf{T}^k$$

the discrete Fourier transform is defined by

$$\hat{\mathcal{Q}}(z) = \sum_{k \in \mathbb{Z}} Q_k z^{-k}$$

Theorem 1: Suppose that condition $\det(\hat{\mathcal{Q}}(z)) \neq 0$ holds for all $z \in \mathbb{S}^1$. Then \mathcal{Q} is invertible and the inverse operator can be represented as

$$\mathcal{Q}^{-1}(\mathbf{T}) = \sum_{k \in \mathbb{Z}} [Q^{-1}]_k \mathbf{T}^k. \quad (5)$$

Furthermore,

$$\lim_{|k| \rightarrow \infty} \|[Q^{-1}]_k\| e^{m|k|} = 0. \quad (6)$$

for $0 < m < \ln(1 + \rho)$ where

$$\rho = \sup\{r : \det(\hat{\mathcal{Q}}(z)) \neq 0 \text{ for all } 1 - r < |z| < 1 + r\}.$$

Proof: This is an immediate application of Theorem 3 in [1]. ■

The following lemma extends the above result to SD operators and gives a similar decay result for the inverse operator.

Lemma 1: Suppose that $\mathcal{Q} \in \mathcal{S}_\tau(\mathcal{C})$ for some $\tau > 0$ has an algebraic inverse on $\mathcal{L}(\ell_2)$. Then $\mathcal{Q}^{-1} \in \mathcal{S}_{\hat{\tau}}(\mathcal{C})$ for some $0 < \hat{\tau} < \tau$.

Proof: It suffices to prove the lemma for positive definite operators. The reason is that for an invertible operator the following relation holds

$$\mathcal{Q}^{-1} = \mathcal{Q}^* (\mathcal{Q}\mathcal{Q}^*)^{-1}$$

Without loss of generality, we may assume that operator \mathcal{Q} is self-adjoint and positive definite. We define a new operator as follows

$$\mathcal{P} = \mathcal{I} - \frac{1}{\|\mathcal{Q}\|_{2/2}} \mathcal{Q} \quad (7)$$

Let denote the spectrum of operator \mathcal{P} by $\sigma(\mathcal{P})$. Using the positive definiteness of operator \mathcal{Q} , one can conclude that $\sigma(\mathcal{P}) \subset [0, 1)$. Thus, the spectral radius of \mathcal{P} satisfies

$$r(\mathcal{P}) := \sup\{|\lambda| : \lambda \in \sigma(\mathcal{P})\} < 1$$

Using the fact that $\mathcal{L}(\ell_2)$ is a C^* -algebra [30], where \mathcal{P}^* denotes the adjoint operator of \mathcal{P} , and the spectral radius formula, it follows that

$$r(\mathcal{P}) = \lim_{n \rightarrow \infty} \|\mathcal{P}^n\|_{2/2}^{\frac{1}{n}} = \|\mathcal{P}\|_{2/2}$$

Therefore, $\|\mathcal{P}\|_{2/2} < 1$. Since $\mathcal{Q} \in \mathcal{S}_\tau(\mathcal{C})$, we have that $\mathcal{P} \in$

$\mathcal{S}_\tau(\mathcal{C})$. According to (7), we have

$$\mathcal{Q}^{-1} = \|\mathcal{Q}\|_{2/2}^{-1} (\mathcal{I} - \mathcal{P})^{-1} \quad (8)$$

By writing the Neumann series for the quantity in the right of (8), it follows that

$$\mathcal{Q}^{-1} = \|\mathcal{Q}\|_{2/2}^{-1} \sum_{k=0}^{\infty} \mathcal{P}^k \quad (9)$$

Now, we consider the convergent Cauchy sequence $\mathcal{W}_n \rightarrow \mathcal{W}$ as $n \rightarrow \infty$ by defining

$$\mathcal{W}_n = \sum_{k=0}^n \mathcal{P}^k \quad \text{and} \quad \mathcal{W} = (\mathcal{I} - \mathcal{P})^{-1}.$$

The Banach algebra $\mathcal{S}_\tau(\mathcal{C})$ is closed under addition and multiplication operations, therefore, it follows that $\mathcal{W}_n \in \mathcal{S}_\tau(\mathcal{C})$. As a result, according to theorem 5 of [1], we have that $\mathcal{W} \in \mathcal{S}_{\hat{\tau}}(\mathcal{C})$ where $0 < \hat{\tau} < \tau$. Thus, from (9) it can be concluded that $\mathcal{Q}^{-1} \in \mathcal{S}_{\hat{\tau}}(\mathcal{C})$. ■

An immediate consequence of lemma 1 is that if $\mathcal{Q} \in \mathcal{S}_\tau(\mathcal{C})$ and $\mathcal{Q}^{-1} \in \mathcal{L}(\ell_2)$, we have $\mathcal{Q}^{-1} \in \mathcal{L}(\ell_p)$ for all $1 \leq p \leq \infty$ (cf. lemma 1 in [1]). The result of lemma 1 shows that the decay margin of the inverse operator is a number $\hat{\tau} > 0$ where $\hat{\tau} < \tau$. The next theorem explicitly quantifies the decay rate $\hat{\tau}$. This result can also be thought of as the extension of results of theorem 1 to SD operators.

Theorem 2: Suppose that $\mathcal{Q} \in \mathcal{S}_\tau(\mathcal{C})$ has an algebraic inverse on $\mathcal{L}(\ell_2)$. Then each nonzero block element of the inverse operator satisfies

$$\|[Q^{-1}]_{ki}\| \leq \frac{c}{\chi_\alpha(\text{dis}(k, i))} \quad (10)$$

for all $\alpha \in [0, \hat{\tau})$ in which $0 < \hat{\tau} < \tau$ and some $c > 0$, where

$$\hat{\tau} = \sup\{\alpha : \Phi(\chi_\alpha) < \|\mathcal{Q}^{-1}\|_{\infty/\infty}^{-1}\} \quad (11)$$

and

$$\Phi(\chi_\alpha) = \sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \|[Q]_{ki}\| (\chi_\alpha(\text{dis}(k, i)) - 1).$$

Proof: We refer to the appendix for a proof. ■

The result of theorem 2 is a direct extension of the result of theorem 1 to SD operators.

In the venue of the above results, if operator \mathcal{Q} in linear equation (4) satisfies the conditions of lemma 1, it follows that

$$x = \mathcal{Q}^{-1}y$$

and for any given $k \in \mathbb{G}$, we have

$$x_k = \sum_{i \in \mathbb{G}} [Q^{-1}]_{ki} y_i$$

According to lemma 1 and theorem 2, we get

$$\lim_{\text{dis}(k, i) \rightarrow \infty} \|[Q^{-1}]_{ki}\| \chi_\alpha(\text{dis}(k, i)) = 0$$

We will apply this result to analyze the structural properties of the optimal solution of the infinite-dimensional quadratic programming (2) in Section VI. In the following, we will show that receding horizon control problem of spatially distributed

systems can be formulated in a compact form as (2).

V. FORMULATION OF RECEDING HORIZON CONTROL PROBLEM

We consider the class of spatially distributed systems which can be described by a discrete-time linear time-invariant model

$$\psi(t+1) = (\mathcal{A}\psi)(t) + (\mathcal{B}u)(t) \quad (12)$$

$$y(t) = (\mathcal{C}\psi)(t) + (\mathcal{D}u)(t) \quad (13)$$

subject to constraints

$$\mathcal{G}u(t) + \mathcal{H}y(t) + \mathcal{F} \preceq 0 \quad (14)$$

for all $t \geq 0$ and with the initial condition $\psi(0) = \psi_0$. All signals are assumed to be in ℓ_2 space: at each time instant $t \in \mathbb{Z}^+$, signals $\psi(t)$, $u(t)$, $y(t)$ are assumed to be in ℓ_2 . The state-space operators $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{G}, \mathcal{H} \in \mathcal{S}_\tau(\mathcal{C})$ for some $\tau > 0$ are assumed to be time-invariant, $\mathcal{F} \in \ell_2$, and the pair $(\mathcal{A}, \mathcal{B})$ stabilizable. Note that the ordering in inequality (14) is defined with respect to the positive cone in ℓ_2 . The following assumption guarantees existence and uniqueness of classical solutions of the system given by (12)-(13) (cf. Chapter 3 of [35]).

Assumption 2: The semigroup generated by \mathcal{A} is *strongly continuous* on ℓ_2 .

The control objective is to regulate the state of system (12)-(13) to zero while satisfying constraints (14). In the sequel, we will explain how to achieve this objective by employing receding horizon control techniques [36].

An equivalent representation of system (12)-(13) can be obtained by using block-composition operation as follows

$$\begin{bmatrix} \psi(t+1) \\ y(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} \psi(t) \\ u(t) \end{bmatrix} \quad (15)$$

Furthermore, we assume that operator \mathcal{A} is the infinitesimal generator of an exponentially stable C_0 -semigroup on ℓ_2 [35]. Otherwise, a stabilizing state feedback law can be found using LQR method to stabilize the system. Moreover, if we assume that the state-space operators in (12)-(13) are SD, the corresponding LQR state feedback control is guaranteed to be SD as well [1]. We refer to [29] and references in there for further discussions on this assumption.

The receding horizon control problem for system (12)-(13) subject to constraint (14) can be formulated as follows

$$\inf_{\mathbf{u}} \mathfrak{J}(\psi_0, \mathbf{u}) \quad (16)$$

subject to:

$$\psi(k+1) = (\mathcal{A}\psi)(k) + (\mathcal{B}u)(k), \quad 0 \leq k \leq N$$

$$\psi(0) = \psi(t)$$

$$\mathcal{G}u(k) + \mathcal{H}y(k) + \mathcal{F} \preceq 0, \quad 0 \leq k \leq N_c$$

$$u(k) = 0, \quad N_u \leq k \leq N-1.$$

The nonnegative integer number N is the state prediction horizon, N_u the control prediction horizon, and N_c the constraint horizon. Furthermore, we assume that $N_u \leq N-1$ and $N_c \leq N-1$ (cf. [29]). For simplicity, we will assume that

$N_u = N_c = N-1$. For $N_u < N-1$ and $N_c < N-1$, optimization is performed only over N_u control variables, and for the rest of the horizon we may use zero control inputs, which in turn will reduce the complexity of the problem.

The functional $\mathfrak{J}(\psi_0, \mathbf{u})$ can be interpreted as the collective performance objective of the entire system which may have one of the following forms:

- 2-norm:

$$\mathfrak{J}(\psi_0, \mathbf{u}) = \langle \psi(N), \mathcal{P}\psi(N) \rangle + \sum_{k=0}^{N-1} \langle \psi(k), \mathcal{Q}\psi(k) \rangle + \langle u(k), \mathcal{R}u(k) \rangle. \quad (17)$$

- Mixed 1/ ∞ -norm:

$$\mathfrak{J}(\psi_0, \mathbf{u}) = \|\mathcal{P}\psi(N)\|_\infty + \sum_{k=0}^{N-1} \|\mathcal{Q}\psi(k)\|_\infty + \|\mathcal{R}u(k)\|_\infty. \quad (18)$$

Assume that the linear operators $\mathcal{Q} \succeq 0$, $\mathcal{R} \succ 0$ are self-adjoint, $(\mathcal{Q}^{1/2}, \mathcal{A})$ detectable, and $\mathcal{Q}, \mathcal{R} \in \mathcal{S}_\tau(\mathcal{C})$.

In this section, we will only consider problem (16) with a quadratic cost function (17). Problem (16) with cost function given by (18) can be formulated as an infinite-dimensional LP problem (cf. [32]). A similar argument to the one in section VI can be applied to the LP case to analyze the problem [2].

In the sequel, we show that similar to the finite-dimensional case [29], the receding horizon control problem (16) with quadratic cost (17) can be represented in the compact form of (2). Moreover, the terminal weighting cost \mathcal{P} can be determined by solving the corresponding Lyapunov equation (cf. [37])

$$\langle \mathcal{A}\phi, \mathcal{P}\mathcal{A}\phi \rangle - \langle \phi, \mathcal{P}\phi \rangle + \langle \phi, \mathcal{Q}\phi \rangle = 0 \quad (19)$$

for all $\phi \in \mathcal{D}(\mathcal{A})$ (domain of the operator). If we assume that $\mathcal{A}, \mathcal{Q} \in \mathcal{S}_\tau(\mathcal{C})$, then the positive definite solution of (19) is $\mathcal{P} \in \mathcal{S}_\tau(\mathcal{C})$. The proof is very similar to the one given in [1] for the continuous-time Lyapunov equation.

The prediction model for system (15) is given by

$$\begin{bmatrix} \Psi \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_p & \mathcal{B}_p \\ \mathcal{C}_p & \mathcal{D}_p \end{bmatrix} \begin{bmatrix} \psi_0 \\ \mathbf{u} \end{bmatrix} \quad (20)$$

where

$$\Psi = \begin{bmatrix} \psi(1) \\ \vdots \\ \psi(N) \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u(0) \\ \vdots \\ u(N-1) \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y(0) \\ \vdots \\ y(N-1) \end{bmatrix}$$

and linear operators $\mathcal{A}_p, \mathcal{B}_p, \mathcal{C}_p, \mathcal{D}_p$ are completely determined from $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and \mathcal{D} by using block-composition operations as follows

$$\mathcal{A}_p = \begin{bmatrix} \mathcal{A} \\ \mathcal{A}^2 \\ \vdots \\ \mathcal{A}^N \end{bmatrix}, \quad \mathcal{B}_p = \begin{bmatrix} \mathcal{B} & 0 & \dots \\ \mathcal{A}\mathcal{B} & \mathcal{B} & \dots \\ \vdots & \vdots & \ddots \\ \mathcal{A}^N\mathcal{B} & \dots & \mathcal{A}\mathcal{B} & \mathcal{B} \end{bmatrix}$$

$$\mathcal{C}_p = \begin{bmatrix} \mathcal{C} \\ \mathcal{CA} \\ \vdots \\ \mathcal{CA}^{N-1} \end{bmatrix}, \quad \mathcal{D}_p = \begin{bmatrix} \mathcal{D} & 0 & \cdots \\ \mathcal{CB} & \mathcal{D} & \cdots \\ \mathcal{CAB} & \mathcal{CB} & \mathcal{D} & \cdots \\ \vdots & & & \ddots \\ \mathcal{CA}^{N-2}\mathcal{B} & \cdots & \mathcal{CB} & \mathcal{D} \end{bmatrix}$$

Therefore, the state prediction model for system (2) is given by

$$\Psi = \mathcal{A}_p \psi_0 + \mathcal{B}_p \mathbf{u} \quad (21)$$

Equation (17) can be rewritten in the following form

$$\mathfrak{J}(\psi_0, \mathbf{u}) = \langle \Psi, \mathcal{Q}_p \Psi \rangle + \langle \mathbf{u}, \mathcal{R}_p \mathbf{u} \rangle \quad (22)$$

in which

$$\mathcal{Q}_p = \begin{bmatrix} \mathcal{Q} & & & \\ & \ddots & & \\ & & \mathcal{Q} & \\ & & & \mathcal{P} \end{bmatrix}, \quad \mathcal{R}_p = \begin{bmatrix} \mathcal{R} & & & \\ & \ddots & & \\ & & \mathcal{R} & \\ & & & \mathcal{R} \end{bmatrix}$$

and the off-diagonal blocks are equal to zero. Substituting (21) into (22) gives us

$$\mathfrak{J}(\psi_0, \mathbf{u}) = \langle \psi_0, (\mathcal{Q} + \mathcal{A}_p^* \mathcal{Q}_p \mathcal{A}_p) \psi_0 \rangle + 2 \langle \psi_0, \mathcal{A}_p^* \mathcal{Q}_p \mathcal{B}_p \mathbf{u} \rangle + \langle \mathbf{u}, (\mathcal{R} + \mathcal{B}_p^* \mathcal{Q}_p \mathcal{B}_p) \mathbf{u} \rangle$$

The input and output constraints in (16) can be rewritten as

$$\mathcal{G}_p \mathbf{u} + \mathcal{H}_p \mathbf{y} + \mathcal{F}_p \leq 0 \quad (23)$$

where

$$\mathcal{G}_p = \begin{bmatrix} \mathcal{G} & & & \\ & \ddots & & \\ & & \mathcal{G} & \\ & & & \mathcal{G} \end{bmatrix}, \quad \mathcal{H}_p = \begin{bmatrix} \mathcal{H} & & & \\ & \ddots & & \\ & & \mathcal{H} & \\ & & & \mathcal{H} \end{bmatrix}, \quad \mathcal{F}_p = \begin{bmatrix} \mathcal{F} \\ \vdots \\ \mathcal{F} \end{bmatrix}$$

and the off-diagonal blocks are equal to zero. From (20), we have

$$\mathbf{y} = \mathcal{C}_p \psi_0 + \mathcal{D}_p \mathbf{u} \quad (24)$$

Substituting (24) into (23), it follows that

$$(\mathcal{G}_p + \mathcal{H}_p \mathcal{D}_p) \mathbf{u} \leq -\mathcal{H}_p \mathcal{C}_p \psi_0 - \mathcal{F}_p$$

Therefore, problem (16) is equivalent to the following infinite-dimensional QP problem

$$\begin{aligned} \inf_{\mathbf{u}} \quad & \frac{1}{2} \langle \mathbf{u}, \mathcal{L}_0 \mathbf{u} \rangle + \langle \mathbf{u}, \mathcal{L}_1 \psi_0 \rangle \\ \text{s.t.} \quad & \mathcal{M} \mathbf{u} \leq \mathcal{N} + \mathcal{E} \psi_0 \end{aligned} \quad (25)$$

where

$$\mathcal{L}_0 := \mathcal{R}_p + \mathcal{B}_p^* \mathcal{Q}_p \mathcal{B}_p, \quad \mathcal{L}_1 := \mathcal{B}_p^* \mathcal{Q}_p \mathcal{A}_p \quad (26)$$

$$\mathcal{M} := \mathcal{G}_p + \mathcal{H}_p \mathcal{D}_p, \quad \mathcal{N} := -\mathcal{F}_p, \quad \mathcal{E} := -\mathcal{H}_p \mathcal{C}_p \quad (27)$$

For a given horizon length N , there is a polyhedral set of initial conditions for which feasible trajectories exists, over which the receding horizon controller is stabilizing (cf. [29], [36], [38]–[40] and the references therein). The set of all initial conditions for which an optimal solution of (16) exist is a polyhedral set \mathbb{X}_0 which can be characterized as follows (with

respect to the positive cone in ℓ_2)

$$\mathbb{X}_0 = \{x \in \ell_2 : \mathcal{A}_{\text{int}} x \leq b_{\text{int}}\},$$

where $\mathcal{A}_{\text{int}} \in \mathcal{L}(\ell_2)$. Formulation in (25) gives a clear picture of the relationship between the control input variables and initial condition ψ_0 . Problem (25) is a multi-parametric optimization problem on \mathbb{G} (the spatial domain) in which ψ_0 is treated as vector of parameters.

Note that Banach algebra $\mathcal{S}_\tau(\mathcal{C})$ is closed under block-composition operation, i.e., if $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathcal{S}_\tau(\mathcal{C})$, then

$$\begin{bmatrix} \mathcal{A} \\ \mathcal{B} \end{bmatrix} \in \mathcal{S}_\tau(\mathcal{C}) \quad \text{and} \quad \begin{bmatrix} \mathcal{C} & \mathcal{D} \end{bmatrix} \in \mathcal{S}_\tau(\mathcal{C})$$

Using this property and the closure under multiplication property of $\mathcal{S}_\tau(\mathcal{C})$, one can conclude that $\mathcal{A}_p, \mathcal{B}_p, \mathcal{C}_p, \mathcal{D}_p, \mathcal{Q}_p, \mathcal{R}_p, \mathcal{G}_p, \mathcal{H}_p \in \mathcal{S}_\tau(\mathcal{C})$. Therefore, it follows that $\mathcal{L}_0, \mathcal{L}_1, \mathcal{M}, \mathcal{E} \in \mathcal{S}_\tau(\mathcal{C})$. From positive definiteness of operator \mathcal{R} , it also follows that $\mathcal{L}_0 \succ 0$. Therefore, problem (25) is a convex infinite-dimensional QP, and that it can be written in the form of (2).

VI. ANALYSIS OF INFINITE-DIMENSIONAL QUADRATIC PROGRAMMING PROBLEM

In this section, we study the structural properties of the optimal solution of the infinite-dimensional QP problem

$$\begin{aligned} \inf_{x \in \ell_2} \quad & \frac{1}{2} \langle x, \mathcal{P}x \rangle + \langle c, x \rangle \\ \text{subject to:} \quad & \mathcal{G}x \leq b \end{aligned} \quad (28)$$

As mentioned earlier, problem (28) can be viewed as a multi-parametric quadratic optimization (MPQP) problem, in which parameters are the components of $b \in \ell_2$. Assume that there exists a compact set $\mathbb{P} \subset \ell_2$ of parameters for which for every $b \in \mathbb{P}$ an optimal solution of (28) exists. When problem (28) is considered on a finite-dimensional vector space, it can be shown that \mathbb{P} can be partitioned into countably many partitions (cf. [29] and references in there) over each of which the optimal solution of (28) is an affine function of the parameters. In the following, we will show that a similar explicit representation exists for the optimal solution when problem (28) is treated in an infinite-dimensional vector space. In this scenario, however, the affine representation is in the form of a convolution sum with some matrix gains appearing as the kernel. We will prove that convolution kernel corresponding to the optimal solution on each partition, exhibit decay in the spatial domain at a rate proportional to the inverse of the corresponding coupling characteristic function of the system. We recall the following theorem which is Theorem 1 in chapter 8 of [2].

Theorem 3: Let X be a vector space, Z a normed space, Ω a convex subset of X , and P the positive cone in Z . Assume that P contains an interior point. Let f be a real-valued convex functional on Ω and G a convex mapping from Ω into Z . Assume the existence of a point $x_1 \in \Omega$ for which $G(x_1) \prec 0$ (i.e., $G(x_1)$ is an interior point of $N = -P$). Let

$$\mu_0 = \inf_{x \in \Omega} f(x) \quad \text{subject to:} \quad G(x) \leq 0 \quad (29)$$

and assume μ_0 is finite. Then

$$\mu_0 = \max_{z^* \succeq 0} \inf_{x \in \Omega} \{f(x) + \langle G(x), z^* \rangle\}. \quad (30)$$

and the maximum is achieved by an element $z_0^* \succeq 0$ in Z^* , where the inequality is defined with respect to the positive cone P^\oplus . Furthermore, if the infimum is achieved in (29) by an $x_0 \in \Omega$, then

$$\langle G(x_0), z_0^* \rangle = 0 \quad (31)$$

and x_0 minimizes $f(x) + \langle G(x), z_0^* \rangle$ with $x \in \Omega$.

The above result provides a precise method to explain primal-dual relationship, and that formulate the necessary optimality conditions for a convex infinite-dimensional optimization problem such as problem (28). The outcome of this theorem can be thought of as the generalized Karush-Kuhn-Tucker conditions in infinite-dimensions. In the following, as we will see the necessary optimality conditions for problem (28) are very similar to its finite-dimensional counterpart. The following theorem is the main result of this paper.

Theorem 4: Assume that $\mathcal{P}, \mathcal{G} \in \mathcal{S}_\tau(\mathcal{C})$, \mathcal{P} is positive definite, and $b, c \in \ell_2$ in problem (28). Suppose that some combination of constraints in (28) are active and the corresponding rows to these active constraints from operator \mathcal{G} form onto operator $\tilde{\mathcal{G}}$. Let $\mathbb{B} \subseteq \mathbb{P}$ be the set of all $b \in \mathbb{P}$ so that such combinations are active at the optimal solution. Then the optimal solution of (28), as well as the corresponding Lagrange multipliers to index $k \in \mathbb{G}$, are

(a) *affine* maps of b over \mathbb{B} , especially

$$\bar{x}_k = \sum_{i \in \mathbb{G}} [\mathcal{K}]_{ki} b_i + \sum_{i \in \mathbb{G}} [\mathcal{K}_0]_{ki} c_i \quad (32)$$

for some linear bounded operators \mathcal{K} and \mathcal{K}_0 .

(b) *spatially distributed*, in the sense that the coupling decays in the spatial domain at a rate proportional to the inverse of the corresponding coupling characteristic function of the system, i.e.,

$$\begin{aligned} \|[\mathcal{K}]_{ki}\| &\leq \frac{\kappa}{\chi_\alpha(\text{dis}(k, i))} \\ \|[\mathcal{K}_0]_{ki}\| &\leq \frac{\kappa_0}{\chi_\alpha(\text{dis}(k, i))} \end{aligned}$$

for some $\kappa, \kappa_0 > 0$ and all $\alpha \in [0, \tau_2)$, where $0 < \tau_2 < \tau$ and τ_2 is determined explicitly in the proof.

Proof: We may assume that \mathbb{P} is nonempty. The quadratic cost functional in (28) is *Fréchet* differentiable [2]. By applying Theorem 3 to (28), we have the following conditions

$$\mathcal{P}\bar{x} + c + \mathcal{G}^*\lambda = 0 \quad (33)$$

$$\lambda_i^j \left(\sum_{k \in \mathbb{G}} [\mathcal{G}]_{ik} \bar{x}_k - b_i \right)^j = 0 \quad (34)$$

$$\begin{aligned} j &= 1, \dots, n \\ \lambda &\succeq 0 \end{aligned} \quad (35)$$

$$\begin{aligned} \mathcal{G}x &\preceq b \\ i &\in \mathbb{G} \end{aligned} \quad (36)$$

where $\bar{x} \in \ell_2$ is the optimal solution, $\lambda = \underset{i \in \mathbb{G}}{\text{cat}} \lambda_i \in \ell_2$ the corresponding Lagrange multipliers, $\lambda_i \in \mathbb{R}^n$, and λ_i^j or $(\cdot)^j$ represents the j^{th} row. Note that the positive cone P in ℓ_2 is defined by

$$P = \{x \in \ell_2 : x = \underset{i \in \mathbb{G}}{\text{cat}} x_i, x_i \geq 0 \text{ for all } i \in \mathbb{G}\}$$

and by definition $P^\oplus = P$. Condition (34) is the so called complementary slackness. It follows from the fact that at optimum we have

$$G(\bar{x}) = \mathcal{G}\bar{x} - b \preceq 0 \quad \text{and} \quad \lambda \succeq 0 \quad (37)$$

and condition (31) can be written as

$$\langle G(\bar{x}), \lambda \rangle = \sum_{i \in \mathbb{G}} \sum_{j=1}^n \lambda_i^j \left(\sum_{k \in \mathbb{G}} [\mathcal{G}]_{ik} \bar{x}_k - b_i \right)^j = 0 \quad (38)$$

According to (37), every term inside the summations in (38) is nonpositive. Therefore, each term has to be zero.

Since \mathcal{P} is bounded and positive definite, it has an algebraic inverse on $\mathcal{L}(\ell_2)$. Equation (33) results in

$$\bar{x} = -\mathcal{P}^{-1} (\mathcal{G}^*\lambda + c) \quad (39)$$

According to equation (34) and (35), all Lagrange multipliers λ_i^j corresponding to inactive constraints are zero, and the Lagrange multipliers corresponding to active constraints, stacked in column vectors $\tilde{\lambda}_i$ (accordingly, vectors \tilde{b}_i can be formed using elements of b_i), are nonnegative numbers. Therefore, we can form linear operator $\tilde{\mathcal{G}}$ whose block elements $[\tilde{\mathcal{G}}]_{ik}$ are obtained by deleting rows corresponding to the inactive constraints from block elements $[\mathcal{G}]_{ik}$ of \mathcal{G} . We may equivalently represent this operation as follows

$$\tilde{\mathcal{G}} = \mathcal{E}\mathcal{G}$$

where \mathcal{E} is a bounded linear operator that is obtained by deleting rows from \mathcal{I} (identity operator) which correspond to the active constraints. Note that $\mathcal{E} \in \mathcal{S}_\tau(\mathcal{C})$. From (34) and our assumptions, for every $b \in \mathbb{B}$ we have the following equation

$$\tilde{\mathcal{G}} \bar{x} - \tilde{b} = 0 \quad (40)$$

in which $\tilde{b} = \underset{i \in \mathbb{G}}{\text{cat}} \tilde{b}_i$ or, equivalently, $\tilde{b} = \mathcal{E}b$. This equation allows us to solve it along with (39) for $\tilde{\lambda}$ where $\tilde{\lambda} = \underset{i \in \mathbb{G}}{\text{cat}} \tilde{\lambda}_i$ or, equivalently, $\tilde{\lambda} = \mathcal{E}\lambda$. Using (39), it follows that

$$\bar{x} = -\mathcal{P}^{-1} \tilde{\mathcal{G}}^* \tilde{\lambda} - \mathcal{P}^{-1} c \quad (41)$$

Substituting (41) into (40), results in

$$\tilde{\mathcal{G}} \mathcal{P}^{-1} \tilde{\mathcal{G}}^* \tilde{\lambda} = -\tilde{b} - \tilde{\mathcal{G}} \mathcal{P}^{-1} c$$

Operator $\tilde{\mathcal{G}}$ is onto and $\tilde{\mathcal{G}}^*$ is (1-1). Thus, linear operator $\tilde{\mathcal{G}} \mathcal{P}^{-1} \tilde{\mathcal{G}}^*$ is invertible and positive definite, and that it has an algebraic inverse on $\mathcal{L}(\ell_2)$. Therefore, we have

$$\tilde{\lambda} = -(\tilde{\mathcal{G}} \mathcal{P}^{-1} \tilde{\mathcal{G}}^*)^{-1} \tilde{b} - (\tilde{\mathcal{G}} \mathcal{P}^{-1} \tilde{\mathcal{G}}^*)^{-1} \tilde{\mathcal{G}} \mathcal{P}^{-1} c \quad (42)$$

and

$$\bar{x} = \mathcal{K} b + \mathcal{K}_0 c \quad (43)$$

where

$$\begin{aligned}\mathcal{K} &:= \mathcal{P}^{-1}\tilde{\mathcal{G}}^* \left(\tilde{\mathcal{G}}\mathcal{P}^{-1}\tilde{\mathcal{G}}^*\right)^{-1} \mathcal{E} \\ \mathcal{K}_0 &:= \mathcal{P}^{-1}\tilde{\mathcal{G}}^* \left(\tilde{\mathcal{G}}\mathcal{P}^{-1}\tilde{\mathcal{G}}^*\right)^{-1} \tilde{\mathcal{G}}\mathcal{P}^{-1} - \mathcal{P}^{-1}\end{aligned}$$

Equation (43) can be written in the form of convolution sums as follows

$$\bar{x}_k = \sum_{i \in \mathbb{G}} [\mathcal{K}]_{ki} b_i + \sum_{i \in \mathbb{G}} [\mathcal{K}_0]_{ki} c_i$$

for all $k \in \mathbb{G}$. This proves part (a) of the theorem that the optimal solution \bar{x} is an affine function of parameters b_i for all $i \in \mathbb{G}$.

Since $\mathcal{G}, \mathcal{E} \in \mathcal{S}_\tau(\mathcal{C})$, we have $\tilde{\mathcal{G}} \in \mathcal{S}_\tau(\mathcal{C})$. According to lemma 1, $\mathcal{P}^{-1} \in \mathcal{S}_{\tau_1}(\mathcal{C})$ for some $0 < \tau_1 < \tau$. It follows that $\tilde{\mathcal{G}}\mathcal{P}^{-1}\tilde{\mathcal{G}}^* \in \mathcal{S}_{\tau_1}(\mathcal{C})$. Applying lemma 1 one more time, it results in

$$\left(\tilde{\mathcal{G}}\mathcal{P}^{-1}\tilde{\mathcal{G}}^*\right)^{-1} \in \mathcal{S}_{\tau_2}(\mathcal{C})$$

for some $0 < \tau_2 < \tau_1$. Therefore, using the close under multiplication property of Banach algebra $\mathcal{S}_{\tau_2}(\mathcal{C})$, it concludes that the gain operator \mathcal{K} satisfies

$$\mathcal{K} \in \mathcal{S}_{\tau_2}(\mathcal{C}).$$

Note that theorem 2 can be used to quantify the value of the decay margin τ_2 . Thus, it follows that

$$\|[\mathcal{K}]_{ki}\| \leq \frac{\kappa}{\chi_\alpha(\text{dis}(k, i))}$$

for some $\kappa > 0$ and all $\alpha \in [0, \tau_2)$. It is straightforward to show that a similar result holds for operator \mathcal{K}_0 , i.e.,

$$\|[\mathcal{K}_0]_{ki}\| \leq \frac{\kappa_0}{\chi_\alpha(\text{dis}(k, i))}$$

for some $\kappa_0 > 0$ and all $\alpha \in [0, \tau_2)$. ■

The result of theorem 4 can be used to characterize the parameter set \mathbb{B} . The empty set is a compact set by the fact that every finite set is compact. Assume that \mathbb{B} is nonempty. The optimal solution has to satisfy constraint (36) and by (35) the Lagrange multipliers (42) must be nonnegative. Therefore, the parameter set \mathbb{B} can be represented as

$$\mathbb{B} = \{ b \in \mathbb{P} : \mathcal{B}_0 b + d_0 \preceq 0, \mathcal{B}_1 b + d_1 \preceq 0 \}$$

where

$$\begin{aligned}\mathcal{B}_0 &:= \mathcal{G}\mathcal{K} - \mathcal{I}, \quad d_0 := -\mathcal{G}\mathcal{K}_0 c \\ \mathcal{B}_1 &:= \left(\tilde{\mathcal{G}}\mathcal{P}^{-1}\tilde{\mathcal{G}}^*\right)^{-1} \mathcal{E}, \quad d_1 := \left(\tilde{\mathcal{G}}\mathcal{P}^{-1}\tilde{\mathcal{G}}^*\right)^{-1} \tilde{\mathcal{G}}\mathcal{P}^{-1} c\end{aligned}$$

The linear operators \mathcal{B}_0 and \mathcal{B}_1 are bounded, therefore, it concludes that the set \mathbb{B} is compact.

VII. CONCLUSIONS

In this paper we studied the spatial structure of infinite-dimensional quadratic programming problems where the cost functional is defined using a spatially decaying (SD) operator. By applying duality theory and complementary slackness conditions, we proved that the optimal solution of a convex infinite-dimensional quadratic programming is piecewise affine which can be represented as convolution sums.

Furthermore, it was shown that the Banach algebra of SD operators is closed under inversion. Also, an explicit formula was proposed for the decay margin of the inverse operator. We used this to prove that the kernel of each convolution sum decays in the spatial domain. These results suggest that in large-scale multi-parametric quadratic optimization problems, the optimal centralized solution is an inherently local function of the parameters, as the influence of farther away nodes decays in spatial domain. This raises and justifies the possibility of spatial truncation in the optimal solution without loss of performance. Future research will be focused on designing robust distributed algorithms based on tools developed in this paper to solve infinite-dimensional linear and quadratic programming problems. We suspect that similar results can be extended to more general class of infinite-dimensional optimization problems such as those whose cost functional is defined in terms of a p norm of an SD operator where p is not necessarily equal to 1, 2 or ∞ . Furthermore, we suspect that similar results might be true in large-scale Semidefinite programming problems

VIII. APPENDIX : PROOF OF THEOREM 2

Proof: According to lemma 1, the inverse operator is SD and that is bounded on ℓ_p for all $1 \leq p \leq \infty$. Consider the following equation

$$\mathcal{Q} x = y. \quad (44)$$

Fix $k \in \mathbb{G}$, y can be selected as

$$|y_i| \leq c_1 \frac{1}{\gamma(\text{dis}(i, k))}. \quad (45)$$

for some coupling characteristic function $\gamma \in \mathcal{C}$ and $c_1 > 0$. It follows that

$$\|y\|_\infty \leq c_1.$$

Pick a neighborhood of index k with radius $R > 0$ and define the auxiliary quantity

$$\omega_i = x_i \chi_\alpha(\min(\text{dis}(i, k), R)). \quad (46)$$

for some $0 \leq \alpha < b$. It is easy to check that

$$\|\omega\|_\infty \leq \|x\|_\infty \chi_\alpha(R).$$

and that

$$\|\omega\|_\infty \leq a_0 c_1. \quad (47)$$

where $a_0 = \chi_\alpha(R) \|\mathcal{Q}^{-1}\|_{\infty/\infty}$. Also, we have

$$[\mathcal{Q}\omega]_i = \sum_{j \in \mathbb{G}} [\mathcal{Q}]_{ij} x_j \chi_\alpha(\min(\text{dis}(j, k), R)). \quad (48)$$

From (44) we have $[\mathcal{Q}x]_i = y_i$, subtract this from (48),

$$\begin{aligned}[\mathcal{Q}\omega]_i &= y_i \chi_\alpha(\min(\text{dis}(i, k), R)) + \\ &\sum_{j \in \mathbb{G}} [\mathcal{Q}]_{ij} x_j (\chi_\alpha(\min(\text{dis}(j, k), R)) - \chi_\alpha(\min(\text{dis}(i, k), R))) \\ &= y_i \chi_\alpha(\min(\text{dis}(i, k), R)) + \\ &\sum_{j \in \mathbb{G}} [\mathcal{Q}]_{ij} \omega_j \left(1 - \frac{\chi_\alpha(\min(\text{dis}(i, k), R))}{\chi_\alpha(\min(\text{dis}(j, k), R))}\right).\end{aligned}$$

it follows that

$$\begin{aligned} \|\mathcal{Q}\omega\|_\infty &\leq \sup_{i \in \mathbb{G}} |y_i| \chi_\alpha(\min(\text{dis}(i, k), R)) \\ &+ \sup_{i \in \mathbb{G}} \left| \sum_{j \in \mathbb{G}} [\mathcal{Q}]_{ij} \omega_j \left(1 - \frac{\chi_\alpha(\min(\text{dis}(i, k), R))}{\chi_\alpha(\min(\text{dis}(j, k), R))} \right) \right|. \end{aligned} \quad (49)$$

Since χ_α is a nondecreasing function, relation $\chi_\alpha(\min(\text{dis}(i, k), R)) \leq \chi_\alpha(\text{dis}(i, k))$ holds, and for all $\chi_\alpha \preceq \gamma$, it follows that

$$\sup_{i \in \mathbb{G}} |y_i| \chi_\alpha(\min(\text{dis}(i, k), R)) \leq c_1 \quad (50)$$

One can justify the following inequality

$$\min(\text{dis}(i, k), R) - \min(\text{dis}(j, k), R) \leq \text{dis}(i, j)$$

and by definition, we have

$$\frac{\chi_\alpha(\min(\text{dis}(i, k), R))}{\chi_\alpha(\min(\text{dis}(j, k), R))} \leq \chi_\alpha(\text{dis}(i, j)) \quad (51)$$

Apply this to the second term of (49) to get

$$\begin{aligned} &\sup_{i \in \mathbb{G}} \left| \sum_{j \in \mathbb{G}} [\mathcal{Q}]_{ij} \omega_j \left(1 - \frac{\chi_\alpha(\min(\text{dis}(i, k), R))}{\chi_\alpha(\min(\text{dis}(j, k), R))} \right) \right| \\ &\leq \left(\sup_{i \in \mathbb{G}} \sum_{j \in \mathbb{G}} \|[\mathcal{Q}]_{ij}\| (\chi_\alpha(\text{dis}(i, j)) - 1) \right) \|\omega\|_\infty \end{aligned} \quad (52)$$

(50) and (52) gives us

$$\|\mathcal{Q}\omega\|_\infty \leq \Psi(\chi_\alpha) \|\omega\|_\infty + c_1 \quad (53)$$

in which

$$\Psi(\chi_\alpha) = \sup_{i \in \mathbb{G}} \sum_{j \in \mathbb{G}} \|[\mathcal{Q}]_{ij}\| (\chi_\alpha(\text{dis}(i, j)) - 1).$$

Ψ is a bounded continuous function of α on $[0, b)$. Applying (47) to (53) results in

$$\|\mathcal{Q}\omega\|_\infty \leq (\Psi(\chi_\alpha) a_0 + 1) c_1$$

and that

$$\|\omega\|_\infty \leq a_1 c_1 \quad (54)$$

where $a_1 = \|\mathcal{Q}^{-1}\|_{\infty/\infty} (\Psi(\chi_\alpha) a_0 + 1)$. By repeating this process, we get the following iterative equation

$$a_{k+1} = \|\mathcal{Q}^{-1}\|_{\infty/\infty} (\Psi(\chi_\alpha) a_k + 1) \quad (55)$$

for $k = 0, 1, \dots$. Note that different selection for R only changes the initial condition of (55). Indeed, in the following we will show that the analysis is independent of choice of R , and that we can select R large enough to cover a reasonable part of the graph around the node k (even for finite graphs the entire graph). In what follows, we will prove that equation (55) has a unique fixed point. $\Psi(\chi_\alpha)$ is a continuous function of α and $\Psi(\chi_0) = \Psi(\mathbf{1}) = 0$. Therefore, there exists $\hat{b} > 0$ such that $\Psi(\chi_\alpha) < \|\mathcal{Q}^{-1}\|_{\infty/\infty}^{-1}$ for all $0 \leq \alpha < \hat{b}$. On the

other hand,

$$\frac{\partial a_{k+1}}{\partial a_k} = \Psi(\chi_\alpha) \|\mathcal{Q}^{-1}\|_{\infty/\infty} < 1$$

for all $0 \leq \alpha < \hat{b}$. Therefore, (55) has a unique fixed point

$$a^*(\chi_\alpha) = \frac{1}{\|\mathcal{Q}^{-1}\|_{\infty/\infty}^{-1} - \Psi(\chi_\alpha)}$$

This leads us to the final inequality that is independent of R

$$\|\omega\|_\infty \leq c_1 a^*(\chi_\alpha)$$

for all $\{\alpha : \chi_\alpha \preceq \gamma\} \cap \{0 \leq \alpha < \hat{b}\}$. For all $i \in \mathbb{G}$, we have

$$|\omega_i| \leq c_1 a^*(\chi_\alpha)$$

and substituting (46) results in

$$|x_i| \leq c_1 a^*(\chi_\alpha) \frac{1}{\chi_\alpha(\text{dis}(i, k))} \quad (56)$$

for $\{\alpha : \chi_\alpha \preceq \gamma\} \cap \{0 \leq \alpha < \hat{b}\}$. By selecting y as in (45) to have only one nonzero component in the the k^{th} entry, and that to be

$$y_k^* = \arg \sup_{y_k \in \mathbb{R}^n} \frac{|[\mathcal{Q}^{-1}]_{ik} y_k|}{|y_k|}$$

with $\chi_{\hat{b}} \prec \gamma$ and $c_1 = |y_k^*|$. Thus from (56) we will have

$$\|[\mathcal{Q}^{-1}]_{ik}\| \leq a^*(\chi_\alpha) \frac{1}{\chi_\alpha(\text{dis}(i, k))}.$$

for all $0 \leq \alpha < \hat{b}$. This completes the proof. \blacksquare

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