

# Optimal Control of Spatially Distributed Systems

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**Abstract**—In this paper, we study the structural properties of optimal control of spatially distributed systems. Such systems consist of an infinite collection of possibly heterogeneous linear control systems that are spatially interconnected via certain distant dependent coupling functions over arbitrary graphs. The key idea of the paper is the introduction of a special class of operators called spatially decaying (SD) operators. Such operators can be thought of as generalization of translation invariant operators used in the study of spatially invariant systems as well as operators defined based on nearest neighbor couplings in networked dynamic systems. We study the structural properties of infinite-horizon linear quadratic optimal controllers for such systems by analyzing the spatial structure of the solution to the corresponding operator Lyapunov and Riccati equations. We prove that the kernel of the optimal feedback of each subsystem decays in the spatial domain at a rate proportional to the inverse of the corresponding coupling function of the system. When the coupling is nearest neighbor based or exponentially decaying, we show that the coupling in the optimal controls decays exponentially in space. In the case when the systems are coupled using an algebraic function, it is proven that the coupling in the optimal controls decays algebraically. Our theoretical results are verified by numerical simulations.

## I. INTRODUCTION

Analysis and synthesis of distributed coordination and control algorithms for networked dynamic systems has become a vibrant part of control theory research. From consensus and agreement problems to formation control and sensing and coverage problems, researchers have been interested in control algorithms that are spatially distributed and designed to achieve a global objective such as consensus or coverage using only local interactions [1]–[9]. Other examples of research on distributed dynamic systems include distributed optimization-based control unmanned aerial vehicles [10], automated highway systems [11]–[12], cross-directional control systems for industrial paper machines [13], large segmented telescopes [14], power distribution systems [15], and arrays of microcantilevers for massively parallel data storage [16]—only to name a few of these applications.

In addition to the above results, several authors have studied the problem of optimal control of certain classes of spatially distributed systems with symmetries in their spatial structure (cf. Figures 1 and 2). In [17], Bamieh *et al.* used spatial Fourier transforms and operator theory to study optimal control of linear spatially invariant systems with standard  $\mathcal{H}_2$  (LQ), and  $\mathcal{H}_\infty$  criteria. It was shown that such problems can be tackled by solving a parameterized family of finite-dimensional

problems in Fourier domain. Furthermore, the authors show that the resulting optimal controllers have an inherent spatial locality similar to the underlying system.

In [18], the authors developed conditions for well-posedness, stability, and performance of spatially interconnected systems whose model consist of homogeneous units on a discrete group (e.g. over a one-dimensional or two-dimensional lattice or ring), in terms of linear matrix inequalities (LMI). These results were later extended to systems with certain types of boundary conditions [19]. In [20], the applicability of results of [18] is extended to a larger class of interconnection topologies with arbitrary discrete symmetry groups. Another related result is reported in [21], where the authors use a scaled small gain theorem, to give a unified interpretation to the analysis results of [20]. Another related result is that of [22], where the problem of distributed controller design with a “funnel causality” constraint is shown to be a convex problem, provided that the plant has a similar funnel-causality structure, and the propagation speed in the controller is at least as fast as those in the plant. Another relevant result is due to the authors in [23] in which a decentralized control approach to spatially invariant systems is studied and the interactions between agents are modeled as disturbances satisfying certain magnitude bounds. In [24], the backstepping approach is utilized to address stabilization, regulation, and asymptotic tracking of nominal systems and systems with parametric uncertainties on lattices. It is also shown that the designed controllers have the same architecture as the original plant. Another much older but related work on this subject was reported in [25] where homogeneous interconnected systems are studied using  $\mathcal{Z}$ -transform analysis. Furthermore, it is shown that many homogeneous large-scale systems can be reasonably approximated by an infinite number of coupled identical subsystems.

The authors in [26] employ tools from dissipativity theory to synthesize optimal controllers for spatially interconnected systems with non-identical units over an arbitrary graph. Another interesting work in this area is reported in [27] where the authors use operator theoretic tools, motivated by results of [28] to analyze time-varying systems, and design optimal controllers for heterogeneous systems which are not shift invariant with respect to spatial or temporal variables. In [29] and [13], the problem of synthesizing and implementing a distributed controller with practical/industrial applications is considered. Another recent work in this area is reported in [30] in which the authors introduce the notion of *quadratic invariance* for a constraint set (e.g. sparsity constraints on communication structure of plant and controller). Using this notion, the authors show that the problem of constructing optimal controllers with certain sparsity patterns on the information structure can be cast as a convex optimization problem. In all of these

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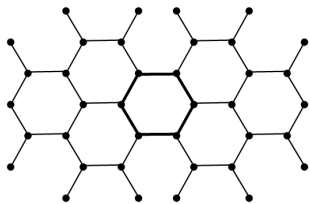


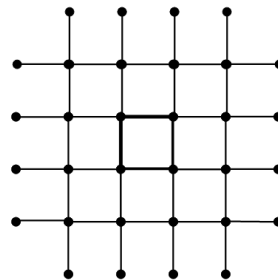
Fig. 1. Spatial invariance on a hexagonal array.

papers, except for [17], [30], a synthesis-based approach is taken to develop a control design method which yields a distributed controller with possibly the same architecture as the underlying plant.

This paper is very close in spirit to [17]. The objective of this paper is to analyze the spatial structure of infinite horizon optimal controllers of spatially distributed systems. Here, we extend the results of [17] to *heterogeneous* systems with *arbitrary* spatial structure and show that quadratically optimal controllers inherit the same spatial structure as the original plant. The key point of departure from [17] is that the systems considered in this work are not spatially invariant and the corresponding operators are not translation invariant either. The spatial structures studied in [17] are Locally Compact Abelian (LCA) groups [31] such as  $(\mathbb{Z}, +)$  and  $(\mathbb{Z}_n, \oplus)$ . As a result, the group operation naturally induces a translation operator for functions defined on the group.

However, when the dynamics of individual subsystems are not identical and the spatial structure does not necessarily enjoy the symmetries of LCA groups, standard tools such as Fourier analysis cannot be used to analyze the system.

To address this issue, a new class of linear operators, called *spatially decaying* (SD) operators, are introduced that are natural extension of linear translation invariant operators. Roughly speaking an operator is SD with respect to an induced norm if a certain auxiliary operator formed by blockwise exponential (or algebraic) inflation of the operator remains bounded. It is shown that such operators exhibit a localized behavior in spatial domain, i.e., the norm of blocks in the matrix representation of the operator decay exponentially or algebraically in space. A trivial example of an SD operator are those representing nearest neighbor coupling such as the graph *Laplacian*, which has received significant attention in cooperative control literature recently [1], [5], [6], [8]. Another important subclass includes any bounded translation invariant operator. It turns out that the coupling between subsystems in many well-known cooperative control and networked control problems can be characterized by an SD operator. A linear control system is called *spatially decaying* if the operators in its state-space representation are SD. It is shown that the space of SD operators  $\mathcal{S}(\mathcal{G})$  is a normed vector space with respect to a specific operator-norm which is not induced and is denoted by  $\|\cdot\|_{\mathcal{S}}$ . Furthermore such operators equipped with the norm form a Banach algebra. Using this result, we prove certain closure properties of this space with respect to convergent sequences on  $\ell_p$ . This will then enable us to prove that the unique solution of *Lyapunov* and algebraic *Riccati* equations (ARE) corresponding to SD system are

Fig. 2. Spatially invariant system over a two-dimensional lattice  $\mathbb{G} = \mathbb{Z} \times \mathbb{Z}$ 

indeed SD themselves. As a result, the corresponding optimal controllers are SD and spatially localized, meaning that in the optimal controller, the gain of subsystems that are “farther away” from a given subsystem decays in space and the resulting controller is inherently localized. Specifically, we show that when the coupling function between subsystems is exponential, the size of the kernel of corresponding optimal controller decays exponentially, and when the coupling is algebraic, optimal controllers decay algebraically in space. For the specific case of nearest neighbor coupling, the decay is shown to be exponential.

The machinery developed in this paper can be used to analyze the spatial structure of a broader range of optimal control problems such as constrained, finite horizon control or Model Predictive Control as well as . This problem has been analyzed in detail in [32], and [33]. This paper is organized as follows. We introduce the notation and the basic concepts used throughout the paper in Section II. The optimal control problem for spatially distributed linear systems and the motivation of the paper are presented in Section III. The concept of spatially decaying operators is introduced in Section V. Three important type of SD operators are studied in details in Section VI. Results of Section VII which is about the properties of SD operators are utilized in Section VIII to show that the solutions of Lyapunov equations and Algebraic Riccati Equations ARE inherits spatial locality. Simulation results are included in Section IX, in which we demonstrate the decay of spatial coupling in the optimal solution by considering interconnection of a large network of interconnected units on an arbitrary connected graph. Finally, our concluding remarks are presented in Section X.

## II. PRELIMINARIES

The notation used in this paper is fairly standard.  $\mathbb{R}$  denotes the set of real numbers,  $\mathbb{R}^+$  the set of nonnegative real numbers,  $\mathbb{C}$  the set of complex numbers, and  $\mathbb{S}^1$  the unit circle in  $\mathbb{C}$ . Consider the Hilbert spaces  $\mathbb{H}_i$  equipped with inner products  $\langle \cdot, \cdot \rangle_{\mathbb{H}_i}$  for  $i \in \mathbb{G}$  where  $\mathbb{G}$  is an index set. We refer to  $\mathbb{G}$  as the spatial domain (see Fig. 1 and 2 for two specific examples). The inner product on each Hilbert space  $\mathbb{H}_i$  induces the norm  $\|x_i\|_{\mathbb{H}_i} = \sqrt{\langle x_i, x_i \rangle_{\mathbb{H}_i}}$  for all  $x_i \in \mathbb{H}_i$ . Whenever it is clear from the context, all induced norms of linear maps between two Hilbert spaces  $\mathbb{H}_i$  and  $\mathbb{H}_j$  are simply denoted by  $\|\cdot\|$ . The Banach space  $\ell_p(\mathbb{G})$  for  $1 \leq p < \infty$  is defined to be the set of all sequences  $x = (x_i)_{i \in \mathbb{G}}$  in which

$x_i \in \mathbb{H}_i$  satisfying

$$\sum_{i \in \mathbb{G}} \|x_i\|_{\mathbb{H}_i}^p < \infty$$

endowed with the norm

$$\|x\|_p := \left( \sum_{i \in \mathbb{G}} \|x_i\|_{\mathbb{H}_i}^p \right)^{\frac{1}{p}}.$$

The Banach space  $\ell_\infty(\mathbb{G})$  denotes the set of all bounded sequences endowed with the norm

$$\|x\|_\infty := \sup_{i \in \mathbb{G}} \|x_i\|_{\mathbb{H}_i}.$$

Throughout the paper, we will use the shorthand notation  $\ell_p$  for  $\ell_p(\mathbb{G})$ . The space  $\ell_2$  is a Hilbert space with inner product

$$\langle x, y \rangle := \sum_{i \in \mathbb{G}} \langle x_i, y_i \rangle_{\mathbb{H}_i}$$

for all  $x, y \in \ell_2$ . An operator  $\mathcal{Q} : \ell_p \rightarrow \ell_q$  for  $1 \leq p, q \leq \infty$  is bounded if it has a finite induced norm, i.e., the following quantity

$$\|\mathcal{Q}\|_{\ell_p \rightarrow \ell_q} := \sup_{\|x\|_p=1} \|\mathcal{Q}x\|_q \quad (1)$$

is bounded. The identity operator is denoted by  $\mathcal{I}$ . The set of all bounded linear operators of  $\ell_p$  into  $\ell_q$  for some  $1 \leq p, q \leq \infty$  is denoted by  $\mathcal{L}(\ell_p, \ell_q)$ . The space  $\mathcal{L}(\ell_p, \ell_q)$  equipped with norm (1) is a Banach space (cf. [34]), and in the case where the initial and target spaces are both  $\ell_p$ , we use the notation  $\mathcal{L}(\ell_p)$ . An operator  $\mathcal{Q} \in \mathcal{L}(\ell_p)$  has an *algebraic* inverse if it has an inverse  $\mathcal{Q}^{-1}$  in  $\mathcal{L}(\ell_p)$  [34]:

$$\mathcal{Q}\mathcal{Q}^{-1} = \mathcal{Q}^{-1}\mathcal{Q} = \mathcal{I}.$$

The adjoint operator of  $\mathcal{Q} \in \mathcal{L}(\ell_2)$  is the operator  $\mathcal{Q}^*$  in  $\mathcal{L}(\ell_2)$  such that  $\langle \mathcal{Q}x, y \rangle = \langle x, \mathcal{Q}^*y \rangle$  for all  $x, y \in \ell_2$ . An operator  $\mathcal{Q}$  is self-adjoint if  $\mathcal{Q} = \mathcal{Q}^*$ . An operator  $\mathcal{Q} \in \mathcal{L}(\ell_2)$  is *positive definite*, shown as  $\mathcal{Q} \succ 0$ , if there exists a number  $\alpha > 0$  such that

$$\langle x, \mathcal{Q}x \rangle > \alpha \|x\|_2^2$$

for all nonzero  $x \in \ell_2$ .

The set of all functions from  $A \subseteq \mathbb{R}$  into  $\mathbb{R}$  is a vector space  $\mathcal{F}$  over  $\mathbb{R}$ . For  $f_1, f_2 \in \mathcal{F}$ , the notation  $f_1 \preceq f_2$  will be used to mean the pointwise inequality  $f_1(s) \leq f_2(s)$  for all  $s \in A$ . A family of *seminorms* on  $\mathcal{F}$  is defined as  $\{\|\cdot\|_T \mid T \in \mathbb{R}^+\}$  in which

$$\|f\|_T := \sup_{s \leq T} |f(s)|$$

for all  $f \in \mathcal{F}$ . The topology generated by all open  $\|\cdot\|_T$ -balls is called the topology generated by the family of seminorms and is denoted by  $\|\cdot\|_T$ -topology. Continuity of a function in this topology is equivalent to continuity in every seminorm in the family.

*Remark 1:* Although the results of section III is set up in a general framework, in this paper we are interested in linear operators  $\mathcal{Q} : \ell_p \rightarrow \ell_q$  for some  $1 \leq p, q \leq \infty$  which have

matrix representations

$$\mathcal{Q} \mapsto \begin{bmatrix} \ddots & & & \\ & \ddots & & \\ & & [\mathcal{Q}]_{k_i} & \\ & & & \ddots \end{bmatrix}$$

where the block element  $[\mathcal{Q}]_{k_i}$  is a linear map from  $\mathbb{H}_i$  into  $\mathbb{H}_k$ . Furthermore, it is assumed that  $\mathbb{H}_i = \mathbb{C}^n$  where  $n > 0$  is an integer.

### III. OPTIMAL CONTROL OF SPATIALLY DISTRIBUTED SYSTEMS

We begin by considering a continuous-time linear model for spatially distributed systems over a discrete spatial domain  $\mathbb{G}$  described by

$$\frac{d}{dt}\psi(t) = (\mathcal{A}\psi)(t) + (\mathcal{B}u)(t) \quad (2)$$

$$y(t) = (\mathcal{C}\psi)(t) + (\mathcal{D}u)(t) \quad (3)$$

with the initial condition  $\psi(0) = \psi_0$ . All signals are assumed to be in  $L_2([0, \infty); \ell_2)$  space: at each time instant  $t \in [0, \infty)$ , signals  $\psi(t)$ ,  $u(t)$ ,  $y(t)$  are assumed to be in  $\ell_2$ . The state-space operators  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$ ,  $\mathcal{D}$  are assumed to be time-invariant linear operators from  $\ell_2$  to itself. The following assumption guarantees existence and uniqueness of classical solutions of the system given by (2)-(3) (cf. Chapter 3 of [35] for more details).

*Assumption 1:* The semigroup generated by  $\mathcal{A}$  is *strongly continuous* on  $\ell_2$ .

The following is an example of a spatially distributed system on  $\mathbb{G} = \mathbb{Z}$ .

*Example 1:* Consider the general one-dimensional heat equation for a bi-infinite bar [36]

$$\frac{\partial}{\partial t}\psi(x, t) = \frac{\partial}{\partial x} \left( c(x) \frac{\partial}{\partial x} \psi(x, t) \right) + b(x)u(x, t)$$

where  $x$  is the spatial independent variable,  $t$  is the temporal independent variable,  $\psi(x, t)$  is the temperature of the bar, and  $u(x, t)$  is a distributed heat source. The thermal conductivity  $c$  is only a function of  $x$  and is differentiable with respect to  $x$ . The boundary conditions are assumed to be  $\psi(\infty, t) = \psi(-\infty, t) = 0$ . By inserting finite difference approximation for the spatial partial derivatives, the following continuous-time, discrete-space model can be obtained:

$$\begin{aligned} \frac{\partial}{\partial t}\psi(x_k, t) &= c'(x_k) \left( \frac{\psi(x_{k-1}, t) - \psi(x_k, t)}{\delta} \right) \\ &+ c(x_k) \left( \frac{\psi(x_{k-1}, t) - 2\psi(x_k, t) + \psi(x_{k+1}, t)}{\delta^2} \right) \\ &+ b(x_k)u(x_k, t) \end{aligned}$$

where  $c'(x) = \frac{d}{dx}c(x)$ . The discretization is performed with equal spacing  $\delta = x_k - x_{k-1}$  of the points  $x_k$  such that there is an integer number of points in space. Hence, after discretization the spatial domain is  $\mathbb{G} = \mathbb{Z}$ . This model can be represented as

$$\frac{d}{dt}\psi(t) = (\mathcal{A}\psi)(t) + (\mathcal{B}u)(t)$$

in which the infinite-tuples  $\psi(t) = (\psi(x_k, t))_{k \in \mathbb{G}}$  and  $u(t) = (u(x_k, t))_{k \in \mathbb{G}}$  are the state and control input variables of the infinite-dimensional system and the block elements of the state-space operators  $\mathcal{A}$  and  $\mathcal{B}$  are defined as follows for every

$$[\mathcal{A}]_{ki} = \begin{cases} \frac{c'(x_k)\delta + c(x_k)}{\delta^2} & , i = k - 1 \\ -\frac{c'(x_k)\delta + 2c(x_k)}{\delta^2} & , i = k \\ \frac{c(x_k)}{\delta^2} & , i = k + 1 \\ 0 & , \text{otherwise} \end{cases}$$

and

$$[\mathcal{B}]_{ki} = \begin{cases} b(x_k) & , i = k \\ 0 & , \text{otherwise} \end{cases}$$

for all  $k \in \mathbb{G}$ . One can show that  $\mathcal{A}$  is a bounded operator on  $\ell_2$  and as a result, it generates a uniformly continuous semigroup. Furthermore the generated semigroup is strongly continuous on  $\ell_2$ .

In the next section, we study the exponential stability problem for autonomous systems of the form (4) as well as linear quadratic optimal control problems for systems described by (2)-(3). The focus of this paper would be on LQR problems, however, results are valid for general  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  optimal control problems as well. The key ingredient of the result is to prove spatial locality of solutions of Lyapunov and Riccati equations.

### A. Exponential Stability

Consider the following autonomous system over  $\mathbb{G}$

$$\frac{d}{dt}\psi(t) = (\mathcal{A}\psi)(t) \quad (4)$$

with initial condition  $\psi(0) = \psi_0$ . Suppose that  $\mathcal{A}$  generates a strongly continuous  $C_0$ -semigroup on  $\ell_2$ , denoted by  $\mathcal{T}(t)$ . Exponential stability can be defined as follows.

*Definition 1:* The system (4) is *exponentially stable* if

$$\|\mathcal{T}(t)\|_{\ell_2 \rightarrow \ell_2} \leq M e^{-\alpha t} \quad \text{for } t \geq 0$$

for some  $M, \alpha > 0$ .

Similar to the finite dimensional case, one can define a similar Lyapunov equation in an operator theoretic framework for infinite-dimensional systems. The following theorem from [35] is standard and provides such an extension.

*Theorem 1:* Let  $\mathcal{A}$  be the infinitesimal generator of the  $C_0$ -semigroup  $\mathcal{T}(t)$  on  $\ell_2$  and  $\mathcal{Q}$  a positive definite operator. Then  $\mathcal{T}(t)$  is exponentially stable if and only if the *Lyapunov equation*

$$\langle \mathcal{A}\phi, \mathcal{P}\phi \rangle + \langle \mathcal{P}\phi, \mathcal{A}\phi \rangle + \langle \phi, \mathcal{Q}\phi \rangle = 0 \quad (5)$$

for all  $\phi \in \mathcal{D}(\mathcal{A})$ , has a positive definite solution  $\mathcal{P} \in \mathcal{L}(\ell_2)$ .

Solving the Lyapunov operator equation (5) can be a tedious task in general. However, the complexity of the problem will reduce significantly if the underlying system is spatially invariant with respect to  $\mathbb{G}$  (cf. Section III.B of [17]). This will be discussed in more detail later on.

### B. LQR control of infinite dimensional systems

We now review the basics of linear quadratic regulator theory for infinite-dimensional systems. While the main results of this paper are proven for LQ optimal controllers, similar results can be proven for  $\mathcal{H}_\infty$  and  $\mathcal{H}_2$  problems. In general, the solutions to these problems can be formulated in terms of two operator AREs. Such problems have been addressed in the literature for general classes of distributed parameter systems [35], [37]. A complete and elegant analysis for the spatially invariant case can be found in [17]. From now on, we will only focus on the structure of the solution to LQR problems for systems described by (2)-(3). Such problems have been well studied in the distributed parameter system literature [35], [37]. Similar to the finite-dimensional case, optimal solutions to infinite-dimensional LQR can be formulated in terms of an operator Riccati equation. Consider the quadratic cost functional given by

$$\mathfrak{J} = \int_0^\infty \langle \psi(t), \mathcal{Q}\psi(t) \rangle + \langle u(t), \mathcal{R}u(t) \rangle dt. \quad (6)$$

The system (2)-(3) with cost (6) is said to be *optimizable* if for every initial condition  $\psi(0) = \psi_0 \in \ell_2$ , there exists an input function  $u \in L_2([0, \infty); \ell_2)$  such that the value of (6) is finite [35]. Note that if  $(\mathcal{A}, \mathcal{B})$  is *exponentially stabilizable*, then the system (2)-(3) is optimizable. The following is a standard result from [35].

*Theorem 2:* Let operators  $\mathcal{Q} \succeq 0$  and  $\mathcal{R} \succ 0$  be in  $\mathcal{L}(\ell_2)$ . If the system (2)-(3) with cost functional (6) is optimizable and  $(\mathcal{A}, \mathcal{Q}^{1/2})$  is exponentially detectable, then there exists a unique nonnegative, self-adjoint operator  $\mathcal{P} \in \mathcal{L}(\ell_2)$  satisfying the ARE

$$\langle \varphi, \mathcal{P}\mathcal{A}\phi \rangle + \langle \mathcal{P}\mathcal{A}\phi, \phi \rangle + \langle \varphi, \mathcal{Q}\phi \rangle - \langle \mathcal{B}^*\mathcal{P}\varphi, \mathcal{R}^{-1}\mathcal{B}^*\mathcal{P}\phi \rangle = 0 \quad (7)$$

for all  $\varphi, \phi \in \mathcal{D}(\mathcal{A})$  such that  $\mathcal{A} - \mathcal{B}\mathcal{R}^{-1}\mathcal{B}^*\mathcal{P}$  generates an exponentially stable  $C_0$ -semigroup. Moreover, the optimal control  $\tilde{u} \in L_2([0, \infty); \ell_2)$  is given by the feedback law

$$\tilde{u}(t) = -\mathcal{R}^{-1}\mathcal{B}^*\mathcal{P}\tilde{\psi}(t)$$

where  $\tilde{\psi}$  is the solution of

$$\frac{d}{dt}\tilde{\psi}(t) = (\mathcal{A} - \mathcal{B}\mathcal{R}^{-1}\mathcal{B}^*\mathcal{P})\tilde{\psi}(t) \quad (8)$$

with initial condition  $\psi_0$ .

Similar to the case of operator Lyapunov equations, Equation (7) is difficult to solve in general. When the system (2)-(3) is spatially invariant, equation (7) reduces to a parameterized family of finite-dimensional LQR problems [17].

As mentioned in the introduction, the main objective of this paper is to analyze the spatial structure of the solutions of operator equations (5) and (7) rather than solving them explicitly. In section VIII, it will be shown that the unique solution of these operator equations (under the appropriate standard assumptions) have an inherent spatial locality property.

## IV. SPATIALLY INVARIANT SYSTEMS

In order to motivate our results on structure of optimal control for general spatially distributed systems, we first consider the important subclass of spatially invariant systems on discrete groups  $(\mathbb{Z}, +)$ . Note that this problem has been studied extensively in [17] with emphasis on continuous group  $(\mathbb{R}, +)$ . We will mention these results and modify them when necessary for the discrete group  $\mathbb{G} = \mathbb{Z}$ .

In what follows, the Banach space  $\ell_p$  is defined exactly the same as in Section II with an additional assumption that all  $u_i$  in the infinite-tuples  $(u_i)_{i \in \mathbb{Z}} \in \ell_p$  belong to the same Hilbert space  $\mathbb{C}^n$ . We begin by introducing the unit translation operator to the right with respect to the group operation '+' as follows

$$\mathbf{T}u = \mathbf{T}(\dots, |u_i, u_{i+1}, \dots) = (\dots, |u_{i-1}, u_i, \dots).$$

One can verify that  $\|\mathbf{T}\|_{\ell_p \rightarrow \ell_p} = 1$  for all  $1 \leq p \leq \infty$ . Higher order translation operators can be defined iteratively by  $\mathbf{T}^s = \mathbf{T}^{s-1}\mathbf{T}$  for all  $s \in \mathbb{Z}$ . We are now ready to define translation invariant operators.

*Definition 2:* Operator  $\mathcal{Q} : \mathcal{D}(\mathcal{Q}) \rightarrow \ell_p$  with domain  $\mathcal{D}(\mathcal{Q}) \subseteq \ell_p$  is *translation invariant* if it commutes with every translation operator  $\mathbf{T}^s : \mathcal{D}(\mathcal{Q}) \rightarrow \mathcal{D}(\mathcal{Q})$ , i.e.,  $\mathbf{T}^s \mathcal{Q} = \mathcal{Q} \mathbf{T}^s$  for all  $s \in \mathbb{Z}$ .

It can be shown that all linear translation invariant operators on  $\ell_p$  can be characterized by forming linear combinations of higher order translation operators of the form

$$\mathcal{Q}(\mathbf{T}) = \sum_{k \in \mathbb{Z}} Q_k \mathbf{T}^k \quad (9)$$

with  $Q_k \in \mathbb{C}^{n \times n}$ .

*Definition 3:* The system described by equations (2)-(3) is called *spatially invariant* if the state-space operators  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$ ,  $\mathcal{D}$  are translation invariant.

For every  $x \in \ell_2$ , the discrete Fourier transform is defined by

$$\hat{x}(z) = \sum_{k \in \mathbb{Z}} x_k z^{-k}$$

where  $z \in \mathbb{S}^1$ . Using this definition, one can compute the discrete Fourier transform of a translation invariant operator. We will assume that the Fourier transform of all operators are continuous. A translation invariant operator  $\mathcal{Q}$  is bounded on  $\ell_2$  [31] if and only if

$$\|\mathcal{Q}\|_{\ell_2 \rightarrow \ell_2} = \sup_{z \in \mathbb{S}^1} \|\hat{\mathcal{Q}}(z)\| < \infty. \quad (10)$$

It can be shown (see Theorem ?? in the appendix) that for translation invariant operators boundedness on  $\ell_2$  implies boundedness on all  $\ell_p$  spaces for  $1 \leq p \leq \infty$ . The following decay result for discrete group  $\mathbb{Z}$  is similar to that of Theorem 7.4.2 of [38] for continuous group  $\mathbb{R}$  (see also Theorem 5 of [17] for the continuous space version).

*Theorem 3:* Let  $\mathcal{Q}$  be defined by (9) on  $\mathbb{Z}$  with discrete Fourier transform  $\hat{\mathcal{Q}}(z)$ . If  $\mathcal{Q} \in \mathcal{L}(\ell_2)$ , then the coefficients of operator  $\mathcal{Q}$  decay exponentially in the spatial domain, i.e., for

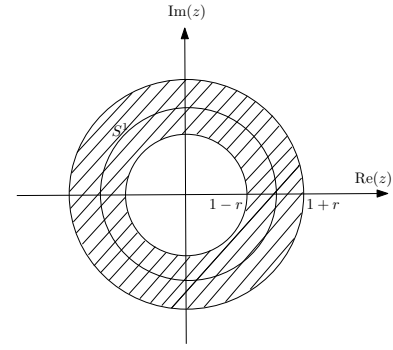


Fig. 3. Analytic continuation to annulus  $\Omega$  when  $\mathbb{G} = \mathbb{Z}$ .

all  $k \in \mathbb{Z}$

$$\|Q_k\| \leq \alpha e^{-\beta|k|} \quad (11)$$

for some  $\alpha > 0$  and  $0 < \beta < \ln(1+r)$  where  $r$  is the distance of the closest pole of  $\hat{\mathcal{Q}}(z)$  to  $\mathbb{S}^1$ .

*Proof:* According to (10) if  $\mathcal{Q} \in \mathcal{L}(\ell_2)$ , then  $\hat{\mathcal{Q}}(z)$  has no pole on  $\mathbb{S}^1$ , and it has analytic continuation to some annulus

$$\Omega = \{z \in \mathbb{C} : 1-r < |z| < 1+r, r > 0\} \quad (12)$$

where  $r$  is the distance of the closest pole of  $\hat{\mathcal{Q}}(z)$  to  $\mathbb{S}^1$ . Now consider the modified operator  $\tilde{\mathcal{Q}}$  which is defined by  $\tilde{\mathcal{Q}}_k = Q_k \zeta^k$ . One can see that  $\tilde{\mathcal{Q}}$  is also a translation invariant operator. From (10), it follows that

$$\|\hat{\mathcal{Q}}(\zeta e^{i\omega})\| < \infty$$

for all  $1-r < \zeta < 1+r$ . Therefore, by using the inequality

$$\|\tilde{\mathcal{Q}}_k\| \leq \|\tilde{\mathcal{Q}}\|_{\ell_2 \rightarrow \ell_2}$$

for all  $k \in \mathbb{Z}$ , the decay result (11) can be obtained immediately.  $\blacksquare$

As shown in [17], the solutions of operator ARE for a spatially invariant system reduces to the following parameterized equation

$$\hat{A}^* \hat{\mathcal{P}} + \hat{\mathcal{P}} \hat{A} - \hat{\mathcal{P}} \hat{B} \hat{R}^{-1} \hat{B}^* \hat{\mathcal{P}} + \hat{\mathcal{Q}} = 0 \quad (13)$$

which is evaluated on  $\mathbb{S}^1$ . The spatial frequency domain indeterminant  $z$  has been dropped from the above equation for notational simplicity. Assuming that all conditions of theorem 2 are satisfied, equation (13) has a unique bounded solution  $\mathcal{P}$  on  $\ell_2$ . Furthermore, if the Fourier transform of all operators  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{Q}$ ,  $\mathcal{R}$  have analytic continuation to some annulus around the unit circle, a similar argument as in Section V.B.1 of [17] can be used to show that the Fourier transform of  $\mathcal{P}$  also has analytic continuation to an annulus around the unit circle. This, in combination with Theorem 3, guarantees that the coefficients of the translation invariant operator  $\mathcal{P}$ , decay exponentially in the spatial domain, i.e.,

$$\|P_k\| \leq \alpha e^{-\beta|k|} \quad (14)$$

for some  $\alpha, \beta > 0$ . Note that the spatial decay of the solution in (14) is identical to that of [17] for continuous group  $\mathbb{R}$  with the minor difference that additional assumptions on growth

bounds for  $\hat{\mathcal{P}}(z)$  are not required (See appendix B of [17] for more details). This is due to the fact that the annulus is a compact set in  $\mathbb{C}$ , and  $\hat{\mathcal{P}}(z)$  is a continuous function (in the case of a continuous group, a strip around the imaginary axis is not bounded). Therefore, the extreme points are attained on the set.

In summary, given a bounded translation invariant operator on  $\ell_2$ , analytic continuity of its Fourier transform guarantees *spatial locality* of the operator by guaranteeing that the operator decays exponentially in space.

Unfortunately, the applicability of this result is limited to systems that are highly symmetric (such as identical dynamics on a lattice). The main question that we are trying to answer here is whether these concepts can be extended to a larger class of operators which are not necessarily spatially invariant.

This question is answered in a rigorous fashion in the next section. It turns out that the notion of spatial locality can be extended from translation invariant operators to a larger class of linear operators. This requires extending the notion of spatial decay in a natural way from linear translation invariant operators to a larger class of linear operators called *spatially decaying* or SD for short

## V. SPATIALLY DECAYING OPERATORS

The key difficulty in extending the results of previous section is that the notion of spatial invariance was critical in being able to use Fourier methods which greatly simplified the analysis. Simply put, if we replace “space” with “time”, we get a more familiar analogue of this problem: Fourier methods can not be used directly for analysis of linear time-varying systems.

The key in extending the results of the previous section to systems that are not spatially invariant is to somehow extend the notion of analytic continuity. Consider the bounded translation invariant operator  $\mathcal{Q}$  of form (9) with discrete Fourier transform  $\hat{\mathcal{Q}}(z)$  which has analytic continuation to some annulus  $\Omega$  around the unit circle  $\mathbb{S}^1$ . Suppose that  $\Gamma$  is a circle with radius  $\zeta > 1$  and strictly lies inside  $\Omega$ . By analytic continuity, it follows that

$$\|\hat{\mathcal{Q}}(z)\| < \infty$$

for all  $z \in \Gamma$ . Now consider the following inequality

$$\begin{aligned} \|\hat{\mathcal{Q}}(z)\| &= \left\| \sum_{k \in \mathbb{Z}} Q_k z^k \right\| \\ &= \left\| \sum_{k \in \mathbb{Z}} Q_k \zeta^k e^{i\omega k} \right\| \\ &\leq \sum_{k \in \mathbb{Z}} \|Q_k\| \zeta^{|k|}. \end{aligned} \quad (15)$$

Applying the result (11) to (15), it can be shown (see Section VI-B for details) that the quantity in the right hand side of inequality (15) is bounded. This shows that

$$\sum_{k \in \mathbb{Z}} \|Q_k\| \zeta^{|k|} < \infty \quad (16)$$

if and only if  $\hat{\mathcal{Q}}(z)$  has analytic continuity on some annulus around the unit circle which strictly contains  $\Gamma$ . Consider the

modified linear operator  $\tilde{\mathcal{Q}}(\zeta)$  defined by

$$[\tilde{\mathcal{Q}}(\zeta)]_{ki} = Q_{k-i} \zeta^{|k-i|}.$$

One can see that the modified operator  $\tilde{\mathcal{Q}}(\zeta)$  is also translation invariant. If condition (16) holds, from (10) and (15) we see that  $\|\tilde{\mathcal{Q}}(\zeta)\|_{\ell_2 \rightarrow \ell_2}$  is bounded. Therefore, by applying theorem ?? we have the following result.

*Proposition 1:* The  $\hat{\mathcal{Q}}(z)$  has analytic continuation to some annulus  $\Omega$  around the unit circle which strictly contains the circle  $\Gamma$  of radius  $\zeta > 1$  if and only if  $\tilde{\mathcal{Q}}(\zeta) \in \mathcal{L}(\ell_p)$ .

The above proposition suggests that analytic continuity is equivalent to boundedness of an auxiliary operator  $\tilde{\mathcal{Q}}(\zeta)$ , which is the exponentially weighted version of the original operator.

In the following, we will generalize this idea to a larger class of linear operators by first forming an auxiliary *weighted* operator and imposing boundedness of the modified operator on  $\ell_p$ . Before doing so, we need to review the notion of a distance function.

*Definition 4:* A distance function on a discrete topology with a set of nodes  $\mathbb{G}$  is defined as a single-valued, nonnegative, real function  $\text{dis}(k, i)$  defined for all  $k, i, j \in \mathbb{G}$  which has the following properties:

- (1)  $\text{dis}(k, i) = 0$  iff  $k = i$ .
- (2)  $\text{dis}(k, i) = \text{dis}(i, k)$ .
- (3)  $\text{dis}(k, i) \leq \text{dis}(k, j) + \text{dis}(j, i)$ .

Next, we will define the notion of a coupling characteristic function which will then be used as a weight function in the auxiliary operator.

*Definition 5:* A nondecreasing continuous function  $\chi : \mathbb{R}^+ \rightarrow [1, \infty)$  is called a coupling characteristic function if  $\chi(0) = 1$  and  $\chi(s+t) \leq \chi(s)\chi(t)$  for all  $s, t \in \mathbb{R}^+$ .

The constant coupling characteristic function with unit value everywhere is denoted by  $\mathbf{1}$ .

In order to be able to characterize rates of decay we define a one-parameter family of coupling characteristic functions as follows

*Definition 6:* A one-parameter family of coupling characteristic functions  $\mathcal{C}$  is defined to be the set of all characteristic functions  $\chi_\alpha$  for  $\alpha \in \mathbb{R}^+$  such that

- (i)  $\chi_0 = \mathbf{1}$ .
- (ii) For all  $\chi_\alpha, \chi_\beta \in \mathcal{C}$  with  $\alpha < \beta$ , relation  $\chi_\alpha \prec \chi_\beta$  holds.
- (iii)  $\chi_\alpha$  is a continuous function of  $\alpha$  in  $\|\cdot\|_T$ -topology, i.e., for every  $T > 0$  and any given  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$\|\chi_\alpha - \chi_\beta\|_T < \varepsilon$$

for all  $|\alpha - \beta| < \delta$ .

A simple example of such a one-parameter family is the family of exponential functions  $e^{\alpha x}$  for  $x, \alpha \in \mathbb{R}^+$ . As one can see, this family satisfies the above definitions. Using this definition, we can now formally define a spatially decaying (SD) operator.

*Definition 7:* Suppose that a distance function  $\text{dis}(\cdot, \cdot)$  and a

one-parameter family of parameterized coupling characteristic functions  $\mathcal{C}$  are given. A linear operator  $\mathcal{Q} \in \mathcal{L}(\ell_p)$  is SD with respect to  $\mathcal{C}$  if there exists  $b > 0$  such that the auxiliary operator  $\tilde{\mathcal{Q}}(\alpha)$ , defined block-wise as

$$[\tilde{\mathcal{Q}}(\alpha)]_{ki} = [\mathcal{Q}]_{ki} \chi_\alpha(\text{dis}(k, i))$$

is bounded on  $\ell_p$  for all  $0 \leq \alpha < b$ . The number  $b$  is referred to as the *decay margin*.

In general, determining the boundedness of the auxiliary operator depends on the choice of  $p$ . When  $p = \infty$  boundedness can be easily characterized. The following result gives us a simple sufficient condition in terms of  $\ell_\infty$  norm.

*Lemma 1:* A linear operator  $\mathcal{Q} \in \mathcal{L}(\ell_\infty)$  is SD with respect to the one-parameter family of coupling characteristic functions  $\mathcal{C}$  on  $\ell_p$  for all  $1 \leq p \leq \infty$  if there exists  $b > 0$  such that the following holds

$$\sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \|[\mathcal{Q}]_{ki}\| \chi_\alpha(\text{dis}(k, i)) < \infty \quad (17)$$

for all  $0 \leq \alpha < b$ .

*Proof:* First, we will show that the auxiliary operator  $\tilde{\mathcal{Q}}(\alpha)$  is simultaneously in  $\mathcal{L}(\ell_1)$  and  $\mathcal{L}(\ell_\infty)$ . For a fixed  $\alpha \in [1, b)$ ,  $\tilde{\mathcal{Q}}(\alpha) \in \mathcal{L}(\ell_1) \cap \mathcal{L}(\ell_\infty)$  if the following quantities

$$\|\tilde{\mathcal{Q}}(\alpha)\|_{\ell_\infty \rightarrow \ell_\infty} \leq \sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \|[\mathcal{Q}]_{ki}\| \chi_\alpha(\text{dis}(k, i))$$

and

$$\|\tilde{\mathcal{Q}}(\alpha)\|_{\ell_1 \rightarrow \ell_1} \leq \sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \|[\mathcal{Q}]_{ki}\| \chi_\alpha(\text{dis}(k, i))$$

are bounded. Finally, using the Riesz-Thorin theorem, it follows that  $\mathcal{Q}$  is also bounded on all intermediate spaces  $\ell_p$  where  $1 \leq p \leq \infty$ . ■

Examples of SD operators in  $\mathcal{L}(\ell_\infty)$  appear naturally in many applications. For every operator defined on a finite graph with a finite number of nodes, condition (17) always holds. Intuitively, we may interpret the norm of each block element  $[\mathcal{Q}]_{ki}$  as the coupling strength between subsystems  $k$  and  $i$ . Given the one-parameter family of coupling characteristic functions  $\mathcal{C}$ , fix a value for  $\alpha \in (0, b)$ . If nodes  $k$  and  $i$  are taken far away from each other, the coupling strength between them will weaken so as to make the quantity in (17) bounded. Indeed, the decay will be proportional to the inverse of the coupling characteristic function  $\chi_\alpha$ . For example, if the coupling characteristic function is chosen to be exponential, the coupling strength will decay exponentially.

For an infinite graph, if we fix a node  $k$  and move on the graph away from node  $k$ , the coupling strength decays proportional to the inverse of the coupling characteristic function  $\chi_\alpha$  so that relation (17) holds. The notion of an SD operator will be key in proving spatial locality of optimal controllers.

*Remark 2:* Throughout the rest of the paper, by SD we mean SD in  $\ell_\infty$ .

One can easily see that all spatially invariant systems are indeed SD with respect to exponential coupling characteristic functions. We finish this section by introducing the notion of an SD systems using the concept of an SD operator.

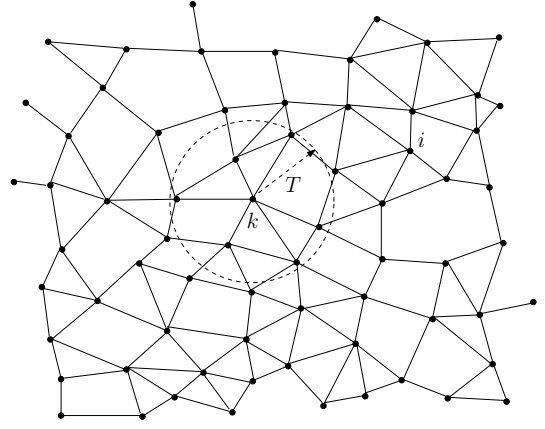


Fig. 4. Topology of the system on an arbitrary connected graph. Coupling between two agents is shown by an undirected edge between them.

*Definition 8:* The system (2)-(3) is called spatially decaying (SD) if the state-space operators  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$ ,  $\mathcal{D}$  are SD with respect to the one-parameter family of coupling characteristic functions  $\mathcal{C}$ .

## VI. EXAMPLES OF SPATIALLY DECAYING OPERATORS

The following class of operators which are used extensively in cooperative and distributed control are interesting special classes of SD operators.

### A. Spatially Truncated Operators

These are operators with finite range couplings. Given the coupling range  $T > 0$ , the following class of linear operators are SD with respect to every coupling characteristic functions  $\chi_\alpha$

$$[\mathcal{Q}]_{ki} = \begin{cases} Q_{ki} & \text{if } \text{dis}(k, i) \leq T \\ 0 & \text{if } \text{dis}(k, i) > T \end{cases} \quad (18)$$

where  $Q_{ki} \in \mathbb{C}^{n \times n}$ . For this case, some common choices for the distance function are Euclidean distance and geodesic or minimum hop count distance (i.e., hop count on the shortest path). For such operators and every given node  $k \in \mathbb{G}$ , we have that

$$\sum_{i \in \mathbb{G}} \|[\mathcal{Q}]_{ki}\| \chi_\alpha(\text{dis}(k, i)) = \sum_{i \sim k} \|[\mathcal{Q}]_{ki}\| \chi_\alpha(T) < \infty. \quad (19)$$

The relation  $\sim$  is the neighborhood relation defined as  $i \sim k$  if and only if  $\text{dis}(k, i) \leq T$ . Inequality (19) shows that  $\mathcal{Q}$  is SD with respect to every  $\mathcal{C}$  and the decay margin is any real number  $b > 0$ .

Examples of such operators arise in motion coordination of autonomous agents such as the *Laplacian* operator [39]. Suppose that  $\mathcal{G}$  is a connected distance dependent *proximity* graph with the set of nodes  $\mathbb{G}$  and the set of edges  $\mathbb{E}$ . Furthermore, suppose that edges are weighted with a given weighting function  $w : \mathbb{E} \rightarrow \mathbb{C}^{n \times n}$ , for example,  $w_{ki}$  is the weight on edge connecting nodes  $k$  and  $i$ . Let  $f : \mathbb{G} \rightarrow \mathbb{C}$  be a function mapping vertices to complex numbers. Then the

discrete Laplacian operator  $\mathcal{L}_w$  is defined by

$$(\mathcal{L}_w f)(k) = \sum_{i \sim k} w_{ik} (f(i) - f(k)) \quad (20)$$

in which the neighborhood relation  $\sim$  is defined as above. By selecting the minimum hop count distance function, the matrix representation of Laplacian operator will be

$$[\mathcal{L}_w]_{ki} = \begin{cases} -d_k w_{kk} & \text{if } k = i \\ w_{ik} & \text{if } \text{dis}(k, i) = 1 \\ 0 & \text{otherwise} \end{cases} \quad (21)$$

in which  $d_k$  is the degree of node  $k$ . Such an operator is obviously SD.

### B. Exponentially Decaying Operators

Consider the one-parameter family of coupling characteristic functions  $\mathcal{C}_E$  defined by

$$\chi_\zeta(s) = (1 + \zeta)^s \quad (22)$$

where  $\zeta \in \mathbb{R}^+$ . Operator  $\mathcal{Q}$  is said to be *exponentially* SD if condition (17) holds with respect to  $\mathcal{C}_E$  defined by (22) for all  $\zeta \in [0, b)$ . Here  $b > 0$  is the decay margin.

The first obvious example of such operators is operators with finite range couplings such as the one defined by (18) for some  $T > 0$ . For every node  $k \in \mathbb{G}$ , we have

$$\sum_{i \in \mathbb{G}} \|[\mathcal{Q}]_{ki}\| \chi_\zeta(\text{dis}(k, i)) \leq \sum_{i \sim k} \|[\mathcal{Q}]_{ki}\| (1 + \zeta)^T. \quad (23)$$

The right hand side of (23) is a polynomial in terms of  $\zeta$  with maximum degree  $T$ . Since every polynomial is finite on any interval  $[0, b)$  with  $b > 0$ , the right hand side of (23) is bounded, and that  $\mathcal{Q}$  is exponentially SD with decay margin  $b = +\infty$ .

Another important example of exponentially SD operators are the class of translation invariant operators. The result of theorem 3 along with the immediate application of lemma 1 shows that a translation invariant operator in  $\mathcal{L}(\ell_2)$  is exponentially SD. In this case, since the interconnection topology is assumed to be a lattice, the suitable choice of a distance function is  $\text{dis}(k, i) = |k - i|$ . Applying the results of theorem 3, for every  $k \in \mathbb{G}$ , by selecting any  $\zeta \in [0, e^\beta - 1)$  where  $0 < \beta < \ln(1 + r)$ , it follows that

$$\begin{aligned} \sum_{i \in \mathbb{Z}} \|[\mathcal{Q}]_{ki}\| (1 + \zeta)^{\text{dis}(k, i)} &\leq \alpha \sum_{i \in \mathbb{Z}} e^{-\beta|k-i|} (1 + \zeta)^{|k-i|} \\ &\leq \alpha \left( \frac{e^\gamma + 1}{e^\gamma - 1} \right) < \infty \end{aligned}$$

where  $\gamma = \beta - \ln(1 + \zeta)$  is a positive number. The decay margin of  $\mathcal{Q}$  is equal to  $r$ , the distance of the nearest pole of  $\hat{\mathcal{Q}}(z)$  to the unit circle.

### C. Algebraically Decaying Operators

Consider the parameterized family of characteristic functions  $\mathcal{C}_A$  defined as

$$\chi_\nu(s) = (1 + \lambda s)^\nu \quad (24)$$

in which  $\lambda > 0$  and  $\nu \in \mathbb{R}^+$ . Operator  $\mathcal{Q}$  is said to be *algebraically* SD if condition (17) holds with respect to  $\mathcal{C}_A$  defined by (24) for all  $\nu \in [0, b)$  where  $b > 0$  is the decay margin. Such functions are often used as pair-wise potentials among agents in flocking and cooperative control problems [40]. The intuition behind such a coupling is that the interaction between two subsystems in a networked system can be modelled as a non-increasing function of their Euclidean distance. Another example of such coupling functions arises in wireless networks. The coupling between nodes, which is considered as the power of the communication signal between agents, decays with the inverse fourth power law [41], i.e.,

$$\frac{1}{\text{dis}(k, i)^4}.$$

In such applications, the Laplacian operator  $\mathcal{L}_w$  (or the corresponding adjacency operator) can be defined with the following weighting function

$$w_{ki} = \frac{1}{\chi_\nu(\text{dis}(k, i))}$$

for some  $\nu \geq 0$  and  $\lambda > 0$ .

## VII. PROPERTIES OF SD OPERATORS

*Theorem 4:* Given a one-parameter family of coupling characteristic functions  $\mathcal{C}$ , the set of all linear operators that are SD with respect to  $\mathcal{C}$  with decay margin at least  $b > 0$  forms a *Banach Algebra*.

*Proof:* See appendix XI-A for a proof.  $\blacksquare$

For a comprehensive discussion on Banach algebras we refer the reader to any Functional Analysis textbook, for example [34]. This Banach algebra is denoted by

$$\mathcal{S}_b(\mathcal{C}) = \{\mathcal{Q} : \|\mathcal{Q}\|_b^* < \infty\}$$

in which the operator norm

$$\|\mathcal{Q}\|_b^* = \sup_{\alpha \in [0, b)} \sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \|[\mathcal{Q}]_{ki}\| \chi_\alpha(\text{dis}(k, i))$$

satisfies the usual conditions, i.e., for all  $\mathcal{Q}, \mathcal{P} \in \mathcal{S}_b(\mathcal{C})$  and  $c \in \mathbb{C}$ ,

- (1)  $\|\mathcal{Q}\|_b^* \geq 0$  and  $\|\mathcal{Q}\|_b^* = 0$  iff  $\mathcal{Q} = 0$ ,
- (2)  $\|c \mathcal{Q}\|_b^* = |c| \|\mathcal{Q}\|_b^*$ ,
- (3)  $\|\mathcal{Q} + \mathcal{P}\|_b^* \leq \|\mathcal{Q}\|_b^* + \|\mathcal{P}\|_b^*$ .

Furthermore, it is submultiplicative,

- (4)  $\|\mathcal{Q}\mathcal{P}\|_b^* \leq \|\mathcal{Q}\|_b^* \|\mathcal{P}\|_b^*$ .

According to the definition, the operator space  $\mathcal{S}_b(\mathcal{C})$  equipped with norm  $\|\cdot\|_b^*$  is a Banach algebra.

The above theorem is a key ingredient in proving that optimal controllers of SD systems are SD. However, we first need the following result on closure under limit property of  $\mathcal{S}_b(\mathcal{C})$ . This result is required for studying the properties of the unique solution of Lyapunov equation and ARE through the study of associated differential equations. In the next section, using the closure under limit property of SD operators, it is

shown that the solution of differential Lyapunov and Riccati equations converge to an SD operator.

*Theorem 5:* Let  $\mathcal{C}$  be a one-parameter family of coupling characteristic functions with decay margin  $b > 0$ . Consider the one-parameter family of operator-valued functions  $\mathcal{P}(t) : \mathbb{R}^+ \rightarrow \mathcal{L}(\ell_p)$  with the following properties:

- (1)  $\lim_{t \rightarrow \infty} \|\mathcal{P}(t) - \mathcal{P}\|_{\ell_p \rightarrow \ell_p} = 0$ ,
- (2)  $\mathcal{P}(t) \in \mathcal{S}_b(\mathcal{C})$  for all  $t \geq 0$ .

Then  $\lim_{t \rightarrow \infty} \|\mathcal{P}(t) - \mathcal{P}\|_b^* = 0$ . Furthermore,  $\mathcal{P} \in \mathcal{S}_b(\mathcal{C})$ .

*Proof:* See appendix XI-B for a proof. ■

To summarize, we have shown that operator space  $\mathcal{S}_b(\mathcal{C})$  is closed under addition, multiplication, and limit properties. Furthermore, it turns out that if an SD operator has an algebraic inverse on  $\mathcal{L}(\ell_\infty)$ , the inverse operator  $\mathcal{Q}^{-1}$  is also SD [?].

*Remark 3:* Using the above results, it is straightforward to check that the serial and parallel composition of two SD systems are SD. Furthermore, if the feedback interconnection of two SD systems

$$\begin{aligned} \frac{d}{dt} \psi_i(t) &= (\mathcal{A}_i \psi_i)(t) + (\mathcal{B}_i u_i)(t) \\ y_i(t) &= (\mathcal{C}_i \psi_i)(t) + (\mathcal{D}_i u_i)(t) \end{aligned}$$

for  $i = 1, 2$ , is *well-posed* [42] and if operator  $\mathcal{I} - \mathcal{D}_2 \mathcal{D}_1$  has an algebraic inverse in  $\mathcal{L}(\ell_\infty)$ , then the feedback interconnection of these two systems is also SD.

## VIII. STRUCTURE OF QUADRATICALLY OPTIMAL CONTROLLERS

As discussed in section III, our aim is not to solve the Lyapunov equation (5) and ARE (7) explicitly but to study the spatial structure of the solution of these algebraic equations by means of tools developed in the previous sections. In the following, it is shown that the solution of equations (5) and (7) have an inherent spatial locality and the characteristics of the coupling function will determine the degree of localization.

### A. Lyapunov Equations

In the following, it is shown that for SD systems that are stable and are described by (4), solution of the Lyapunov equation  $\mathcal{P}$  is also SD. The proof sketch is as follows: first, we consider the corresponding Lyapunov differential equation and show that the solution is SD for every time instant. Then using the closure properties of the Banach Algebra  $\mathcal{S}_b(\mathcal{C})$ , it is concluded that  $\mathcal{P}$  is SD with respect to  $\mathcal{C}$ .

*Theorem 6:* Assume that operators  $\mathcal{A}, \mathcal{Q} \in \mathcal{S}_b(\mathcal{C})$  and  $\mathcal{Q}$  is positive definite. If  $\mathcal{A}$  is the infinitesimal generator of an exponentially stable  $C_0$ -semigroup  $\mathcal{T}(t)$  on  $\ell_2$ , then the unique positive definite solution of Lyapunov equation

$$\langle \mathcal{A}\phi, \mathcal{P}\phi \rangle + \langle \mathcal{P}\phi, \mathcal{A}\phi \rangle + \langle \phi, \mathcal{Q}\phi \rangle = 0 \quad (25)$$

for all  $\phi \in \ell_2$ , satisfies  $\mathcal{P} \in \mathcal{S}_b(\mathcal{C})$ .

*Proof:* First, we will prove that the  $C_0$ -semigroup  $\mathcal{T}(t)$  with infinitesimal generator  $\mathcal{A}$  is SD with respect to  $\mathcal{C}$ . The

following is a standard result from [35]:

$$\frac{d}{dt} \mathcal{T}(t)\phi = \mathcal{A} \mathcal{T}(t)\phi \quad (26)$$

with  $\mathcal{T}(0) = \mathcal{I}$  and for all  $\phi \in \ell_2$  and  $t > 0$ . Therefore, for all  $k, i \in \mathbb{G}$  we have

$$\frac{d}{dt} [\mathcal{T}(t)]_{ki} = \sum_{j \in \mathbb{G}} [\mathcal{A}]_{kj} [\mathcal{T}(t)]_{ji}. \quad (27)$$

For a differentiable matrix  $X(t) \in \mathbb{C}^{n \times n}$  for  $t \geq 0$ , we have the following inequality

$$\begin{aligned} \frac{d}{dt} \|X(t)\| &= \lim_{\tau \rightarrow 0} \frac{\|X(t+\tau)\| - \|X(t)\|}{\tau} \\ &\leq \lim_{\tau \rightarrow 0} \left\| \frac{X(t+\tau) - X(t)}{\tau} \right\| \\ &\leq \left\| \lim_{\tau \rightarrow 0} \frac{X(t+\tau) - X(t)}{\tau} \right\| \\ &\leq \left\| \frac{d}{dt} X(t) \right\|. \end{aligned} \quad (28)$$

Assume that  $\mathcal{T}(t)$  is a solution of (26), using inequality (28), we have

$$\begin{aligned} \frac{d}{dt} \sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \|[\mathcal{T}(t)]_{ki}\| \chi_\alpha(\text{dis}(k, i)) \\ &\leq \sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \left\| \frac{d}{dt} [\mathcal{T}(t)]_{ki} \right\| \chi_\alpha(\text{dis}(k, i)) \\ &\leq \sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \left\| \sum_{j \in \mathbb{G}} [\mathcal{A}]_{kj} [\mathcal{T}(t)]_{ji} \right\| \chi_\alpha(\text{dis}(k, i)). \end{aligned}$$

Using the fact that  $\|\cdot\|_b^*$  is submultiplicative, from the above inequality we can conclude that

$$\frac{d}{dt} \|\mathcal{T}(t)\|_b^* \leq \|\mathcal{A}\|_b^* \|\mathcal{T}(t)\|_b^*$$

and it follows that

$$\|\mathcal{T}(t)\|_b^* \leq \|\mathcal{T}(0)\|_b^* e^{\|\mathcal{A}\|_b^* t} \quad (29)$$

for all  $t \geq 0$ . Note that  $\|\mathcal{T}(0)\|_b^* = 1$ . Because operator  $\mathcal{A}$  is SD with respect to  $\mathcal{C}$  with decay margin  $b$ , according to (29) the family of one-parameter operators satisfy  $\mathcal{T}(t) \in \mathcal{S}_b(\mathcal{C})$  for all  $t \geq 0$ . Now consider the differential form of the Lyapunov function

$$\left\langle \phi, \frac{d}{dt} \mathcal{P}(t)\phi \right\rangle = \langle \mathcal{A}\phi, \mathcal{P}\phi \rangle + \langle \mathcal{P}\phi, \mathcal{A}\phi \rangle + \langle \phi, \mathcal{Q}\phi \rangle$$

with  $\mathcal{P}(0) = \mathcal{Q}$  for all  $\phi \in \ell_2$ . This equation has a solution of the following form (cf. [35])

$$\mathcal{P}(t)\phi = \int_0^t \mathcal{T}(\sigma)^* \mathcal{Q} \mathcal{T}(\sigma)\phi \, d\sigma \quad (30)$$

for all  $\phi \in \ell_2$ . Therefore, for every  $k, i \in \mathbb{G}$ , we have

$$[\mathcal{P}(t)]_{ki} = \int_0^t [\mathcal{T}(\sigma)^* \mathcal{Q} \mathcal{T}(\sigma)]_{ki} \, d\sigma. \quad (31)$$

According to inequality (29) and equation (31) and using the fact that  $\|\cdot\|_b^*$  is submultiplicative, we get

$$\|\mathcal{P}(t)\|_b^* \leq \|\mathcal{Q}\|_b^* \frac{(e^{2\|\mathcal{A}\|_b^* t} - 1)}{2\|\mathcal{A}\|_b^*}. \quad (32)$$

Therefore,  $\mathcal{P}(t) \in \mathcal{S}_b(\mathcal{C})$  for all  $t \geq 0$ . On the other hand, solution of the differential Lyapunov equation (30) converges to the unique solution of (25), i.e.,

$$\lim_{t \rightarrow \infty} \|\mathcal{P}(t) - \mathcal{P}\|_{\ell_2 \rightarrow \ell_2} = 0.$$

According to Theorem 5, it follows that

$$\lim_{t \rightarrow \infty} \mathcal{P}(t) = \mathcal{P}$$

uniformly in  $\mathcal{S}_b(\mathcal{C})$ . Therefore,  $\mathcal{P} \in \mathcal{S}_b(\mathcal{C})$ .  $\blacksquare$

### B. Algebraic Riccati Equation

As discussed in Section III, the key ingredient in solution of linear optimal control problems in finite and infinite-dimensional case is finding the solution to the corresponding Riccati equation. In the following, we will show that the solution of the Riccati equation for SD systems as well as the kernel of optimal feedback associated to an SD system is itself an SD operator,

$$\mathcal{K} = -\mathcal{R}^{-1}\mathcal{B}^*\mathcal{P}. \quad (33)$$

Without loss of generality, we will assume that  $\mathcal{R} = \mathcal{I}$ . Otherwise, by only assuming that  $\mathcal{R}$  has an algebraic inverse on  $\mathcal{L}(\ell_\infty)$ , it can be shown that  $\mathcal{R}^{-1}$  is SD [?]. According to the closure under multiplication property of SD operators, if  $\mathcal{P}$  is SD, then  $\mathcal{K}$  will be SD. The proof of the following theorem, is more or less similar to the proof of the theorem 6.

*Theorem 7:* Let  $\mathcal{A}, \mathcal{B}, \mathcal{Q} \in \mathcal{S}_b(\mathcal{C})$  and  $\mathcal{Q} \succeq 0$ . Moreover, assume that conditions of theorem 2 hold. Then the unique positive definite solution of the following ARE

$$\langle \varphi, \mathcal{P}\mathcal{A}\phi \rangle + \langle \mathcal{P}\mathcal{A}\varphi, \phi \rangle + \langle \varphi, \mathcal{Q}\phi \rangle - \langle \mathcal{B}^*\mathcal{P}\varphi, \mathcal{B}^*\mathcal{P}\phi \rangle = 0$$

for all  $\varphi, \phi \in \ell_2$ , satisfies  $\mathcal{P} \in \mathcal{S}_b(\mathcal{C})$ .

*Proof:* Consider the following Differential Riccati Equation (DRE)

$$\begin{aligned} \frac{d}{dt} \langle \varphi, \mathcal{P}(t)\phi \rangle &= \langle \varphi, \mathcal{P}(t)\mathcal{A}\phi \rangle + \langle \mathcal{P}(t)\mathcal{A}\varphi, \phi \rangle + \langle \varphi, \mathcal{Q}\phi \rangle \\ &\quad - \langle \mathcal{B}^*\mathcal{P}(t)\varphi, \mathcal{B}^*\mathcal{P}(t)\phi \rangle \end{aligned}$$

with  $\mathcal{P}(0) = 0$ . We denote the unique solution of this differential Riccati equation in the class of strongly continuous, self-adjoint operators in  $\mathcal{L}(\ell_2)$  by the one-parameter family of operator-valued function  $\mathcal{P}(t)$  for  $t \geq 0$ . The nonnegative operator  $\mathcal{P}$ , the unique solution of ARE, is the strong limit of  $\mathcal{P}(t)$  on  $\ell_2$  as  $t \rightarrow \infty$  (see theorem 6.2.4 of [35]). Therefore, we have that

$$\lim_{t \rightarrow \infty} \|\mathcal{P}(t) - \mathcal{P}\|_{\ell_2 \rightarrow \ell_2} = 0. \quad (34)$$

From the differential Riccati equation, it follows that

$$\frac{d}{dt} [\mathcal{P}(t)]_{ki} = [\mathcal{A}^*\mathcal{P}(t) + \mathcal{P}(t)\mathcal{A} - \mathcal{P}(t)\mathcal{B}\mathcal{B}^*\mathcal{P}(t) + \mathcal{Q}]_{ki}$$

for all  $k, i \in \mathbb{G}$ . Using inequality (28), we have

$$\begin{aligned} \frac{d}{dt} \|\mathcal{P}(t)\|_b^* &= \sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \|\mathcal{P}(t)\|_{ki} \chi_\alpha(\text{dis}(k, i)) \\ &\leq \sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \left\| \frac{d}{dt} [\mathcal{P}(t)]_{ki} \right\| \chi_\alpha(\text{dis}(k, i)) \\ &\leq \|\mathcal{A}^*\mathcal{P}(t) + \mathcal{P}(t)\mathcal{A} - \mathcal{P}(t)\mathcal{B}\mathcal{B}^*\mathcal{P}(t) + \mathcal{Q}\|_b^*. \end{aligned}$$

For simplicity in notations, denote  $\pi(t) = \|\mathcal{P}(t)\|_b^*$ . Using the triangle inequality and the fact that norm  $\|\cdot\|_b^*$  is submultiplicative, we have the following differential inequality

$$\dot{\pi}(t) \leq 2\|\mathcal{A}\|_b^* \pi(t) + (\|\mathcal{B}\|_b^*)^2 \pi(t)^2 + \|\mathcal{Q}\|_b^* \quad (35)$$

with initial condition  $\pi(0) = 0$  and constraint  $\pi(t) \geq 0$  for all  $t \geq 0$ .

All coefficients  $\|\mathcal{A}\|_b^*, \|\mathcal{B}\|_b^*, \|\mathcal{Q}\|_b^*$  in the right hand side of the inequality (35) are finite numbers. If  $\pi(t)$  for  $t \geq 0$  is a solution of the differential inequality (35), then it is also a solution of the following differential inequality

$$\dot{\pi}(t) \leq \lambda (\pi(t) + 1)^2 \quad (36)$$

with initial condition  $\pi(0) = 0$ , in which

$$\lambda = \max(\|\mathcal{A}\|_b^*, (\|\mathcal{B}\|_b^*)^2, \|\mathcal{Q}\|_b^*).$$

In other words, the set of feasible solutions of (35) is a subset of solutions of (36). From (36), we have

$$-\frac{d}{dt} \left( \frac{1}{\pi(t) + 1} \right) \leq \lambda$$

which has the following set of solutions

$$\frac{1}{\pi(t) + 1} \geq \frac{e^{-\lambda t}}{\pi(0) + 1}.$$

Using the fact that  $\pi(t) \geq 0$  for all  $t \geq 0$  and  $\pi(0) = 0$ , it follows that

$$\pi(t) \leq e^{\lambda t} - 1.$$

The above inequality is feasible, i.e., there exists at least one sequence of solutions satisfying  $\pi(t) \geq 0$  for all  $t \geq 0$ . The above inequality also proves that  $\pi(t) < \infty$  for all  $t \geq 0$ . Therefore, we have that  $\mathcal{P}(t) \in \mathcal{S}_b(\mathcal{C})$  for all  $t \geq 0$ . According to theorem 5, we can use this result and (34) to conclude that  $\mathcal{P} \in \mathcal{S}_b(\mathcal{C})$ . This completes the proof.  $\blacksquare$

## IX. SIMULATION

We consider a large network of  $N$  linear subsystems coupled on a one-dimensional chain which can be described by

$$\frac{d}{dt} \psi(t) = (\mathcal{A}\psi)(t) + (\mathcal{B}u)(t).$$

The coupling characteristic function is  $\chi$  and the system operators are given by

$$[\mathcal{A}]_{ki} = \frac{10}{\chi(\text{dis}(k, i))}$$

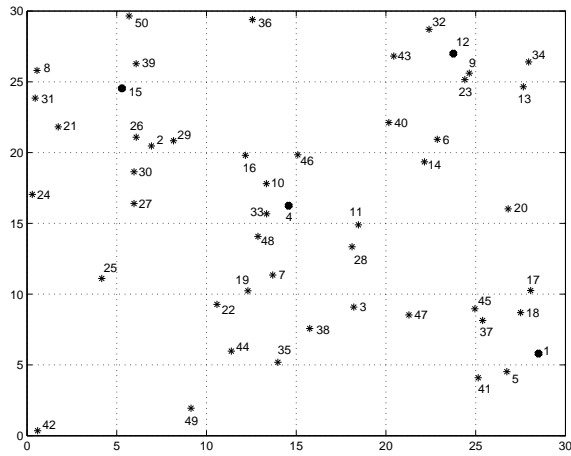


Fig. 5.  $N=50$  nodes are randomly and uniformly distributed in a region of area  $30 \times 30$  (units)<sup>2</sup>. Each node is a linear subsystem which is coupled to other subsystems through their dynamic and a central cost function by a given coupling characteristic function.

and  $\mathcal{B} = \mathcal{I}$ . The distance function is Euclidean. We will study the LQR problem discussed in Section III with weighting operators  $\mathcal{R} = \mathcal{I}$  and  $\mathcal{Q}$  being the corresponding graph Laplacian given by

$$[\mathcal{Q}]_{ki} = \begin{cases} N-1 & \text{if } k=i \\ -1 & \text{if } k \neq i. \end{cases}$$

The corresponding ARE is given by

$$\mathcal{A}^* \mathcal{P} + \mathcal{P} \mathcal{A} - \mathcal{P}^2 + \mathcal{Q} = 0. \quad (37)$$

Then the LQR optimal feedback is given by

$$\mathcal{K} = -\mathcal{P}. \quad (38)$$

In the following simulations, it is assumed that  $N = 50$  nodes are randomly and uniformly distributed in a region of area  $30 \times 30$  (units)<sup>2</sup>. Each node is assumed to be a linear system which is coupled through its dynamic and the LQR cost functional to other subsystems. In the sequel, three different scenarios are considered for the coupling characteristic function.

#### A. Algebraical Decay

The first simulation is done based on the coupling characteristic functions of algebraical type given by (24) with parameters  $\lambda = 0.1$  and  $\nu = 4$ . In Figure 6, the norm of the LQR feedback gains (38) corresponding to agents  $k = 1, 4, 12, 15$  (their locations are marked by bold stars in Figure 5) is depicted versus the distance of other subsystems to subsystem  $k$ . As seen from these simulations, for every subsystem  $k$  the norm of the optimal feedback kernel  $[\mathcal{K}]_{ki}$  is upper bounded by function  $\frac{\|[\mathcal{K}]_{kk}\|}{\chi_\nu(\text{dis}(k,i))}$ . Therefore, the spatial decay rate of the optimal controller can be determined priori only using the information of the coupling characteristic function  $\chi_\nu(\text{dis}(k,i))$ . As seen in Figure 6, for each subsystem  $k$ , the corresponding optimal controller is effectively coupled only to those subsystems (with index  $i$ 's) for which  $\text{dis}(k,i) \leq 10$  (units). This suggests the possibility of formulating the optimal control problem in a distributed fashion, rather than solving

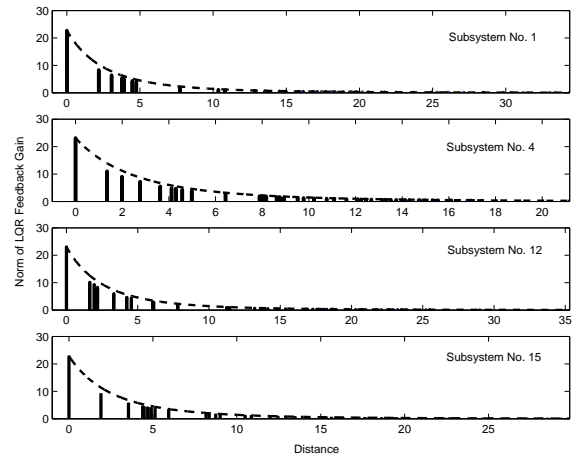


Fig. 6. Norm of LQR feedback gain  $\|[\mathcal{K}]_{ki}\|$  (bar) and  $\frac{\|[\mathcal{K}]_{kk}\|}{\chi_\nu(\text{dis}(k,i))}$  when  $\lambda = 0.1$  and  $\nu = 4$  (dashed) for subsystems  $k = 1, 4, 12, 15$ , respectively, from top to bottom.

a centralized high-dimension algebraic equation such as (37) (see [27]).

#### B. Exponential Decay

In the next simulation, the exponential coupling characteristic functions given by (22) are investigated. The simulation parameter is selected as  $\zeta = e - 1$ . Figure 7 shows the norm of LQR feedback gains (38) corresponding to agents  $k = 1, 4, 12, 15$  versus the distance of other subsystems to subsystem  $k$ . For subsystem number  $k$ , it can be seen that norm of the optimal feedback gains are upper bounded by function  $\frac{\|[\mathcal{K}]_{kk}\|}{\chi_\zeta(\text{dis}(k,i))}$  which implies that  $\|[\mathcal{K}]_{ki}\|$  decays exponentially as distance increases. According to simulations, the coupling strength between subsystems are negligible for those which are beyond distance of approximately 5 (units) from each other. Therefore, spatial truncation can be performed around each subsystem beyond radius of 5 (units).

#### C. Nearest Neighbor Coupling

In the last simulation, the nearest neighbor coupling case with  $T = 5$  is studied, as discussed in Section VI. Figure 8 represents the norm of LQR feedback gains (38) corresponding to agents  $k = 1, 4, 12, 15$  versus the distance of other subsystems to subsystem  $k$ . In this case, the coupling strength is negligible between dynamically uncoupled subsystems, i.e., subsystems with a distance greater than  $T = 5$  (units). In fact, by selecting a nearest neighbor coupling policy, we impose a specific architecture on the network. Simulation results affirm that the optimal controller inherits the same architecture as the underlying system. Naturally the underlying solution is SD both with respect to algebraic and exponential coupling characteristic function

#### D. Spatial Truncation

Let  $\mathcal{K}_T$  be the spatially truncated operator defined as follows

$$[\mathcal{K}_T]_{ki} = \begin{cases} [\mathcal{K}]_{ki} & \text{if } \text{dis}(k,i) \leq T \\ 0 & \text{if } \text{dis}(k,i) > T. \end{cases}$$

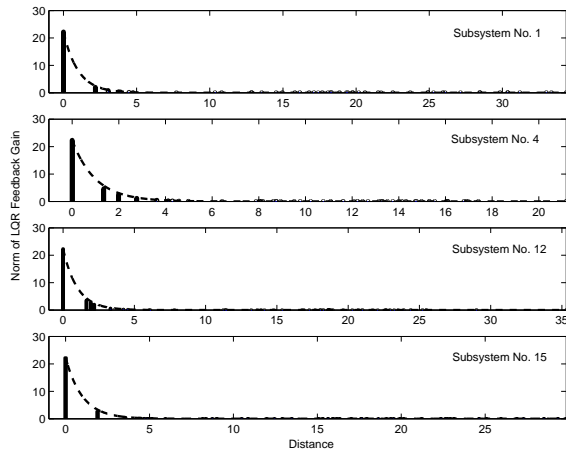


Fig. 7. Norm of LQR feedback gain  $\|[\mathcal{K}]_{ki}\|$  (bar) and  $\frac{\|[\mathcal{K}]_{kk}\|}{\chi_{\zeta}(\text{dis}(k,i))}$  when  $\zeta = e - 1$  (dashed) for subsystems  $k = 1, 4, 12, 15$ , respectively, from top to bottom.

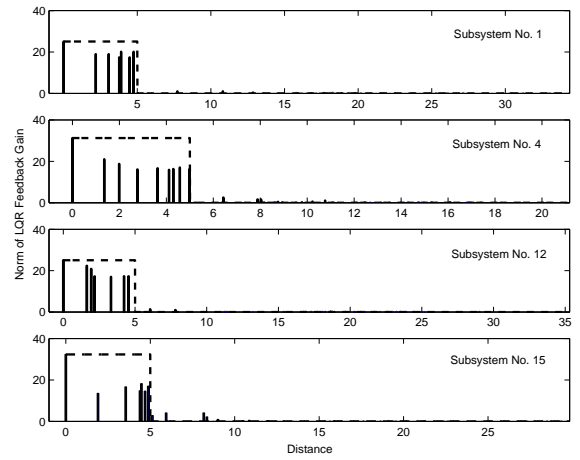


Fig. 8. Norm of LQR feedback gain  $\|[\mathcal{K}]_{ki}\|$  (bar) and  $\|[\mathcal{K}]_{ki}\| \times$  pulse function with length  $T = 5$  (dashed) for subsystems  $k = 1, 4, 12, 15$ , respectively, from top to bottom.

Applying the small-gain stability argument, it is straightforward to verify that in the above simulations the truncated feedback  $\mathcal{K}_T$  is stabilizing if  $T \geq T_s$  in which:

- $T_s = 4.5$  for algebraical decay.
- $T_s = 0$  for exponential decay.
- $T_s = 5$  for nearest neighbor coupling.

Figure 9 illustrates the performance loss percentage defined as

$$\left| \frac{\lambda_{\max}(\mathcal{P} - \mathcal{P}_T)}{\lambda_{\max}(\mathcal{P})} \right| \times 100$$

versus different values of  $T \geq T_s$  for different coupling characteristic functions. As seen from Figure 9, the larger values of truncation length  $T$  ensue better closed-loop performance. For example, a 10% performance loss trade off results in to the following stabilizing truncation lengths:

- $T = 13.2$  (units) for algebraical decay.
- $T = 2.1$  (units) for exponential decay.
- $T = 5.2$  (units) for nearest neighbor coupling.

### X. CONCLUSIONS

In this paper we studied the spatial structure of infinite horizon optimal controllers for spatially distributed systems. By introducing the notion of SD operators we extended the notion of analytic continuity to operators that are not spatially invariant. Furthermore, we proved that SD operators form a Banach algebra. We used this to prove that solutions of Lyapunov and Riccati equations for SD systems are themselves SD. This result was utilized to show that the kernel of optimal LQ feedback is also SD. Although these results were proven for LQ problems, they can be easily extended to general  $\mathcal{H}_2$ , and  $\mathcal{H}_{\infty}$  optimal control problems as the key enabling property is the spatial decay of solution of the corresponding Riccati equations. One major implication of these results is that the optimal control problem for spatially decaying systems lends itself to distributed solutions without too much loss in performance as even the centralized solutions are inherently localized. These results have been extended to the case of

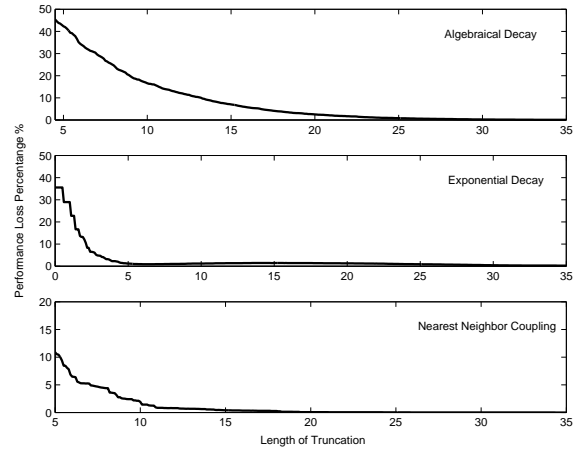


Fig. 9. Performance Loss percentage of LQR controller after spatial truncation for different types of couplings: (i) algebraical decay (ii) exponential decay (iii) nearest neighbor coupling.

constrained finite horizon optimal control problems by blending the ideas developed here with Multi Parametric Quadratic Progaming [32], [33]. One important future research direction is to further study the case of SD operators with finite support (e.g., systems with nearest neighbor coupling). It would be interesting to find out under what extra conditions the solutions are themselves finite support, as opposed to just being spatially decaying. This would provide an interesting connection between these results and those of [30].

### XI. APPENDIX

#### A. Banach Algebra $S_b(\mathcal{C})$

*Proof:* Properties (1) and (2) are immediate from the definition. To prove (3), we use the following chain of in-

equalities:

$$\begin{aligned} \|\mathcal{Q} + \mathcal{P}\|_b^* &= \sup_{\alpha \in [0, b]} \sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \|[\mathcal{Q} + \mathcal{P}]_{ki}\| \chi_\alpha(\text{dis}(k, i)) \\ &\leq \sup_{\alpha \in [0, b]} \sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} (\|[\mathcal{Q}]_{ki}\| + \|[\mathcal{P}]_{ki}\|) \chi_\alpha(\text{dis}(k, i)) \\ &\leq \|\mathcal{Q}\|_b^* + \|\mathcal{P}\|_b^*. \end{aligned}$$

To show property (4), we proceed as follows

$$\begin{aligned} \|\mathcal{QP}\|_b^* &\leq \sup_{\alpha \in [0, b]} \sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \|[\mathcal{QP}]_{ki}\| \chi_\alpha(\text{dis}(k, i)) \\ &= \sup_{\alpha \in [0, b]} \sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \sum_{j \in \mathbb{G}} [\mathcal{Q}]_{kj} [\mathcal{P}]_{ji} \chi_\alpha(\text{dis}(k, i)). \end{aligned}$$

Using the fact that the induced norm of linear maps is submultiplicative, we obtain the following

$$\|\mathcal{QP}\|_b^* \leq \sup_{\alpha \in [0, b]} \sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \sum_{j \in \mathbb{G}} \|[\mathcal{Q}]_{kj}\| \|[\mathcal{P}]_{ji}\| \chi_\alpha(\text{dis}(k, i))$$

For every  $j \in \mathbb{G}$ , we have that  $\text{dis}(k, i) \leq \text{dis}(k, j) + \text{dis}(j, i)$ . Applying this inequality and using the definition 5, the following chain of inequalities hold

$$\begin{aligned} \chi_\alpha(\text{dis}(k, i)) &\leq \chi_\alpha(\text{dis}(k, j) + \text{dis}(j, i)) \\ &\leq \chi_\alpha(\text{dis}(k, j)) \chi_\alpha(\text{dis}(j, i)) \end{aligned}$$

therefore,

$$\begin{aligned} \|\mathcal{QP}\|_b^* &\leq \sup_{\alpha \in [0, b]} \sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \sum_{j \in \mathbb{G}} \|[\mathcal{Q}]_{kj}\| \|[\mathcal{P}]_{ji}\| \chi_\alpha(\text{dis}(k, j) + \text{dis}(j, i)) \\ &\leq \sup_{\alpha \in [0, b]} \sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \sum_{j \in \mathbb{G}} \|[\mathcal{Q}]_{kj}\| \|[\mathcal{P}]_{ji}\| \chi_\alpha(\text{dis}(k, j)) \chi_\alpha(\text{dis}(j, i)). \end{aligned}$$

Finally, we can write

$$\|\mathcal{QP}\|_b^* \leq \sup_{\alpha \in [0, b]} \left( \sup_{k \in \mathbb{G}} \sum_{j \in \mathbb{G}} \|[\mathcal{Q}]_{kj}\| \chi_\alpha(\text{dis}(k, j)) \times \sup_{j \in \mathbb{G}} \sum_{i \in \mathbb{G}} \|[\mathcal{P}]_{ji}\| \chi_\alpha(\text{dis}(j, i)) \right)$$

From this, we get the final result

$$\|\mathcal{QR}\|_b^* \leq \|\mathcal{Q}\|_b^* \|\mathcal{R}\|_b^*.$$

The last step of proof is to show that  $\mathcal{S}_b(\mathcal{C})$  is complete. Consider the Cauchy sequence  $\{\mathcal{P}(t) \in \mathcal{S}_b(\mathcal{C}) : t \geq 0\}$  and the corresponding sequence of continuous functions

$$\varphi_t(\alpha) = \sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \|[\mathcal{P}(t)]_{ki}\| \chi_\alpha(\text{dis}(k, i))$$

defined on interval  $[0, b)$ . According to definition, we have that

$$\lim_{t, s \rightarrow \infty} \|\mathcal{P}(t) - \mathcal{P}(s)\|_b^* = 0. \quad (39)$$

Since  $\mathcal{S}_b(\mathcal{C}) \subset \mathcal{L}(\ell_\infty)$ , we may assume that  $\lim_{t \rightarrow \infty} \mathcal{P}(t) = \mathcal{P}$  in which  $\mathcal{P} \in \mathcal{L}(\ell_\infty)$ . It follows that

$$\lim_{t \rightarrow \infty} \|[\mathcal{P}(t)]_{ki}\| = \|[\mathcal{P}]_{ki}\|.$$

Hence,  $\varphi_t(\alpha) \rightarrow \varphi(\alpha)$  pointwise as  $t \rightarrow \infty$  where

$$\varphi(\alpha) = \sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \|[\mathcal{P}]_{ki}\| \chi_\alpha(\text{dis}(k, i)).$$

By applying the triangle inequality, we have

$$\begin{aligned} &\sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \|[\mathcal{P}(t)]_{ki}\|_2 \chi_\alpha(\text{dis}(k, i)) - \\ &\sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \|[\mathcal{P}(s)]_{ki}\|_2 \chi_\alpha(\text{dis}(k, i)) \\ &\leq \sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \|[\mathcal{P}(t) - \mathcal{P}(s)]_{ki}\|_2 \chi_\alpha(\text{dis}(k, i)) \end{aligned}$$

therefore,

$$|\varphi_t(\alpha) - \varphi_s(\alpha)| \leq \sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \|[\mathcal{P}(t) - \mathcal{P}(s)]_{ki}\| \chi_\alpha(\text{dis}(k, i))$$

for all  $\alpha \in [0, b)$ . Hence,

$$\|\varphi_t - \varphi_s\|_\infty = \sup_{\alpha \in [0, b)} |\varphi_t(\alpha) - \varphi_s(\alpha)| \leq \|\mathcal{P}(t) - \mathcal{P}(s)\|_b^*.$$

By applying (39), we have

$$\lim_{t, s \rightarrow \infty} \|\varphi_t - \varphi_s\|_\infty = 0. \quad (40)$$

According to the Cauchy criteria (cf. Theorem 7.3.1 of [43]), (40) is equivalent to the fact that  $\varphi_t$  converges uniformly to the limit function  $\varphi$  on  $[0, b)$ , i.e.,

$$\lim_{t \rightarrow \infty} \|\varphi_t - \varphi\|_\infty = 0.$$

Hence, we have that  $\|\varphi\|_\infty < \infty$ . Furthermore,  $\varphi$  is continuous on  $[0, b)$  (see theorem 7.3.2 of [43]). This proves that  $\mathcal{P} \in \mathcal{S}_b(\mathcal{C})$  and this completes the proof. ■

## B. Proof of Theorem 5

*Proof:* From property (1), it follows that

$$\lim_{t, s \rightarrow \infty} \|\mathcal{P}(t) - \mathcal{P}(s)\|_{\ell_p \rightarrow \ell_p} = 0.$$

In the following, we will prove that

$$\lim_{t, s \rightarrow \infty} \|\mathcal{P}(t) - \mathcal{P}(s)\|_b^* = 0.$$

In section VII, it is shown that  $\mathcal{S}_b(\mathcal{C})$  is a normed vector space. The norm  $\|\cdot\|_b^*$  is a continuous function on  $\mathcal{S}_b(\mathcal{C})$ . Therefore, the function  $\|\cdot\|_b^* : \mathcal{S}_b(\mathcal{C}) \cap \mathcal{L}(\ell_p) \rightarrow \mathbb{R}^+$  is continuous. Consider the sequence of operators  $\mathcal{P}(t)$  in  $\mathcal{S}_b(\mathcal{C}) \cap \mathcal{L}(\ell_p)$ . By assumption

$$\lim_{t, s \rightarrow \infty} \|(\mathcal{P}(t) - \mathcal{P}(s))\phi\|_p = 0$$

for all  $\phi \in \ell_p$ . Therefore, we have

$$\lim_{t \rightarrow \infty} \|[\mathcal{P}(t) - \mathcal{P}(s)]_{ki}\| = 0$$

for all  $k, i \in \mathbb{G}$ . From the continuity property of the norm  $\|\cdot\|_b^*$  on  $\mathcal{S}_b(\mathcal{C})$ , it follows that

$$\begin{aligned} \lim_{t, s \rightarrow \infty} \|\mathcal{P}(t) - \mathcal{P}(s)\|_b^* &= \lim_{t, s \rightarrow \infty} \|(\mathcal{P}(t) - \mathcal{P}(s))\|_b^* \\ &= \sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} \left( \lim_{t \rightarrow \infty} \|[\mathcal{P}(t) - \mathcal{P}(s)]_{ki}\| \right) \chi_\alpha(\text{dis}(k, i)) = 0. \end{aligned}$$

This result shows that  $\{\mathcal{P}(t) : t \in \mathbb{R}^+\}$  is a Cauchy sequence in  $\mathcal{S}_b(\mathcal{C})$ . Therefore, using the fact that  $\mathcal{S}_b(\mathcal{C})$  is a Banach Algebra, we conclude that  $\mathcal{P} \in \mathcal{S}_b(\mathcal{C})$ . ■

## REFERENCES

- [1] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. on Automatic Control*, vol. 48, no. 6, pp. 988–1001, June 2003.
- [2] R. W. Beard and V. Stepanyan, "Synchronization of information in distributed multiple vehicle coordinated control," in *Proc. of IEEE Conference on Decision and Control*, December 2003.
- [3] P. Ogren, E. Fiorelli, and N. Leonard, "Cooperative control of mobile sensing networks: Adaptive gradient climbing in a distributed environment," *IEEE Transaction on Automatic Control*, vol. 49(8), pp. 1292–1302, August 2004.
- [4] J. Cortes, S. Martinez, T. Karatas, and F. Bullo, "Coverage control for mobile sensing networks," *IEEE Trans. on Robotics and Automation*, vol. 20, no. 2, pp. 243–255, February 2004.
- [5] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, September 2004.
- [6] J. A. Fax and R. M. Murray, "Graph Laplacians and stabilization of vehicle formations," *15th IFAC Congress, Barcelona, Spain*, 2002.
- [7] E. Justh and P. Krishnaprasad, "Equilibria and steering laws for planar formations," *Systems and Control letters*, vol. 52, no. 1, pp. 25–38, May 2004.
- [8] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Trans. on Automatic Control*, vol. 50, no. 2, pp. 169–182, February 2005.
- [9] Z. Lin, M. Brouke, and B. Francis, "Local control strategies for groups of mobile autonomous agents," *IEEE Trans. on Automatic Control*, vol. 49, no. 4, pp. 622–629, April 2004.
- [10] W. B. Dunbar and R. M. Murray, "Distributed receding horizon control with application to multi-vehicle formation stabilization," *Automatica*, vol. 2, no. 4, pp. 549–558.
- [11] P. Varaiya, "Smart cars on smart roads: problems of control," *IEEE Tran. on Automatic Control*, vol. 38, no. 2, pp. 195 – 207, 1993.
- [12] H. Raza and P. Ioannou, "Vehicle following control design for automated highway systems," *IEEE Control Sys. Mag.*, vol. 16, no. 6, pp. 43 – 60, Dec. 1996.
- [13] G. Stewart, D. Gorinevsky, and G. Dumont, "Feedback controller design for a spatially-distributed system: The paper machine problem," *IEEE Trans. on Control Systems Tech.*, vol. 11, no. 5, pp. 612–628, 2003.
- [14] S. Jiang, P. Voulgaris, L. Holloway, and L. Thompson, "Distributed control of large segmented telescopes," in *Proc. of the American Control Conference*, Minneapolis, 2006, pp. 1942 – 1947.
- [15] E. Camponogara, D. Jia, B. Krogh, and S. Talukdar, "Distributed model predictive control," *IEEE Control Systems Magazine*.
- [16] M. Napoli, B. Bamieh, and M. Dahleh, "Optimal control of arrays of microcantilevers," *Trans. ASME J. Dyna. Syst., Measure, Control*, vol. 121, no. 4, pp. 686–690, 1999.
- [17] B. Bamieh, F. Paganini, and M. A. Dahleh, "Distributed control of spatially invariant systems," *IEEE Trans. Automatic Control*, vol. 47, no. 7, pp. 1091–1107, 2002.
- [18] R. D'Andrea and G. Dullerud, "Distributed control design for spatially interconnected systems," *IEEE Trans. Automatic Control*, vol. 48, no. 9, pp. 1478–1495, 2003.
- [19] C. Langbort, R. Chandra, and R. D'Andrea, "Distributed control design for systems interconnected over an arbitrary graph," *IEEE Trans. on Automatic Control*, vol. 49, no. 9, pp. 1502–1519, 2004.
- [20] B. Recht and R. D'Andrea, "Distributed control of systems over discrete groups," *IEEE Trans. on Automatic Control*, vol. 49, no. 9, pp. 1446–1452, 2004.
- [21] R. Chandra and R. D'Andrea, "A scaled small gain theorem with applications to spatially interconnected systems," *IEEE Tran. on Automatic Control*, vol. 51, no. 3, pp. 465–469, 2006.
- [22] B. Bamieh and P. Voulgaris, "A convex characterization of distributed control problems in spatially invariant systems with communication constraints," *Systems and Control Letters*, vol. 54, no. 6, pp. 575–583, 2005.
- [23] J. Shamma and G. Arslan, "A decomposition approach to distributed control of spatially invariant systems," *IEEE Trans. on Automatic Control*, vol. 51, no. 4, pp. 701–707, 2006.
- [24] M. R. Jovanović and B. Bamieh, "Lyapunov-based distributed control of systems on lattices," *IEEE Tran. on Automatic Control*, vol. 50, no. 4, pp. 422–433, April 2005.
- [25] M. L. El-Sayed and P. Krishnaprasad, "Homogeneous interconnected systems : An example," *IEEE Trans. on Automatic Control*, vol. AC-26, no. 4, 1981.
- [26] C. Langbort and R. D'Andrea, "Distributed control of spatially reversible interconnected systems with boundary conditions," *SIAM J. Control Optim.*, vol. 44, no. 1, pp. 1–28.
- [27] G. Dullerud and R. D'Andrea, "Distributed control of heterogeneous systems," *IEEE Trans. on Automatic Control*, vol. 49, no. 12, pp. 2113–2128, 2004.
- [28] G. Dullerud and S. Lall, "A new approach for analysis and synthesis of time-varying systems," *IEEE Trans. on Automatic Control*, vol. 44, no. 8, pp. 1486 – 1497, 1999.
- [29] G. Hagen, I. Mezic, and B. Bamieh, "Distributed control design for parabolic evolution equations: Application to compressor stall control," *IEEE Trans. Automatic Control*, vol. 49, no. 8, 2004.
- [30] M. Rotkowitz and S. Lall, "A characterization of convex problems in decentralized control," *IEEE Tran. on Automatic Control*, vol. 51, no. 2, pp. 274 – 286, 2006.
- [31] W. Rudin, *Fourier Analysis on Groups*. Interscience, 1962.
- [32] N. Motee and A. Jadbabaie, "Distributed receding horizon control of spatially invariant systems," in *Proc. of the American Control Conference*, Minneapolis, MN, 2006.
- [33] —, "Receding horizon control of spatially distributed systems over arbitrary graphs," in *Proc. 45th IEEE Conference on Decision and Control*, San Diego, CA, 2006.
- [34] P. Lax, *Functional Analysis*. John Wiley, 2002.
- [35] R. Curtain and H. Zwart, *An introduction to Infinite-Dimensional Linear Systems Theory*. Springer-Verlag, 1995.
- [36] S. Kakaç and Y. Yener, *Heat conduction, 3rd Ed.* Taylor & Francis, 1993.
- [37] B. van Keulen,  *$\mathcal{H}_\infty$ -Control for Distributed Parameter Systems: A State-Space Approach*. Boston, MA: Birkhauser, 1993.
- [38] L. Hormander, *The Analysis of Linear Partial Differential Operators I, 2nd Ed.* Springer-Verlag, 1990.
- [39] F. R. K. Chung, *Spectral graph theory*. Regional Conference Series in Mathematics 92, Amer. Math. Soc., Providence, 1997.
- [40] F. Cucker and S. Smale, "Emergent behaviour in flocks," 2005.
- [41] T. S. Rappaport, *Wireless Communications, Principles and Practice*. Upper Saddle River, NJ: Prentice Hall, 1999.
- [42] J. C. Willems, *The Analysis of Feedback Systems*. The MIT Press, 1971.
- [43] R. S. Strichartz, *The Way of Analysis*. Jones and Bartlett Publishers, 2000.