ON THE ISS PROPERTY FOR RECEDING HORIZON CONTROL OF CONSTRAINED LINEAR SYSTEMS

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Abstract: Recent results on receding horizon control of linear systems with state and input constraints have shown that the optimal receding horizon controller is piecewise affine and continuous, with the resulting value function being piecewise quadratic and continuously differentiable. The purpose of this note is to exploit these results to show that the controller renders the closed-loop system globally input-to-state stable (ISS) when the open-loop system is stable, and locally input-to-state stable when the open-loop plant is unstable. While the result is simple in nature, it has interesting implications in utilizing constrained receding horizon scheme in a switching based supervisory control framework.

Keywords: Receding horizon control, constrained control, Input-to-state stability

1. INTRODUCTION

Receding horizon control is the most popular scheme for control of constrained processes. The main idea behind receding horizon control is to solve a finite horizon, open-loop optimal control problem. The first portion of the optimal control sequence is then applied to the plant, until a new state update occurs. This procedure is then repeated, resulting in an implicit feedback policy. It is well known that a naive application of this strategy would lead to instability, even, in the case of unconstrained linear systems, if the horizon length is short. Several necessary ingredients for stability have been identified in the literature. (See (Mayne *et al.* (2000); Mayne (2001)) for an excellent up-to-date survey of these results). While the stability issue has been treated fairly well in the nonlinear case, the robustness issue is not resolved even in the linear case. Two major strategies have been proposed in the literature for addressing the issue of robustness. The first one poses the problem as a min-max optimization, where the maximum is taken over all possible realizations of the disturbance sequence, and the minimum is taken over control sequences. Although some characterizations of this result exist even in the nonlinear case, since they are of an open-loop nature, they tend to be extremely conservative (Mayne (2001)). An alternative choice is to search over *control policies* rather than control sequences. While this is clearly less conservative, it makes the computations prohibitively complex. In the linear case, however, due to the fact that the underlying sets are polytopes, and because relatively efficient numerical techniques exist for set theoretic manipulation of polytopic sets, the problem is tractable to a certain degree (Gilbert and Tan (1991); Kolmanovsky and Gilbert (1998)). The purpose of this paper is to show that the receding horizon controller, at least locally (and globally, when the system is open-loop stable), has some inherent robustness properties without explicitly taking the disturbance into account. Specifically, using recent results in explicit characterization of the receding horizon feedback law and the corresponding value function, we show that the closed-loop system is input-to-state stable (ISS) since it admits an ISS Lyapunov function (Sontag and Yang (1995)).

Roughly speaking, if the system has finite input-state gain, then the system is ISS. In the case where the system is open-loop stable, and only input constraints are present, we show that the closed-loop system with a receding horizon feedback is globally ISS. When the system is unstable, the constrained receding horizon control scheme is stabilizing only over compact subsets of the state space, hence the results are local. In this case, we will use some recent results in Mayne and Langson (2001) to obtain a compact set of initial conditions for which the receding horizon scheme "works" in spite of the disturbances. These results are the first step in developing a switching based supervisory control scheme in which the candidate controllers are defined implicitly, as receding horizon feedback controllers.

This paper is organized as follows: In section 2 we formulate the problem and review the stability results. Section 3 we present our result for open-loop stable linear systems with input constraints. In section 4 we present the local ISS result for open-loop unstable systems. Finally, we present our conclusions in section 5.

2. PROBLEM SETTING

Our notation will be consistent with (Mayne *et al.* (2000)), and we focus on the discrete-time case. The system under consideration is

$$\begin{aligned} x^+ &= Ax + Bu \\ u(k) &\in \mathcal{U}, \;\; x(k) \in \mathcal{X} \;\; k = 0 \cdots, N-1 \end{aligned} \tag{1}$$

where $\mathcal{U}$ is a polytope and $\mathcal{X}$ is a polyhedron, $x \in \mathbb{R}^n$, and $A,\ B$ are matrices of appropriate dimension, and the pair (A, B) is assumed to be stabilizable. We are interested in solving the following optimization problem

$$\begin{aligned} V_N^*(x) = \min_{u \in \mathcal{U}} & \sum_{i=0}^{N-1} (x(k)^T Q x(k) + u^T R u(k)) \\ & + F(x(N)) \end{aligned}$$

where $x(k) = x^u(i; x, 0)$, $Q > 0,\ R > 0$, and $F(\cdot)$ is chosen to be an appropriate terminal cost. The solution to the above optimization problem is the optimal sequence

$$u^o(x) := \{u^o(0;x), u^o(1;x), \cdots, u^o(N-1;x)\}$$

The receding horizon feedback is defined for all initial conditions in a *controllability set* X_N, the set of initial conditions that can be steered into the set $\mathcal{X}_f$ in N steps or less. The set $\mathcal{X}_f$ is a set over which there exist a feasible and stabilizing control, and applying the feasible controller results in feasible state trajectories as well. Using specific choices for $F(\cdot)$ and $\mathcal{X}_f$, one can prove exponential stability of the closed-loop system under the receding horizon feedback $\kappa_N(x) := u^o(0;x)$. If the system is open-loop stable and there are no state constraints, $\mathcal{X}_f = \mathbb{R}^n$, and $\kappa_N(x)$ is globally exponentially stabilizing. In the next section, we will show that the above controller also makes the system ISS.

3. RHC FOR STABLE SYSTEMS WITH INPUT CONSTRAINTS

A stabilizing strategy for receding horizon control of input constrained, stable linear systems was first given in Rawlings and Muske (1993). In that setting, the terminal cost $F(\cdot)$ was chosen to be the cost incurred by flowing along the open-loop trajectory, and is of the form $x^T P x$ where P is the positive definite solution to the following Lyapunov equation

$$A^T P A - P + Q = 0.$$

In the absence of state constraints ($\mathcal{X} = \mathbb{R}^n$), this choice of F will result in global exponential stability, having $\mathcal{X}_f = \mathbb{R}^n$.

In order to analyze the ISS property for this system, we note that recently, an explicit characterization of the solution of the linear constrained receding horizon problem has become available (Bemporad *et al.* (1999)). The solution to the constrained RHC problem has been shown to be continuous and piecewise affine. The resulting value function is shown to be piecewise quadratic and differentiable), with a piecewise affine derivative (under some mild assumptions). The regions in each the control is piecewise affine are the regions where the active constraints do not change.

Using this result we can show that the the closed-loop system is a piecewise affine system which is globally exponentially stable with a piecewise quadratic Lyapunov function $V_N^*(x) = \bar{x}^T P_{i(x)} \bar{x}$ ($\bar{x} = [x \quad 1]$) which is continuously differentiable (Bemporad *et al.* (1999); Mayne (2001)), where $i(\cdot)$ is a switching function that maps the state space to a finite set of indices labeling the polytpoic partitions of the state-space. We that since the resulting controller is piecewise affine instead of linear, we should augment the state with the constant 1 and re-define the A and B matrices accordingly. It should also be noted that there is a region around origin where the controller is piecewise linear, i.e., the constant term is zero. In the interest of clarity, with a slight abuse of notation, we use the same notation as before for the augmented system.

Several results exist for analyzing stability of piecewise linear systems (cf. Liberzon and Morse

(1999); Johansson and Rantzer (1998)). The main point in stability of state dependent piecewise affine systems is that each quadratic value function be a Lyapunov function in their corresponding polytopic set, and a matching condition hold at the switching boundary (Hespanha (2001)), i.e., the value of the Lyapunov function should decrease or remain the same on the switching surfaces, while it should decrease inside each polytopic set. In the case of receding horizon controllers, due to continuous differentiability of the value function (Mayne (2001)), and also because there are only a finite number of partitions when the horizon length is finite [1], these conditions exactly hold. We can therefor state the following theorem:

Theorem 1. The receding horizon scheme globally input-to-state- stabilizes stable linear systems with input constraints, with respect to additive disturbance w.

Proof: Let $x(\cdot)$ be the trajectory of the system when the additive disturbance $w(\cdot)$ is present. The receding horizon feedback $\kappa_N(x)$ is piecewise linear (in the extended state), which makes the closed-loop system a linear difference inclusion with an additive disturbance.

$$\begin{aligned} x^+ &= (A - B\kappa_N(x))x(k) + w(k) \\ &:= \bar{A}_{i(x(k))}x(k) + w(k) \end{aligned} \tag{2}$$

where $\bar{A}_{i(x(k))}$ is the closed-loop matrix corresponding to the ith partition of the state space, and i is the switching function that maps the state space to a finite set of indices corresponding to different polytopic regions where the active constraints do not change. Since the disturbance-free system is globally exponentially stable with the piecewise quadratic value function $V_N^*(\hat{x})$, we have

$$V_N^*(A_{i(x)}x) - V_N^*(x) \leq -C_q||x||^2$$

where $C_q > 0$ is the rate of exponential decay for the closed-loop disturbance-free system. Utilizing the fact that the value function is continuously differentiable,we can write

$$\begin{aligned} V_N^*(x^+) - V_N^*(x) &= V_N^*(\bar{A}_{i(x)}x + w) - V_N^*(x) \\ &= V_N^*(\bar{A}_{i(x)}x + w) - V_N^*(\bar{A}_{i(x)}x) \\ &+ V_N^*(\bar{A}_{i(x)}x) - V_N^*(x). \end{aligned} \tag{3}$$

We now note that the controllability set X_N (which in this case is the whole space $\mathbb{R}^n$) is partitioned into a finite number of polytopes. This, in addition to continuous differentiability of V_N^*, and the fact that the value function is piecewise quadratic, implies

$$||\frac{\partial V_N^*(x)}{x}|| := ||P_{i(x)}x|| \leq L||x||$$

where $L := \max_i \lambda_{\max}(P_i)$ and $\lambda_{\max}$ is the largest singular value. The maximum in the above equation is taken over all possible partitions. Since there are only a finite number of partitions, the maximum exists. We now use the above bound on the norm of the gradient of the value function in conjunction with the mean value theorem to conclude the following

$$\begin{aligned} &||V_N^*(\bar{A}_{i(x)}x + w) - V_N^*(\bar{A}_{i(x)}x)|| \\ &\leq ||\frac{\partial V_N^*(y)}{\partial y}||_{y=\bar{A}_{i(x)}x+sw}||w|| \\ &\leq L||\bar{A}_{i(x)}x + sw||||w|| \\ &\leq L[(\max_i ||\bar{A}_{i(x)}||||x||||w||) + ||w||^2), \end{aligned} \tag{4}$$

where $0 \leq s \leq 1$. Let $C_A := \max_i ||\bar{A}_{i(x)}||$. We note that for any positive ϵ and any two signals x, and w, we have $||x||||w|| \leq \epsilon||x||^2 + \frac{1}{\epsilon}||w^2||$. We can now write (4) as

$$\begin{aligned} &||V_N^*(\bar{A}_{i(x)}x + w) - V_N^*(\bar{A}_{i(x)}x)|| \\ &< LC_A\epsilon||x||^2 + (1 + \frac{LC_A}{\epsilon})||w||^2. \end{aligned} \tag{5}$$

Finally, we use (3) and (5) to conclude the following (ϵ is a small enough constant)

$$\begin{aligned} V_N^*(x^+) - V_N^*(x) \leq (-C_q + \epsilon LC_A)||x(k)||^2 + \\ (1 + \frac{1}{\epsilon})LC_A||w(k)||^2. \end{aligned}$$

Therefore the piecewise quadratic value function is also an ISS Lyapunov function for the closed-loop system. □

Remark 2. In the special case of marginally stable systems, i.e., systems with simple poles on the imaginary axis, one can show that the exponential stability of the receding horizon scheme is *semi global*, i.e., the region of attraction can be enlarged as much as desired. Using a simple extension of the above argument, it is expected that the ISS property would be semi-global as well.

Remark 3. One can minimize an upper bound on the estimate of the $\mathcal{L}_2$ gain of the closed-loop system by solving a semidefinite program.

In the next section, we turn to unstable linear systems and characterize a local version of the above result. Since the mere existence of input constraints results in local rather than global stability arguments, adding state constraints would not change the nature of the result.

[1] It is not clear whether the number of partitions remain finite when the horizon length approaches infinity

4. UNSTABLE LINEAR SYSTEMS WITH STATE AND INPUT CONSTRAINTS

When the open-loop system is unstable, there is no guarantee that the receding horizon optimizations stay feasible for all points in the controllability set X_N in the presence of disturbances. To address this problem, we resort to a recent result of Mayne and Langson (2001), in which the authors provide a compact set of initial conditions $\bar{X}_N \subset X_N$ such that when a slightly modified receding horizon controller of the form $h(x, \hat{x}) := \bar{\kappa}_N(\hat{x}) + K(x - \hat{x})$ is employed, the open-loop optimizations remain feasible. In this setting, $\hat{x}$ is the state of the nominal system, and x is the state of the actual system with additive disturbance w which is assumed to be in a compact set $\mathcal{W}$. The gain matrix K is any stabilizing feedback which renders $A + BK$ stable.

In other words, Mayne and Langson provide a smaller set of initial conditions which is obtained by calculating the controllability set $\bar{X}_N$ for the nominal system, by imposing tighter state and control constraints, however, inside this set, the receding horizon scheme remains feasible for any value of the disturbance in the allowable set $\mathcal{W}$. The new controllability set is the set of all initial conditions which can be steered to a terminal set despite the presence of disturbances. Using the exact same argument as in the previous section, we can show that for all states in the modified controllability set $\bar{X}_N$, and for all disturbances in the disturbance set $\mathcal{W}$, the closed loop system with the modified receding horizon controller $h(\cdot, \cdot)$ in the loop, is input to state stable.

5. CONCLUSIONS

We showed that the implicit receding horizon feedback not only exponentially stabilizes constrained discrete-time linear systems (as was known before), but also renders them Input-to-state stable. While this result is obvious for linear systems, it is not so for a receding horizon controller in the loop.

In the light of new results providing explicit characterizations of the solutions to the constrained linear receding horizon control problem, we have shown that the closed-loop system is indeed input to state stable, without any explicit design for robustness. The results were shown to be global for stable systems. In the case of unstable systems, it is possible to find a compact set of initial conditions in which the receding horizon controller "works", despite the presence of bounded disturbances. The above result would hopefully facilitate the development of switching based supervisory control schemes that switch between candidate receding horizon controllers.

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