Fall 2010 CIS 160

Mathematical Foundations of Computer Science

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Practice Final Exam

December 10, 2010; Solutions will be posted on December 14, 2010

This is an *open book* exam. In solving a Problem, you may use, *without proof*, results from the notes or previous Homework assignments. However, give a precise reference for any result you are quoting and be very clear as to what exactly you are assuming if you don't prove it.

Please, be concise!

Note that this practice exam contains more problems than the actual final will have (about three more!).

Problem 1. (**15 points**) Give an an intuitionistic proof for

$$\neg\neg(P \vee \neg P).$$

Problem 2. (**20 points**) Let X, Y, Z be any three nonempty sets and let $f\colon X \to Y$ be any function. Define the function, $R_f\colon Z^Y \to Z^X$ (R_f, as a reminder that we compose with f on the right), by

$$R_f(h) = h \circ f,$$

for every function, $h\colon Y \to Z$.

Let T be another nonempty set and let $g\colon Y \to T$ be any function.

(a) Prove that

$$R_{g\circ f} = R_f \circ R_g$$

and if $X = Y$ and $f = \mathrm{id}_X$, then

$$R_{\mathrm{id}_X}(h) = h,$$

for every function, $h\colon X \to Z$.

(b) Use (a) to prove that if f is surjective, then R_f is injective and if f is injective, then R_f is surjective.

Problem 3. (**10 points**) Prove that

$$1\cdot 2^0 + 2\cdot 2^1 + 3\cdot 2^2 + \cdots + n\cdot 2^{n-1} = (n-1)2^n + 1,$$

for all $n \geq 1$.

Problem 4. (**10 points**) Recall that a set, A, is infinite iff there is no bijection from $\{1, \ldots, n\}$ onto A, for any natural number, $n \in \mathbb{N}$. Prove that the set of even natural numbers is infinite.

Problem 5. (**15 points**) Consider the sum

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)},$$

with $n \geq 1$.

Which of the following expressions is the sum of the above:

$$(1)\ \frac{1}{n+1} \qquad\qquad (2)\ \frac{n}{n+1}.$$

Justify your answer.

Problem 6. (**10 points**) It has been observed that no living person has more than 500,000 strands of hair. It is also known that New York city has at least 10 million inhabitants.

Prove that there are 20 New Yorkers who have the same number of strands of hair.

Problem 7. (**15 points**) Let R be a relation on a finite set, X, let $R^0 = \mathrm{id}_X$, the identity relation on X and let

$$R^{n+1} = R^n \circ R,$$

for all $n \geq 0$.

(a) The relation, $R \subseteq X \times X$, corresponds to a simple directed graph, $G = (X, R)$, where an edge, (a, b), is in G iff $(a, b) \in R$. Prove that

$$(a, b) \in R^k \quad \text{iff there is a path of length } k \text{ from } a \text{ to } b,$$

where $k \geq 0$.

(b) Give an example of a relation, R, for which $R^k \neq R^{k+1}$, for all $k \geq 0$.

Problem 8. (**15 points**) Consider the sequence defined recursively as follows:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_{n+2} &= F_{n+1} + F_n, \quad n \geq 0. \end{aligned}$$

Compute a few terms of this sequence. Do you recognize this sequence for $n \geq 1$?

Prove the *Cassini identity*:

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n, \qquad n \geq 1.$$

Problem 9. (**10 points**) It has been observed that no living person has more than 500,000 strands of hair. It is also known that Boston has at least 1 million inhabitants.

Prove that there are two people in Boston who have the same number of strands of hair.

Problem 10. (**15 points**) Let G be a finite undirected connected graph.

(a) If we delete some edge from G, is the remaining graph still connected? Give a proof or a counter-example.

(b) Prove that if we delete an edge belonging to a cycle in G, then the remaining graph is still connected.

Problem 11. (**10 points**) Let G be a finite graph. Prove that no edge of G can connect nodes in different connected components of G.