Fall 2010 CIS 160

Mathematical Foundations of Computer Science
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Final Exam (2 hours)

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This is a *closed book* exam but you are allowed four pages of "cheat sheets." In solving a Problem, you may use, *without proof*, results from the notes or previous Homework assignments. However, give a precise reference for any result you are quoting and be very clear as to what exactly you are assuming if you don't prove it.

Please, be concise!

Problem 1. (**20 points**) Prove that the proposition

$$((P \Rightarrow \neg P) \wedge (\neg P \Rightarrow P)) \Rightarrow Q$$

is provable intuitionistically for *every proposition*, Q.

Problem 2. (**20 points**) Let X, Y, Z be any three nonempty sets and let $g\colon Y \to Z$ be any function. Define the function, $L_g\colon Y^X \to Z^X$, (L_g, as a reminder that we compose with g on the left), by

$$L_g(f) = g \circ f,$$

for every function, $f\colon X \to Y$.

(a) Prove that if $Y = Z$ and $g = \mathrm{id}_Y$, then

$$L_{\mathrm{id}_Y}(f) = f,$$

for all $f\colon X \to Y$.

Let T be another nonempty set and let $h\colon Z \to T$ be any function. Prove that

$$L_{h\circ g} = L_h \circ L_g.$$

(b) Use (a) to prove that if g is injective, then $L_g\colon Y^X \to Z^X$ is also injective and if g is surjective, then $L_g\colon Y^X \to Z^X$ is also surjective.

Problem 3. (**10 points**) Recall that a set, A, is infinite iff there is no bijection from $\{1, \ldots, n\}$ onto A, for any natural number, $n \in \mathbb{N}$. Prove that the set of cubes of natural numbers is infinite (the set $\{n^3 \mid n \in \mathbb{N}\}$).

Problem 4. (**10 points**) Prove that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (n-1) \cdot n = \frac{(n-1)n(n+1)}{3}$$

for all $n \geq 1$.

Remark: The sum on the left is considered to be 0 when $n = 1$.

Problem 5. (**15 points**) Let R be a relation on a finite set, X, let $R^0 = \mathrm{id}_X$, the identity relation on X and let

$$R^{n+1} = R^n \circ R,$$

for all $n \geq 0$.

(a) The relation, $R \subseteq X \times X$, corresponds to a simple directed graph, $G = (X, R)$, where an edge, (a, b), is in G iff $(a, b) \in R$. For any $k \geq 0$, let

$$R^{(k)} = \mathrm{id}_X \cup R \cup R^2 \cup \cdots \cup R^k.$$

Observe that $R^{(k)} \subseteq R^{(k+1)}$.

Prove that

$$(a, b) \in R^{(k)} \quad \text{iff there is a path of length at most } k \text{ from } a \text{ to } b,$$

where $k \geq 0$.

(b) Prove that if X has n elements, then

$$R^{(n-1)} = R^{(n)}.$$

Hint. Since $R^{(n-1)} \subseteq R^{(n)}$, it is enough to prove that $R^{(n)} \subseteq R^{(n-1)}$.

Problem 6. (**15 points**) Let S be any set of 38 integers, n_i, with $1 \leq n_i \leq 999$. Prove that there must be two numbers in the set S so that their difference, $n_i - n_j$, is at most 26.

Hint. Consider the differences $n_{38} - n_{37}, n_{37} - n_{36}, \ldots, n_2 - n_1$.

Problem 7. (**15 points**) Consider the sequence, (u_n), defined recursively as follows:

$$u_{n+2} = P u_{n+1} - Q u_n,$$

for all $n \geq 0$, with arbitrary starting elements u_0 and u_1 (where P, Q are integers and u_0, u_1 are non-negative integers).

Prove that

$$u_{n+1} u_{n-1} = u_n^2 + Q^{n-1}(-Q u_0^2 + P u_0 u_1 - u_1^2),$$

for all $n \geq 1$.

Problem 8. (**15 points**) Let G be a finite connected graph with at least two nodes. Prove that G must have a node, v, such that if we delete this node and all the edges incident to it, the resulting graph is still connected.