## Fall 2010 CIS 160

# Mathematical Foundations of Computer Science Jean Gallier 

## Homework 10

December 7, 2010; Due December 14, 2010 by 2:00pm
Problem 1. Consider the recurrence relation

$$
u_{n+2}=3 u_{n+1}-2 u_{n} .
$$

For $u_{0}=0$ and $u_{1}=1$, we obtain the sequence, $\left(U_{n}\right)$, and for $u_{0}=2$ and $u_{1}=3$, we obtain the sequence, $\left(V_{n}\right)$.
(1) Prove that

$$
\begin{aligned}
U_{n} & =2^{n}-1 \\
V_{n} & =2^{n}+1
\end{aligned}
$$

for all $n \geq 0$.
(2) Prove that if $U_{n}$ is a prime number, then $n$ must be a prime number.

Hint. Use the fact that

$$
2^{a b}-1=\left(2^{a}-1\right)\left(1+2^{a}+2^{2 a}+\cdots+2^{(b-1) a}\right)
$$

Remark: The numbers of the form $2^{p}-1$, where $p$ is a prime are known as Mersenne numbers. It is an open problem whether there are infinitely many Mersenne primes.
(3) Prove that if $V_{n}$ is a prime number, then $n$ must be a power of 2 , that is, $n=2^{m}$, for some natural number, $m$.
Hint. Use the fact that

$$
a^{2 k+1}+1=(a+1)\left(a^{2 k}-a^{2 k-1}+a^{2 k-2}+\cdots+a^{2}-a+1\right) .
$$

Remark: The numbers of the form $2^{2^{m}}+1$ are known as Fermat numbers. It is an open problem whether there are infinitely many Fermat primes.

If you can prove that there are infinitely many Mersenne primes or that there are infinitely many Fermat primes you will become famous (in mathematical circles)!


Figure 1: A fan

Problem 2. Give an algorithm (in pseudo-code) for finding the connected components of a finite undirected graph.
Hint. Adapt the algorithm strcomp given in the notes.
Problem 3. Consider the undirected graph (fan) with $n+1$ nodes and $2 n-1$ edges, with $n \geq 1$, shown in Figure 1. The purpose of this problem is to prove that the number of spanning subtrees of this graph is $F_{2 n}$, the $2 n$th Fibonacci number.
(1) Prove that

$$
1+F_{2}+F_{4}+\cdots+F_{2 n}=F_{2 n+1}
$$

for all $n \geq 0$, with the understanding that the sum on the left-hand side is 1 when $n=0$ (as usual, $F_{k}$ denotes the $k$ th Fibonacci number, with $F_{0}=0$ and $F_{1}=1$ ).
(2) Let $s_{n}$ be the number of spanning trees in the fan on $n+1$ nodes $(n \geq 1)$. Prove that $s_{1}=1$ and that $s_{2}=3$.

There are two kinds of spannings trees:
(a) Trees where there is no edge from node $n$ to node 0 .
(b) Trees where there is an edge from node $n$ to node 0 .

Prove that in case (a), the node $n$ is connected to $n-1$ and that in this case, there are $s_{n-1}$ spanning subtrees of this kind, see Figure 2.

Observe that in case (b), there is some $k \leq n$ such that the edges between the nodes $n, n-1, \ldots, k$ are in the tree but the edge from $k$ to $k-1$ is not in the tree and that none of the edges from 0 to any node in $\{n-1, \ldots, k\}$ are in this tree, see Figure 3.

Furthermore, prove that if $k=1$, then there is a single tree of this kind (see Figure 4) and if $k>1$, then there are

$$
s_{n-1}+s_{n-2}+\cdots+s_{1}
$$

trees of this kind.


Figure 2: Spanning trees of type (a)


Figure 3: Spanning trees of type (b) when $k>1$


Figure 4: Spanning tree of type (b) when $k=1$
(3) Deduce from (2) that

$$
s_{n}=s_{n-1}+s_{n-1}+s_{n-2}+\cdots+s_{1}+1,
$$

with $s_{1}=1$. Use (1) to prove that

$$
s_{n}=F_{2 n}
$$

for all $n \geq 1$.
Problem 4. Prove that every undirected simple graph with $n \geq 2$ nodes and (strictly) more than $(n-1)(n-2) / 2$ edges is connected.

Hint. A simple undirected graph with $k$ nodes has at most $k(k-1) / 2$ edges.
Problem 5. Recall that $n!$ ( $n$ factorial) is defined for all $n \in \mathbb{N}$ by $0!=1$ and $(n+1)!=(n+1) n$ !. Define $\binom{n}{k}(n$ choose $k)$ by

$$
\binom{n}{k}=\frac{n!}{k!(n-k)!},
$$

for all $k, n \in \mathbb{N}$ with $0 \leq k \leq n$. Also set

$$
\binom{n}{k}=0
$$

for all $n \in \mathbb{N}$ if $k<0$ or if $k>n$.
(a) Prove that for all $n \in \mathbb{N}$ and for all $k \in \mathbb{Z}$, we have

$$
\begin{aligned}
& \binom{n}{k}=0, \quad \text { if } \quad k \notin\{0, \ldots, n\} \\
& \binom{0}{0}=1 \\
& \binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}, \quad \text { if } \quad n \geq 1
\end{aligned}
$$

The third identity is often called Pascal's identity (or the Pascal triangle identity).
(b) Prove that

$$
\binom{n}{k}=\binom{n}{n-k}
$$

for all $k, n \in \mathbb{N}$ with $0 \leq k \leq n$.
(c) The Fibonacci numbers, $F_{n}$, are defined recursively as follows:

$$
\begin{aligned}
F_{0} & =0 \\
F_{1} & =1 \\
F_{n+2} & =F_{n+1}+F_{n}, \quad n \geq 0 .
\end{aligned}
$$

For example, $0,1,1,2,3,5,8,13,21,34,55, \cdots$ are the first 11 Fibonacci numbers. Prove that

$$
F_{n+1}=\binom{n}{0}+\binom{n-1}{1}+\binom{n-2}{2}+\cdots+\binom{0}{n}
$$

for all $n \geq 0$.
Hint. Use complete induction. Also, consider the two cases, $n$ even and $n$ odd.

