Fall 2010 CIS 160

Mathematical Foundations of Computer Science
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Homework 5

October 26, 2010; Due November 2, 2010

Problem 1. (a) Show that the following are provable intuitionistically

$$\neg\exists t P \equiv \forall t \neg P$$
$$\forall t(P \wedge Q) \equiv \forall t P \wedge \forall t Q$$
$$\exists t(P \vee Q) \equiv \exists t P \vee \exists t Q.$$

Show that $\exists t \neg P \Rightarrow \neg\forall t P$ is provable intuitionistically and that $\neg\forall t P \Rightarrow \exists t \neg P$ is provable classically.

(b) Moreover, show that the propositions $\exists t(P \wedge Q) \Rightarrow \exists t P \wedge \exists t Q$ and $\forall t P \vee \forall t Q \Rightarrow \forall t(P \vee Q)$ are provable in intuitionistic first-order logic (and thus, also in classical first-order logic).

(c) Prove intuitionistically that

$$\exists x \forall y P \Rightarrow \forall y \exists x P.$$

Give an informal argument to the effect that the converse, $\forall y \exists x P \Rightarrow \exists x \forall y P$, is not provable, even classically.

Problem 2. (a) Assume that Q is a formula that does **not** contain the variable t (free or bound). Give a classical proof of

$$\forall t(P \vee Q) \Rightarrow (\forall t P \vee Q).$$

(b) If P is a proposition, write $P(x)$ for $P[x/t]$ and $P(y)$ for $P[y/t]$, where x and y are distinct variables not occurring in the orginal proposition P. Give an intuitionistic proof for

$$\neg\forall x \exists y(\neg P(x) \wedge P(y)).$$

(c) Give a classical proof for

$$\exists x \forall y(P(x) \vee \neg P(y)).$$

Hint. Negate the above, then use some identities we've shown (such as de Morgan) and reduce the problem to part (b).