$\boxed{\textbf{Fall 2010} \quad \textbf{CIS 160}}$

Mathematical Foundations of Computer Science
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Homework 6

November 2, 2010; Due November 9, 2010
Beginning of class

Problem 1. (a) Let $X = \{X_i \mid 1 \leq i \leq n\}$ be a finite family of sets. Prove that if $X_{i+1} \subseteq X_i$ for all i, with $1 \leq i \leq n-1$, then

$$\bigcap X = X_n.$$

Prove that if $X_i \subseteq X_{i+1}$ for all i, with $1 \leq i \leq n-1$, then

$$\bigcup X = X_n.$$

(b) Recall that $\mathbb{N}_+ = \mathbb{N} - \{0\} = \{1, 2, 3, \ldots, n, \ldots\}$. Give an example of an infinite family of sets, $X = \{X_i \mid i \in \mathbb{N}_+\}$, such that

1. $X_{i+1} \subseteq X_i$ for all $i \geq 1$;
2. X_i is infinite for every $i \geq 1$;
3. $\bigcap X$ has a single element.

(c) Give an example of an infinite family of sets, $X = \{X_i \mid i \in \mathbb{N}_+\}$, such that

1. $X_{i+1} \subseteq X_i$ for all $i \geq 1$;
2. X_i is infinite for every $i \geq 1$;
3. $\bigcap X = \emptyset$.

Problem 2. Given any two sets, A, B, prove that for all $a_1, a_2 \in A$ and all $b_1, b_2 \in B$,

$$\{\{a_1\}, \{a_1, b_1\}\} = \{\{a_2\}, \{a_2, b_2\}\}$$

iff

$$a_1 = a_2 \quad \text{and} \quad b_1 = b_2.$$

Problem 3. Let A and be B be any two sets of sets.

(1) Prove that

$$\left(\bigcup A\right) \cup \left(\bigcup B\right) = \bigcup (A \cup B)$$

(2) Assume that A and B are nonempty. Prove that

$$\left(\bigcap A\right) \cap \left(\bigcap B\right) = \bigcap (A \cup B)$$

(3) Assume that A and B are nonempty. Prove that

$$\bigcup (A \cap B) \subseteq \left(\bigcup A\right) \cap \left(\bigcup B\right)$$

and give a counter-example of the inclusion

$$\left(\bigcup A\right) \cap \left(\bigcup B\right) \subseteq \bigcup (A \cap B).$$

Hint. Reduce the above questions to the provability of certain formulae that you have already proved in a previous assignment (you need **not** reprove these formulae!).

Problem 4. Let A be any nonempty set. Prove that the definition

$$X = \{a \in A \mid a \notin X\}$$

yields a "set", X, such that X is empty iff X is nonempty and therefore does not define a set, after all.