Fall 2010 CIS 160

Mathematical Foundations of Computer Science
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Homework 7

November 11, 2010; Due November 18, 2010
Beginning of class

Problem 1. Consider the following table:

$$\begin{array}{rcl} 1 & = & 1^3 \\ 3+5 & = & 2^3 \\ 7+9+11 & = & 3^3 \\ 13+15+17+19 & = & 4^3 \\ 21+23+25+27+29 & = & 5^3 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots & & \end{array}$$

(a) If we number the rows starting from $n = 1$, prove that the leftmost number on row n is $1 + (n-1)n$. Then, prove that the sum of the numbers on row n (the n consecutive odd numbers beginning with $1 + (n-1)n$) is n^3.

(b) Use the triangular array in (a) to give a geometric proof of the identity

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2.$$

Hint. Recall that

$$1 + 3 + \cdots + 2n - 1 = n^2.$$

Problem 2. Recall that the triangular numbers, Δ_n, are given by the formula

$$\Delta_n = \frac{n(n+1)}{2},$$

with $n \in \mathbb{N}$.

(a) Prove that

$$\Delta_n + \Delta_{n+1} = (n+1)^2$$

and

$$\Delta_1 + \Delta_2 + \Delta_3 + \cdots + \Delta_n = \frac{n(n+1)(n+2)}{6}.$$

(b) The numbers,

$$T_n = \frac{n(n+1)(n+2)}{6}$$

are called *tetrahedral numbers*, due to their geometric interpretation as $3D$ stacks of triangular numbers. Prove that

$$T_1 + T_2 + \cdots + T_n = \frac{n(n+1)(n+2)(n+3)}{24}.$$

(c) Prove that

$$T_n + T_{n+1} = 1^2 + 2^2 + \cdots + (n+1)^2,$$

and from this, derive the formula

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Problem 3. Consider the sequences of integers arising from the recurrence

$$u_{n+2} = u_{n+1} + u_n, \quad n \geq 0,$$

where u_0 and u_1 are some given initial values. The *Fibonacci sequence*, (F_n), arises for $u_0 = 0$ and $u_1 = 1$ and the *Lucas sequence*, (L_n), arises for $u_0 = 2$ and $u_1 = 1$. The Fibonacci sequence begins with

$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610$

and the Lucas sequence begins with

$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364.$

(a) Prove the identities

$$\begin{aligned} L_n &= F_{n-1} + F_{n+1} \\ 5F_n &= L_{n-1} + L_{n+1}, \end{aligned}$$

for all $n \geq 1$.

(b) Prove that

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n,$$

for all $k \geq 1$ and all $n \geq 0$.

(c) Prove that

$$\begin{aligned} F_{2n} &= F_n L_n \\ F_{2n+1} &= F_{n+1}^2 + F_n^2, \end{aligned}$$

for all $n \geq 1$.

(d) Prove that

$$\begin{aligned} L_n L_{n+2} &= L_{n+1}^2 + 5(-1)^n \\ L_{2n} &= L_n^2 - 2(-1)^n \\ L_{2n+1} &= L_n L_{n+1} - (-1)^n, \end{aligned}$$

for all $n \geq 0$.

Problem 4. (a) Give an example of a function, $f\colon A \to A$, such that $f^2 = f \circ f = f$ and f is not the identity function.

(b) Prove that if a function, $f\colon A \to A$, is not the identity function and $f^2 = f$, then f is not invertible.

(c) Give an example of an invertible function, $f\colon A \to A$, such that $f^3 = f \circ f \circ f = f$, yet $f \circ f \neq f$.

(d) Give an example of a non-invertible function, $f\colon A \to A$, such that $f^3 = f \circ f \circ f = f$, yet $f \circ f \neq f$.