Fall 2010   CIS 160

Mathematical Foundations of Computer Science
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Homework 8

November 18, 2010; Due November 30, 2010

Problem 1. Let $f\colon X \to Y$ be any function. (a) Prove that for any two subsets, $A, B \subseteq X$, we have

$$\begin{aligned} f(A \cup B) &= f(A) \cup f(B) \\ f(A \cap B) &\subseteq f(A) \cap f(B). \end{aligned}$$

Give an example of a function f and of two subsets A, B such that

$$f(A \cap B) \neq f(A) \cap f(B).$$

Prove that if $f\colon X \to Y$ is injective, then

$$f(A \cap B) = f(A) \cap f(B).$$

(b) For any two subsets, $C, D \subseteq Y$, prove that

$$\begin{aligned} f^{-1}(C \cup D) &= f^{-1}(C) \cup f^{-1}(D) \\ f^{-1}(C \cap D) &= f^{-1}(C) \cap f^{-1}(D). \end{aligned}$$

Problem 2. (a) Prove that the composition of two injective functions is injective. Prove that the composition of two surjective functions is surjective.

(b) Prove that a function, $f\colon A \to B$, is injective iff for all functions, $g, h\colon C \to A$,

$$\text{if} \quad f \circ g = f \circ h, \quad \text{then} \quad g = h.$$

(c) Prove that a function, $f\colon A \to B$, is surjective iff for all functions, $g, h\colon B \to C$,

$$\text{if} \quad g \circ f = h \circ f, \quad \text{then} \quad g = h.$$

Problem 3. Prove that the set of natural numbers, $\mathbb{N}$, is infinite. (Recall, a set, X, is finite iff there is a bijection from X to $[n] = \{1, \ldots, n\}$, where $n \in \mathbb{N}$ is a natural numbers, with $[0] = \emptyset$. Thus, a set, X, is infinite iff there is no bijection from X to any $[n]$, with $n \in \mathbb{N}$.)

Problem 4. Let $R \subseteq A \times A$ be a relation. Prove that if $R \circ R = \mathrm{id}_A$, then R is the graph of a bijection whose inverse is equal to itself.

Problem 5. Let $f\colon A \to A'$ and $g\colon B \to B'$ be two functions and define $h\colon A \times B \to A' \times B'$ by

$$h(\langle a, b\rangle) = \langle f(a), g(b)\rangle,$$

for all $a \in A$ and $b \in B$.

(a) Prove that if f and g are injective, then so is h.

Hint. Use the definition of injectivity, not the existence of a left inverse and do not proceed by contradiction.

(b) Prove that if f and g are surjective, then so is h.

Hint. Use the definition of surjectivity, not the existence of a right inverse and do not proceed by contradiction.

Problem 6. Let $[0, 1]$ and $(0, 1)$ denote the set of real numbers

$$\begin{aligned} [0,1] &= \{x \in \mathbb{R} \mid 0 \leq x \leq 1\} \\ (0,1) &= \{x \in \mathbb{R} \mid 0 < x < 1\}. \end{aligned}$$

(a) Give a bijection $f\colon [0, 1] \to (0, 1)$.

Hint. There are such functions which are the identity almost everywhere but for a countably infinite set of points in $[0, 1]$.

(b) Consider the open square, $(0, 1) \times (0, 1)$ and the closed square $[0, 1] \times [0, 1]$. Give a bijection $f\colon [0, 1] \times [0, 1] \to (0, 1) \times (0, 1)$.