Fall 2010 CIS 160

Mathematical Foundations of Computer Science Jean Gallier

Homework 9

November 30, 2010; Due December 7, 2010

Problem 1. Let $S = \{a_1, \ldots, a_n\}$ be any nonempty set of *n* positive natural numbers. Prove that there is a nonempty subset of *S* whose sum is divisible by *n*.

Hint. Consider the numbers, $b_1 = a_1, b_2 = a_1 + a_2, \ldots, b_n = a_1 + a_2 + \cdots + a_n$.

Problem 2. (i) In 3-dimensional space, \mathbb{R}^3 , the sphere, S^2 , is the set of points of coordinates (x, y, z) such that $x^2 + y^2 + z^2 = 1$. The point N = (0, 0, 1) is called the *north pole*, and the point S = (0, 0, -1) is called the *south pole*. The *stereographic projection map*, $\sigma_N: (S^2 - \{N\}) \to \mathbb{R}^2$, is defined as follows: For every point $M \neq N$ on S^2 , the point $\sigma_N(M)$ is the intersection of the line through N and M and the equatorial plane of equation z = 0.

Prove that if M has coordinates (x, y, z) (with $x^2 + y^2 + z^2 = 1$), then

$$\sigma_N(M) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

Hint. Recall that if $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ are any two distinct points in \mathbb{R}^3 , then the unique line, (AB), passing through A and B has parametric equations

$$\begin{aligned} x &= (1-t)a_1 + tb_1 \\ y &= (1-t)a_2 + tb_2 \\ z &= (1-t)a_3 + tb_3, \end{aligned}$$

which means that every point (x, y, z) on the line (AB) is of the above form, with $t \in \mathbb{R}$. Find the intersection of a line passing through the North pole and a point $M \neq N$ on the sphere S^2 .

Prove that σ_N is bijective and that its inverse is given by the map, $\tau_N \colon \mathbb{R}^2 \to (S^2 - \{N\})$, with

$$(x,y) \mapsto \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right).$$

Hint. Find the intersection of a line passing through the North pole and some point, P, of the equatorial plane, z = 0, with the sphere of equation

$$x^2 + y^2 + z^2 = 1.$$

(ii) Give a bijection between the sphere, S^2 , and the equatorial plane of equation z = 0.

Hint. Use the stereographic projection and the method used in HW8, Problem 6, to define a bijection between [0, 1] and (0, 1).

Problem 3. Recall that given any two sets, X, Y, every function, $f: X \to Y$, induces a function, $f: 2^X \to 2^Y$, such that for every subset, $A \subseteq X$,

$$f(A) = \{ f(a) \in Y \mid a \in A \}$$

and a function, $f^{-1}: 2^Y \to 2^X$, such that, for every subset, $B \subseteq Y$,

$$f^{-1}(B) = \{ x \in X \mid f(x) \in B \}.$$

(a) Prove that if $f: X \to Y$ is injective, then so is $f: 2^X \to 2^Y$.

(b) Prove that if f is bijective then $f^{-1}(f(A)) = A$ and $f(f^{-1}(B)) = B$, for all $A \subseteq X$ and all $B \subseteq Y$. Deduce from this that $f: 2^X \to 2^Y$ is bijective.

(c) Prove that for any set, A, there is an injection from the set, A^A , of all functions from A to A to $2^{A \times A}$, the power set of $A \times A$. If A is infinite, prove that there is an injection from A^A to 2^A .

Problem 4. Given a group of people (say the CIS160 students), a person A may know the email address of a person B (different from A). We assume that each individual has a unique email address and that if A knows B's email then B may not know A's email. Prove that if the number of people in the group is odd, then there must be a person, A, in the group such that the sum of the total number of people (different from A) who know A's email plus the total number of people (different from A) whose email is known by A is even (possibly zero).

Problem 5. If G_1 and G_2 are isomorphic finite directed graphs, then prove that for every $k \ge 0$, the number of nodes, u, in G_1 such that $d_{G_1}^-(u) = k$, is equal to the number of nodes, $v \in G_2$, such that $d_{G_2}^-(v) = k$ (resp. the number of nodes, u, in G_1 such that $d_{G_1}^+(u) = k$, is equal to the number of nodes, $v \in G_2$, such that $d_{G_2}^+(v) = k$). Give a counter-example showing that the converse property is false.