CIS160
Mathematical Foundations of Computer Science
Some Notes

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Contents

Chapter 1

Mathematical Reasoning, Proof Principles and Logic

1.1 Problems, Motivations

Problem 1. Find formulae for the sums

$$\begin{aligned} 1+2+3+\cdots+n &= \ ? \\ 1^2+2^2+3^2+\cdots+n^2 &= \ ? \\ 1^3+2^3+3^3+\cdots+n^3 &= \ ? \\ \cdots\cdots\cdots\cdots & \\ 1^k+2^k+3^k+\cdots+n^k &= \ ? \end{aligned}$$

Jacob Bernoulli (1654-1705) discovered the formulae listed below:

If

$$S_k(n) = 1^k + 2^k + 3^k + \cdots + n^k$$

then

$$\begin{aligned}
S_0(n) &= 1n \\
S_1(n) &= \frac{1}{2}n^2 + \frac{1}{2}n \\
S_2(n) &= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \\
S_3(n) &= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \\
S_4(n) &= \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n \\
S_5(n) &= \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2 \\
S_6(n) &= \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n \\
S_7(n) &= \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2 \\
S_8(n) &= \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n \\
S_9(n) &= \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{3}{20}n^2 \\
S_{10}(n) &= \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + n^5 - \frac{1}{2}n^3 + \frac{5}{66}n
\end{aligned}$$

Is there a pattern?

What are the mysterious numbers

$$1 \quad \frac{1}{2} \quad \frac{1}{6} \quad 0 \quad -\frac{1}{30} \quad 0 \quad \frac{1}{42} \quad 0 \quad -\frac{1}{30} \quad 0 \quad \frac{5}{66} \quad ?$$

The next two are

$$0 \qquad -\frac{691}{2730}$$

Why?

It turns out that the answer has to do with the *Bernoulli polynomials*, $B_k(x)$, with

$$B_k(x) = \sum_{i=0}^{k} \binom{k}{i} x^{k-i} B^i,$$

where the B^i are the *Bernoulli numbers*.

There are various ways of computing the Bernoulli numbers, including some recurrence formulae.

Amazingly, the Bernoulli numbers show up in very different areas of mathematics, in particular, algebraic topology!


Figure 1.1: Funny spheres (in 3D)

Figure 1.2: A plane (granite slab!)

Problem 2. Prove that a sphere and a plane in 3D have *the same number of points*.

More precisely, find a one-to-one and onto mapping of the sphere onto the plane (a *bijection*)

Actually, there are also bijections between the sphere and a (finite) rectangle, with or without its boundary!

Problem 3. Counting the number of *derangements* of n elements.

A *permutation* of the set $\{1, 2, \ldots, n\}$ is any one-to-one function, f, of $\{1, 2, \ldots, n\}$ into itself. A permutation is characterized by its image: $\{f(1), f(2), \ldots, f(n)\}$.

For example, $\{3, 1, 4, 2\}$ is a permutation of $\{1, 2, 3, 4\}$.

It is easy to show that there are $n! = n \cdot (n-1) \cdots 3 \cdot 2$ distinct permutations of n elements.

A *derangements* is a permutation that leaves no element fixed, that is, $f(i) \neq i$ for all i.

$\{3, 1, 4, 2\}$ is a derangement of $\{1, 2, 3, 4\}$ but
$\{3, 2, 4, 1\}$ is *not* a derangement since 2 is left fixed.

What is the *number of derangements*, p_n, of n elements?

The number $p_n/n!$ can be interpreted as a *probability*.

Say n people go to a restaurant and they all check their coat. Unfortunately, the cleck loses all the coat tags. Then, $p_n/n!$ is the probability that nobody gets her or his coat back!

Interestingly, $p_n/n!$ has limit $\frac{1}{e} \approx \frac{1}{3}$ as n goes to infinity, a surprisingly large number.

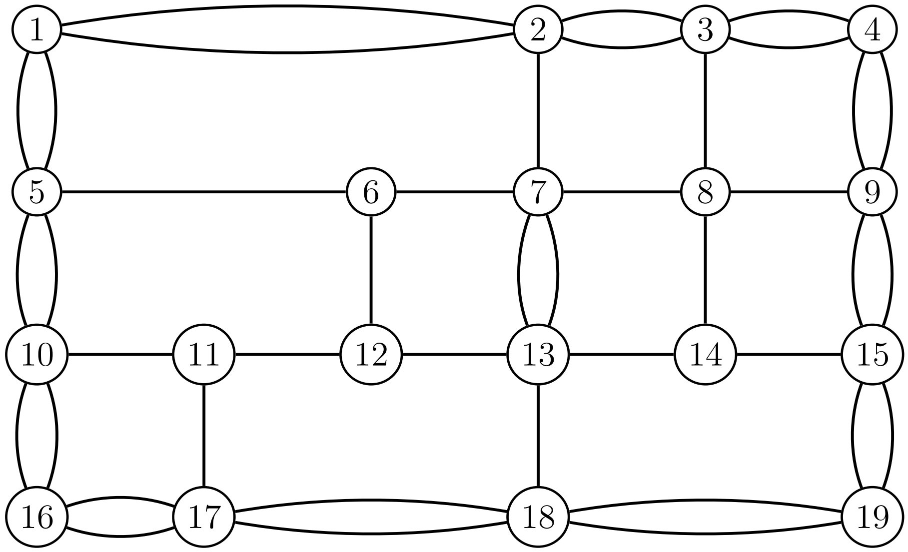

Figure 1.3: An undirected graph modeling a city map

Problem 4. Finding *strongly connected components* in a directed graph.

The *undirected graph* of Figure 1.3 represents a map of some busy streets in a city.

The city decides to improve the traffic by making these streets *one-way* streets.

However, a good choice of orientation should allow one to travel between any two locations. We say that the resulting *directed graph* is *strongly connected*.

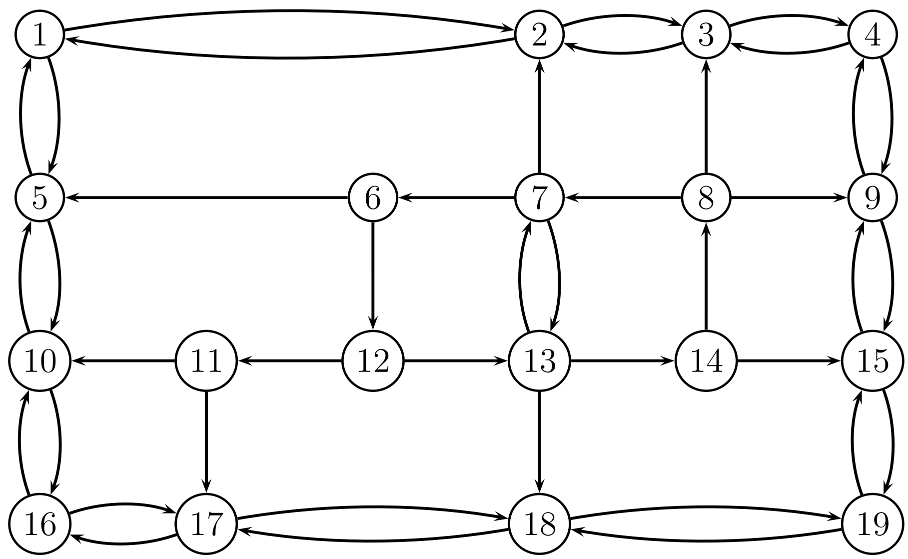

Figure 1.4: A choice of one-way streets

A possibility of orienting the streets is shown in Figure 1.4.

Is the above graph strongly connected?

If not, how do we find its *strongly connected components*?

How do we use the strongly connected components to find an orientation that solves our problem?

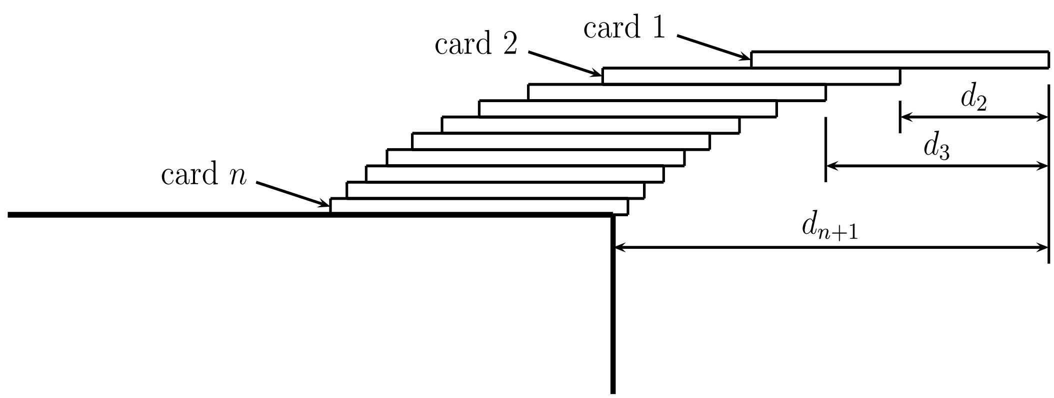

Figure 1.5: Stack of overhanging cards

Problem 5. The *maximum overhang* problem.

How do we stack n cards on the edge of a table, respecting the law of gravity, and achieving a maximum overhang.

We assume each card is 2 units long.

Is it possible to achieve any desired amount of overhang or is there a fixed bound?

How many cards are needed to achieve an overhang of d units?

Problem 6. *Ramsey Numbers*

A version of *Ramsey's Theorem* says that for every pair, (r, s), of positive natural numbers, there is a least positive natural number, $R(r, s)$, such that for every coloring of the edges of the complete (undirected) graph on $R(r, s)$ vertices using the colors *blue* and *red*, either there is a complete subgraph with r vertices whose edges are all *blue* or there is a complete subgraph with s vertices whose edges are all *red*.

So, $R(r, r)$, is the smallest number of vertices of a complete graph whose edges are colored either *blue* or *red* that must contain a complete subgraph with r vertices whose edges are all of the same color.

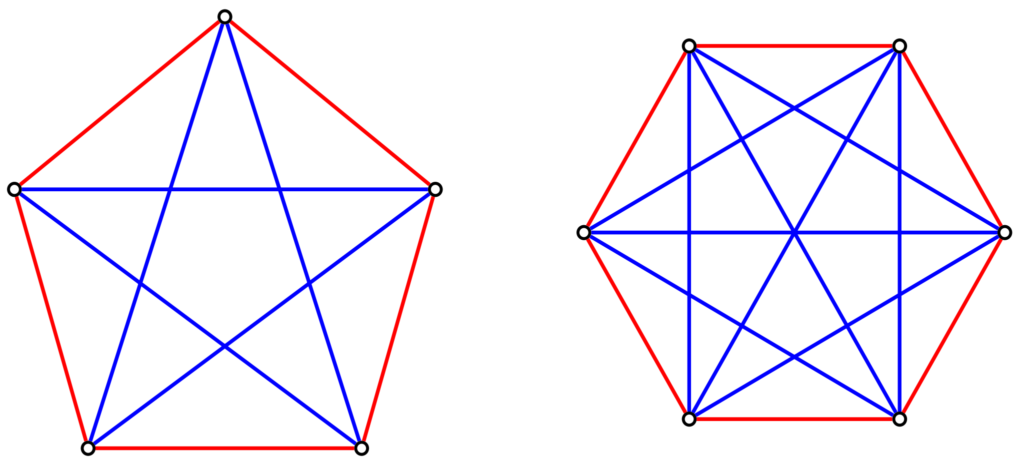

Figure 1.6: Left: A 2-coloring of K_5 with no monochromatic K_3; Right: A 2-coloring of K_6 with several monochromatic K_3's

The graph shown in Figure 1.6 (left) is a complete graph on 5 vertices with a coloring of its edges so that there is no complete subgraph on 3 vertices whose edges are all of the same color.

Thus, $R(3,3) > 5$.

There are

$$2^{15} = 32768$$

2-colored complete graphs on 6 vertices. One of these graphs is shown in Figure 1.6 (right).

It can be shown that all of them contain a triangle whose edges have the same color, so $R(3,3) = 6$.

The numbers, $R(r, s)$, are called *Ramsey numbers*.

It turns out that there are *very few* numbers r, s for which $R(r, s)$ is known because the number of colorings of a graph grows very fast! For example, there are

$$2^{43\times 21} = 2^{903} > 1024^{90} > 10^{270}$$

2-colored complete graphs with 43 vertices, a huge number!

In comparison, the universe is *only* approximately 14 billions years old, namely 14×10^9 years old.

For example, $R(4, 4) = 18$, $R(4, 5) = 25$, but $R(5, 5)$ *is unknown*, although it can be shown that
$43 \leq R(5, 5) \leq 49$.

Finding the $R(r, s)$, or, at least some sharp bounds for them, is an *open problem*.