The Logic of Rotations
Lie Groups and Homogeneous Spaces

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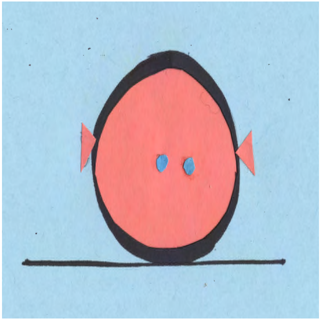
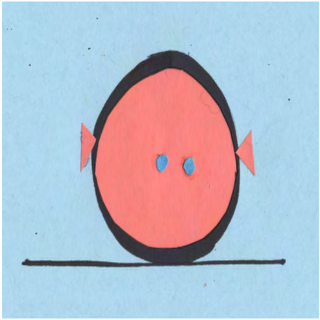
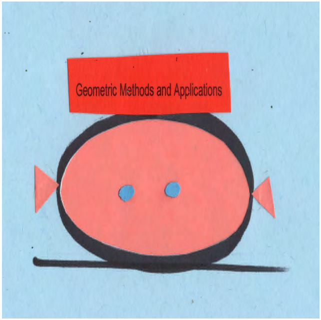
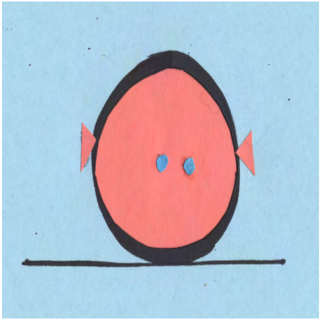
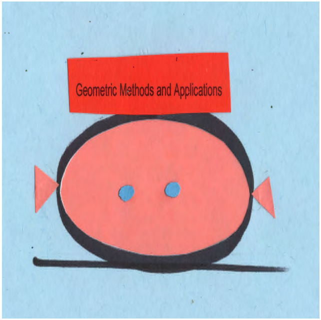
Figure: Dog Logic



 
Geometric Methods and Applications

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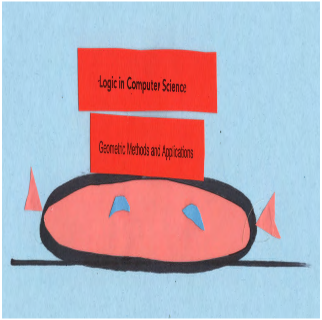
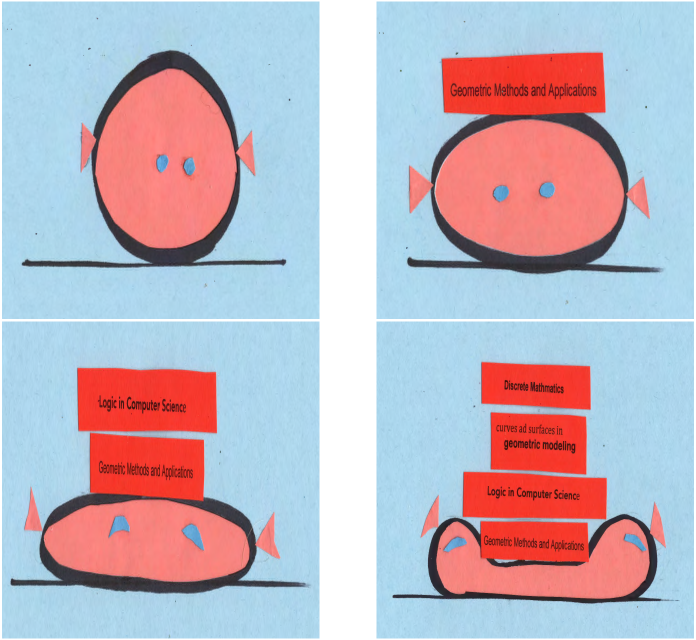

Logic in Computer Science
Geometric Methods and Applications


Geometric Methods and Applications
Logic in Computer Science
Geometric Methods and Applications
Discrete Mathmatics
curves ad surfaces in
geometric modeling
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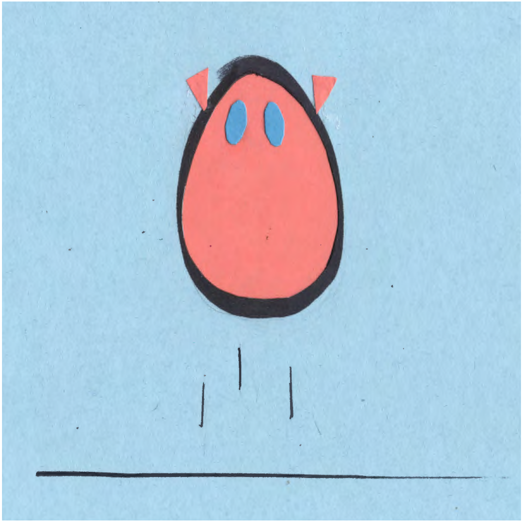
Geometric Methods and Applications



(Thanks to Anne for the cute graphics!)

1. Formalizing Motions and Deformations

In the previous cartoon, we have a sequence of objects

$$\mathcal{B}_0 = \mathcal{B},\ \mathcal{B}_1,\ \mathcal{B}_2,\ \ldots,\ \mathcal{B}_m,$$

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Some transformation $\mathcal{D}_i$ takes $\mathcal{B}$ to $\mathcal{B}_i$.

It is convenient to assume that the transformations $\mathcal{D}_i$ are invertible and belong to some *group* G (nothing "catastrophic" happens).

Motions and Deformations

Then, the motion and deformation of a body (rigid or not) can be described by a *curve* in a *group G of transformations* of a space E (say $\mathbb{R}^n$, $n = 2, 3, \ldots$).

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$$\mathcal{D}\colon [0, T] \to G.$$

The (moved and) deformed body $\mathcal{B}_t$ at time t is given by

$$\mathcal{B}_t = \mathcal{D}(t)(\mathcal{B}).$$

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If $\langle -, - \rangle$ denotes the *Euclidean inner product* on $\mathbb{R}^n$, then $\mathbf{SO}(n)$ consists of all invertible linear maps $f : \mathbb{R}^n \to \mathbb{R}^n$ that preserve $\langle -, - \rangle$:

$$\langle f(x), f(y) \rangle = \langle x, y \rangle, \quad \text{for all } x, y \in \mathbb{R}^n.$$

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The elements of $\mathbf{SO}(n)$ are *rotations* (of $\mathbb{R}^n$). With respect to any orthonormal basis, every rotation is represented by an *orthogonal matrix* R, which means that

$$\begin{aligned} RR^\top = R^\top R &= I \\ \det(R) &= 1. \end{aligned}$$

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The group $\mathbf{SE}(n)$ consists of all invertible *affine maps* $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$, such that

$$\rho(x) = f(x) + u, \quad x \in \mathbb{R}^n,$$

with $f \in \mathbf{SO}(n)$ and $u \in \mathbb{R}^n$ (the *translation component*). The elements of $\mathbf{SE}(n)$ are the *(direct) rigid motions* (or $\mathbb{R}^n$).

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The standard trick is to represent ρ by an $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} R & u \\ 0 & 1 \end{pmatrix} \quad R \in \mathbf{SO}(n),\ u \in \mathbb{R}^n,$$

where $x \in \mathbb{R}^n$ becomes $\begin{pmatrix} x \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1}$.

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We can consider more complicated groups G, as long as they are *Lie groups*. From now on, we will consider groups of matrices.

2. Interpolation

The *interpolation problem* is the following:
given a sequence $g_0, \ldots, g_m$ of deformations $g_i \in G$, with $g_0 = \mathrm{id}$, find a (reasonably smooth) curve $c\colon [0, m] \to G$ such that

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Unfortunately, the naive solution which consists in performing an interpolation

$$(1 - t)g_i + tg_{i+1} \quad (0 \leq t \leq 1)$$

between g_i and g_{i+1} does not work, because $(1 - t)g_i + tg_{i+1}$ *does not belong* to G (in general).

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So, what can we do?

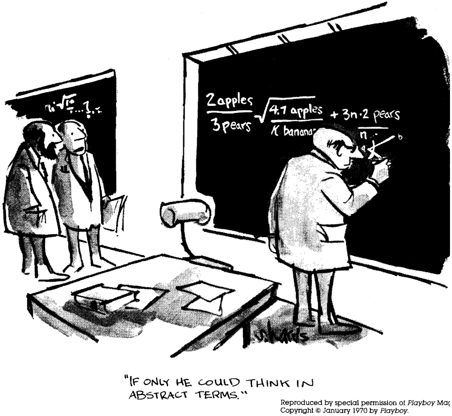

Figure: The power of abstraction

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In the case of matrix groups,

$$[X, Y] = XY - YX.$$

The Lie algebra $\mathfrak{so}(n)$ of $\mathbf{SO}(n)$ consists of all $n \times n$ *skew symmetric matrices*; matrices B such that

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$$\begin{pmatrix} B & u \\ 0 & 0 \end{pmatrix} \quad B \in \mathfrak{so}(n),\ u \in \mathbb{R}^n.$$

The Lie algebra $\mathfrak{sim}(n)$ of $\mathbf{SIM}(n)$ consists of all $(n+1) \times (n+1)$ matrices of the form

$$\begin{pmatrix} \lambda I_n + B & u \\ 0 & 0 \end{pmatrix} \quad B \in \mathfrak{so}(n),\ u \in \mathbb{R}^n,\ \lambda \in \mathbb{R}.$$

We can think of the Lie algebra $\mathfrak{g}$ as a *linearization* of G. There is a map $\exp\colon \mathfrak{g} \to G$ (the *exponential map*) that brings us back into G. For matrix groups, it is simply

$$\exp(X) = e^X = I + \frac{X}{1!} + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots + \frac{X^k}{k!} + \cdots$$

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This means that we have a *logarithm function* (actually, a multi-valued function) $\log\colon G \to \mathfrak{g}$, such that

$$e^{\log A} = A, \quad A \in G.$$

4. Interpolation in Lie Groups

We can use the maps $\log\colon G \to \mathfrak{g}$ and $\exp\colon \mathfrak{g} \to G$ to interpolate in G as follows: Given the sequence of "snapshots"

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1. Compute logs

$$X_0 = \log g_0,\ X_1 = \log g_1, \ldots,\ X_m = \log g_m, \quad \text{in } \mathfrak{g}$$

2. Find an interpolating curve $X\colon [0, m] \to \mathfrak{g}$, *in* $\mathfrak{g}$
3. Exponentiate, to get the curve

$$c(t) = e^{X(t)}, \quad \text{in } G.$$

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For $\mathfrak{so}(3)$, this is the *Rodrigues formula (1840)*. For $\mathfrak{se}(3)$, there is a variant of Rodrigues formula. Both can be generalized to any $n \geq 2$ (J.G. and Dianna Xu). There is also a formula for $\mathfrak{sim}(3)$.

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Let $\mathcal{S}(n)$ be the set of real matrices whose eigenvalues $\lambda + i\mu$ lie in the horizontal strip $-\pi < \mu < \pi$. Then, $\exp\colon \mathcal{S}(n) \to \exp(\mathcal{S}(n))$ is a bijection onto the set of real matrices with *no negative eigenvalues*.

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There are efficient algorithms for computing such logs using *inverse scaling and squaring* methods.

5. Metrics on Lie Groups

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In the case $G = \mathbf{SO}(n)$, we can use the inner product on $\mathfrak{so}(n)$ given by

$$\langle X, Y \rangle = -\frac{1}{2}\mathrm{tr}(XY) = \frac{1}{2}\mathrm{tr}(X^\top Y).$$

Given a curve $\gamma\colon [0,1] \to G$, the *length* $L(\gamma)$ of γ is defined by

$$L(\gamma) = \int_0^1 \langle \gamma'(t), \gamma'(t)\rangle^{\frac{1}{2}}\, dt.$$

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A *geodesic* through I is a curve $\gamma(t)$ in G such that $\gamma(0) = I$, and the acceleration $\gamma''(t)$ is normal to the tangent space $T_{\gamma(t)} G$ for all t (rigorously, we would need the connection on G induced by the metric).

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It turns out that for every $X \in \mathfrak{so}(n)$, there is a *unique geodesic* through I such that $\gamma'(0) = X$; namely,

$$\gamma(t) = e^{tX}.$$

Furthermore, for every $A \in G = \mathbf{SO}(n)$, there is *some* geodesic from I to A.

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We define the *distance* $d(I, A)$ betwen I and A as

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For any $A, B \in G$, we have

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Since there is always a geodesic from I to A,

$$d(I, A) = \inf_{\gamma} \{L(\gamma) \mid \gamma \text{ is a geodesic joining } I \text{ and } A\}.$$

Theorem 1

The distance between any two rotations $A, B \in \mathbf{SO}(n)$ *is*

$$d(A,B) = \sqrt{\theta_1^2 + \cdots + \theta_m^2},$$

where $e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_m}$ *are the eigenvalues* ($\neq 1$) *of* $A^\top B$, *with* $0 < \theta_i \leq \pi$.

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What about $\mathbf{SE}(n)$?

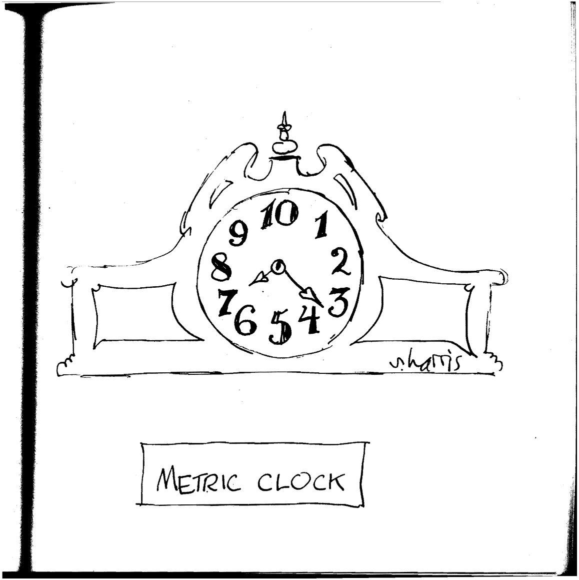

Figure: Metric Clock

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Part of the problem is that $\mathbf{SE}(n)$ is not compact and not semisimple (the Killing form is degenerate). New ideas are needed!

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The solution is to make $\mathbf{SO}(n)$ *act* on $G(k, n)$.

A k-dimensional subspace V is specified by k orthonormal vectors in V, and these vectors constitute a $n \times k$ matrix A with *orthogonal columns* $(A^\top A = I_k)$.

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$$(R, A) \mapsto RA,$$

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The action $\cdot\colon \mathbf{SO}(n) \times G(k, n) \to G(k, n)$ is *transitive* (which means that for any two subspaces $V, W \in G(k, n)$, there is some rotation R such that $R \cdot V = W$).

In such a situation, we look for the *stabilizer* of any subspace V in $G(k, n)$. This is the subgroup K of $\mathbf{SO}(n)$ such that $R \cdot V = V$ for all $R \in K$.

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Then, it can be shown that $G(k, n)$ is *isomorphic to the quotient space* $\mathbf{SO}(n)/K$*, consisting of all cosets* RK*, with* $R \in \mathbf{SO}(n)$ ($R_1 \equiv R_2$ iff $R_1^{-1}R_2 \in K$). Let $\pi\colon G \to \mathbf{SO}(n)/K$ be the canonical projection.

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We find that the stabilizer of $V =$ the first k columns of I_n is $K = S(\mathbf{O}(k) \times \mathbf{O}(n-k))$; that is,

$$K = \left\{ \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \;\middle|\; P \in \mathbf{O}(k),\ Q \in \mathbf{O}(n-k),\ det(P)\det(Q) = 1 \right\},$$

whose Lie algebra $\mathfrak{k}$ is

$$\mathfrak{k} = \left\{ \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} \;\middle|\; S \in \mathbf{so}(k),\ T \in \mathbf{so}(n-k) \right\}.$$

The tangent space $T_I\mathbf{SO}(n) = \mathfrak{so}(n)$ splits as a direct sum

$$\mathfrak{so}(n) = \mathfrak{k} \oplus \mathfrak{m},$$

with

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & -A^\top \\ A & 0 \end{pmatrix} \,\middle|\, A \in \mathrm{M}_{n-k,k} \right\}.$$

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It turns out that *the tangent space* $T_o(\mathbf{SO}(n)/K)$ *to* $\mathbf{SO}(n)/K$ *at* o *(= the coset* K*) is isomorphic to* $\mathfrak{m}$.

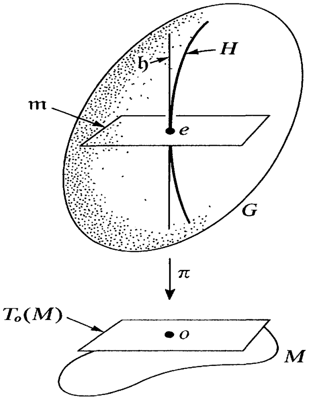

Figure: Reductive homogeneous space, from O'Neill

(In the above Figure, $G = \mathbf{SO}(n)$, $K \mapsto H$, $\mathfrak{k} \mapsto \mathfrak{h}$, $M = \mathbf{SO}(n)/K$).

Furthermore with the metric on $\mathfrak{so}(n)$ given by

$$\langle X, Y \rangle = -\frac{1}{2}\mathrm{tr}(XY) = \frac{1}{2}\mathrm{tr}(X^\top Y),$$

the spaces $\mathfrak{k}$ and $\mathfrak{m}$ are orthogonal complements. $\mathbf{SO}(n)/K$ is a *naturally reductive homogeneous space*. In fact, it is a *symmetric space (Élie Cartan)*.

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Geodesics in $G(k, n) \cong \mathbf{SO}(n)/K$ *are projections of horizontal geodesics in* $\mathbf{SO}(n)$ (geodesics with initial velocity $X \in \mathfrak{m}$).

Theorem 2

The distance between any two subspaces $U, V \in G(k, n)$ *specified by two* $n \times k$ *matrices* A, B *with orthogonal columns is*

$$d(U, V) = \sqrt{\theta_1^2 + \cdots + \theta_k^2},$$

where $(\cos\theta_1, \ldots, \cos\theta_k)$ *are the singular values of* $A^\top B$, *with* $0 \leq \theta_i \leq \pi/2$.

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The angles $\theta_1, \ldots, \theta_k$ are also known as the *principal angles* of the subspaces U and V (Camille Jordan).

Other interesting manifolds, such as $\mathbf{SPD}(n)$ (symmetric, positive, definite matrices) are presented as homogeneous spaces; for example, $\mathbf{SPD}(n) \cong \mathbf{GL}^+(n)/\mathbf{SO}(n)$.

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Dealing with $\mathbf{SE}(n)$ and the Grassmannian of affine subspaces remains an open problem.