The Logic of Rotations
Lie Groups and Homogeneous Spaces

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All cats have four legs.
I have four legs.
Therefore, I am a cat.

Figure: Dog Logic
(Thanks to Anne for the cute graphics!)
1. Formalizing Motions and Deformations

In the previous cartoon, we have a sequence of objects

\[ B_0 = B, B_1, B_2, \ldots, B_m, \]

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Some transformation \( D_i \) takes \( B \) to \( B_i \).

It is convenient to assume that the transformations \( D_i \) are invertible and belong to some group \( G \) (nothing “catastrophic” happens).
Motions and Deformations

Then, the motion and deformation of a body (rigid or not) can be described by a *curve* in a *group G of transformations* of a space $E$ (say $\mathbb{R}^n$, $n = 2, 3, ...$).

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Given an *initial shape* $B \in E$, a *deformation* of $B$ is a (smooth enough) curve

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The (moved and) deformed body $B_t$ at time $t$ is given by

$$B_t = D(t)(B).$$
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Recall that $\text{SO}(n)$ is the group of *direct isometries* of $\mathbb{R}^n$.

If $\langle -, - \rangle$ denotes the *Euclidean inner product* on $\mathbb{R}^n$, then $\text{SO}(n)$ consists of all invertible linear maps $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ that preserve $\langle -, - \rangle$:

$$\langle f(x), f(y) \rangle = \langle x, y \rangle, \quad \text{for all } x, y \in \mathbb{R}^n.$$ 

Furthermore, $\det(f) = +1$. 
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The elements of \( \text{SO}(n) \) are \textit{rotations} (of \( \mathbb{R}^n \)). With respect to any orthonormal basis, every rotation is represented by an \textit{orthogonal matrix} \( R \), which means that

\[
RR^\top = R^\top R = I
\]

\[
\det(R) = 1.
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The group $\mathbf{SE}(n)$ consists of all invertible affine maps $\rho : \mathbb{R}^n \to \mathbb{R}^n$, such that

$$\rho(x) = f(x) + u, \quad x \in \mathbb{R}^n,$$

with $f \in \mathbf{SO}(n)$ and $u \in \mathbb{R}^n$ (the translation component). The elements of $\mathbf{SE}(n)$ are the (direct) rigid motions (or $\mathbb{R}^n$).
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The standard trick is to represent $\rho$ by an $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} R & u \\ 0 & 1 \end{pmatrix} \quad R \in \text{SO}(n), \ u \in \mathbb{R}^n,$$

where $x \in \mathbb{R}^n$ becomes $\begin{pmatrix} x \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1}$. 
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The group $\text{SIM}(n)$ is defined by matrices of the form

$$\begin{pmatrix} \alpha R & u \\ 0 & 1 \end{pmatrix}, \quad R \in \text{SO}(n), \ u \in \mathbb{R}^n, \alpha > 0.$$
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We can consider more complicated groups $G$, as long as they are *Lie groups*. From now on, we will consider groups of matrices.
2. Interpolation

The *interpolation problem* is the following:
given a sequence \( g_0, \ldots, g_m \) of deformations \( g_i \in G \), with \( g_0 = \text{id} \), find a (reasonably smooth) curve \( c: [0, m] \to G \) such that

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c(i) = g_i, \quad i = 0, \ldots, m.
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Unfortunately, the naive solution which consists in performing an interpolation

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(1 - t)g_i + tg_{i+1} \quad (0 \leq t \leq 1)
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between \( g_i \) and \( g_{i+1} \) does not work, because \( (1 - t)g_i + tg_{i+1} \) does not belong to \( G \) (in general).
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So, what can we do?
Figure: The power of abstraction
3. Lie Groups to The Rescue

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This means that they are also *manifolds*. Roughly speaking, locally they “look” like $\mathbb{R}^m$ (for some $m$), and at every point $g$ of the group $G$, there is a *tangent space*, $T_g G$. 

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The tangent space at $I$ (the identity element of $G$), denoted $\mathfrak{g}$, has a special structure. It is a Lie algebra. This means that there is a funny multiplication $[−, −]$ on $\mathfrak{g}$, the Lie bracket.
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In the case of matrix groups,

$$[X, Y] = XY - YX.$$
The Lie algebra $\mathfrak{so}(n)$ of $\text{SO}(n)$ consists of all $n \times n$ skew symmetric matrices; matrices $B$ such that

$$B^\top = -B.$$
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The Lie algebra $\mathfrak{se}(n)$ of $\mathbf{SE}(n)$ consists of all $(n+1) \times (n+1)$ matrices of the form

$$\begin{pmatrix} B & u \\ 0 & 0 \end{pmatrix} \quad B \in \mathfrak{so}(n), \ u \in \mathbb{R}^n.$$ 

The Lie algebra $\mathfrak{sim}(n)$ of $\mathbf{SIM}(n)$ consists of all $(n+1) \times (n+1)$ matrices of the form

$$\begin{pmatrix} \lambda I_n + B & u \\ 0 & 0 \end{pmatrix} \quad B \in \mathfrak{so}(n), \ u \in \mathbb{R}^n, \ \lambda > 0.$$
We can think of the Lie algebra $\mathfrak{g}$ as a *linearization* of $G$. There is a map $\exp: \mathfrak{g} \to G$ (the *exponential map*) that brings us back into $G$. For matrix groups, it is simply

$$\exp(X) = e^X = I + \frac{X}{1!} + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots + \frac{X^k}{k!} + \cdots$$

Fortunately, for all the groups we just considered, the exponential map is *surjective*. This means that we have a logarithm function (actually, a multi-valued function) $\log : G \to \mathfrak{g}$, such that $e^{\log A} = A$, $A \in G$. 
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4. Interpolation in Lie Groups

We can use the maps \( \log : G \to g \) and \( \exp : g \to G \) to interpolate in \( G \) as follows: Given the sequence of “snapshots”

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1. Compute logs

\[ X_1 = \log g_1, \ X_2 = \log g_2, \ldots, \ X_m = \log g_m, \quad \text{in } \mathfrak{g} \]

2. Find an interpolating curve \( X: [0, m] \to \mathfrak{g}, \quad \text{in } \mathfrak{g} \)

3. Exponentiate, to get the curve

\[ c(t) = e^{X(t)}, \quad \text{in } G. \]
Since $\mathfrak{g}$ is a vector space (with an inner product), interpolating in $\mathfrak{g}$ can be done easily using spline curves.

Two problems remain:

1. Computing the logarithm of a matrix.
2. Computing the exponential of a matrix.

Fortunately, we are dealing with special kinds of matrices, and for matrices $X$ in $\mathfrak{so}(3)$, $\mathfrak{se}(3)$, and $\mathfrak{sim}(3)$, there are explicit formulae to compute $e^X$.

For $\mathfrak{so}(3)$, this is the Rodrigues formula (1840). For $\mathfrak{se}(3)$, there is a variant of Rodrigues formula. Both can be generalized to any $n \geq 2$ (J.G. and Dianna Xu). There is also a formula for $\mathfrak{sim}(3)$.
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Let $S(n)$ be the set of real matrices whose eigenvalues $\lambda + i\mu$ lie in the horizontal strip $-\pi < \mu < \pi$. Then, $\exp: S(n) \rightarrow \exp(S(n))$ is a bijection onto the set of real matrices with no negative eigenvalues.
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There are efficient algorithms for computing such logs using inverse scaling and squaring methods.
5. Metrics on Lie Groups

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This can be done by giving $g = T_I G$ an inner product. Then, because $G$ is a group, this metric can be propagated to the tangent space $T_g G$ at any point $g \in G$. We get a *Riemannian metric*. 
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In the case $G = \text{SO}(n)$, we can use the inner product on $\mathfrak{so}(n)$ given by

$$\langle X, Y \rangle = -\frac{1}{2} \text{tr}(XY) = \frac{1}{2} \text{tr}(X^\top Y).$$
Given a curve \( \gamma: [0, 1] \to G \), the \textit{length} \( L(\gamma) \) of \( \gamma \) is defined by

\[
L(\gamma) = \int_0^1 \langle \gamma'(t), \gamma'(t) \rangle^{\frac{1}{2}} dt.
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A *geodesic* through $I$ is a curve $\gamma(t)$ in $G$ such that $\gamma(0) = I$, and the acceleration $\gamma''(t)$ is normal to the tangent space $T_{\gamma(t)} G$ for all $t$ (rigorously, we would need the connection on $G$ induced by the metric).
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It turns out that for every $X \in so(n)$, there is a unique geodesic through $I$ such that $\gamma'(0) = X$; namely,

$$\gamma(t) = e^{tX}.$$
Furthermore, for every $A \in G = \text{SO}(n)$, there is some geodesic from $I$ to $A$. 

We define the distance $d(I, A)$ between $I$ and $A$ as $d(I, A) = \inf_{\gamma} \{ L(\gamma) | \gamma$ joins $I$ and $A \}$. 

For any $A, B \in G$, we have $d(A, B) = d(I, A^{-1}B) = d(I, A^{\top}B)$. 

Since there is always a geodesic from $I$ to $A$, $d(I, A) = \inf_{\gamma} \{ L(\gamma) | \gamma$ is a geodesic joining $I$ and $A \}$. 
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Theorem 1

The distance between any two rotations $A, B \in \text{SO}(n)$ is

$$d(A, B) = \sqrt{\theta_1^2 + \cdots + \theta_m^2},$$

where $e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_m}$ are the eigenvalues ($\neq 1$) of $A^\top B$, with $0 < \theta_i \leq \pi$. 
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What about $\text{SE}(n)$?
Figure: Metric Clock
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However, for $\text{SO}(n)$, the above metric is both \textit{left and right invariant}, but for $\text{SE}(n)$, it is only \textit{left invariant}. In fact, there are \textit{no} left and right invariant metrics on $\text{SE}(n)$. I don’t know of any formula for $d(A, B)$. An unfortunate consequence is that not all geodesics in $\text{SE}(n)$ are given by the exponential. Part of the problem is that $\text{SE}(n)$ is not compact and not semisimple (the Killing form is degenerate). New ideas are needed!
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An unfortunate consequence is that not all geodesics in $\text{SE}(n)$ are given by the exponential.
We can still define a Riemmanian metric on $se(n)$ as before:

$$\langle X, Y \rangle = \frac{1}{2} \text{tr}(X^\top Y).$$

However, for $SO(n)$, the above metric is both \textit{left and right invariant}, but for $SE(n)$, it is only \textit{left invariant}. In fact, there are \textit{no} left and right invariant metrics on $SE(n)$. I don’t know of any formula for $d(A, B)$.

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Part of the problem is that $SE(n)$ is not compact and not semisimple (the Killing form is degenerate). New ideas are needed!
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What do we mean by the distance $d(V, \mathcal{W})$ between two subspaces? Can we give a formula?

The solution is to make $\text{SO}(n)$ act on $G(k, n)$. 

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A rotation $R \in \text{SO}(n)$ acts on $V$ by rotating every vector in $V$; that is, $R$ is applied to the matrix $A$ representing $V$:

$$(R, A) \mapsto RA,$$

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The action $\cdot : \textbf{SO}(n) \times G(k, n) \to G(k, n)$ is *transitive* (which means that for any two subspaces $V, W \in G(k, n)$, there is some rotation $R$ such that $R \cdot V = W$).
In such a situation, we look for the stabilizer of any subspace $V$ in $G(k, n)$. This is the subgroup $K$ of $\text{SO}(n)$ such that $R \cdot V = V$ for all $R \in K$. 
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Then, it can be shown that \( G(k, n) \) is *isomorphic to the quotient space* \( \text{SO}(n)/K \), consisting of all cosets \( RK \), with \( R \in \text{SO}(n) \) (\( R_1 \equiv R_2 \) iff \( R_1^{-1}R_2 \in K \)). Let \( \pi: G \to \text{SO}(n)/K \) be the canonical projection.
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We find that the stabilizer of $V = $ the first $k$ columns of $I_n$ is $K = \text{S}(\text{O}(k) \times \text{O}(n - k))$; that is,

$$K = \left\{ \mqty(P & 0 \\ 0 & Q) \bigg| P \in \text{O}(k), \ Q \in \text{O}(n - k), \ \det(P) \det(Q) = 1 \right\},$$

whose Lie algebra $\mathfrak{k}$ is

$$\mathfrak{k} = \left\{ \mqty(S & 0 \\ 0 & T) \bigg| S \in \text{so}(k), \ T \in \text{so}(n - k) \right\}.$$
The tangent space $T_{\text{SO}(n)} = \mathfrak{so}(n)$ splits as a direct sum

$$\mathfrak{so}(n) = \mathfrak{k} \oplus \mathfrak{m},$$

with

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix} \mid A \in M_{n-k,k} \right\}. $$
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The tangent vectors $X \in \mathfrak{k}$ are \textit{vertical tangent vectors}, and the tangent vectors $X \in \mathfrak{m}$ are \textit{horizontal tangent vectors}.

It turns out that \textit{the tangent space $T_{o}(\mathbf{SO}(n)/K)$ to $\mathbf{SO}(n)/K$ at $o$ (= the coset $K$) is isomorphic to $\mathfrak{m}$}. 
(In the above Figure, $G = \textbf{SO}(n)$, $K \leftrightarrow H$, $\mathfrak{t} \leftrightarrow \mathfrak{h}$, $M = \textbf{SO}(n)/K$).
Furthermore with the metric on $\mathfrak{so}(n)$ given by

$$\langle X, Y \rangle = -\frac{1}{2} \text{tr}(XY) = \frac{1}{2} \text{tr}(X^\top Y),$$

the spaces $\mathfrak{k}$ and $\mathfrak{m}$ are orthogonal complements. $\text{SO}(n)/K$ is a naturally reductive homogeneous space. In fact, it is a symmetric space (Élie Cartan).
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Geodesics in $G(k, n) \cong \text{SO}(n)/K$ are projections of horizontal geodesics in $\text{SO}(n)$ (geodesics with initial velocity $X \in \mathfrak{m}$).
Theorem 2

The distance between any two subspaces $U, V \in G(k, n)$ specified by two $n \times k$ matrices $A, B$ with orthogonal columns is

$$d(U, V) = \sqrt{\theta_1^2 + \cdots + \theta_k^2},$$

where $(\cos \theta_1, \ldots, \cos \theta_k)$ are the singular values of $A^T B$, with $0 \leq \theta_i \leq \pi/2$. 
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The angles $\theta_1, \ldots, \theta_k$ are also known as the principal angles of the subspaces $U$ and $V$ (Camille Jordan).
Other interesting manifolds, such as $\text{SPD}(n)$ (symmetric, positive, definite matrices) are presented as homogeneous spaces; for example, $\text{SPD}(n) \cong \text{GL}^+(n)/\text{SO}(n)$. 
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The ability to compute explicity geodesic on the Grassmannian $G(k, n)$ (also the Stiefel manifolds $S(k, n)$) allows the generalization of optimization methods such as gradient descent and conjugate gradient to $G(k, n)$, $S(k, n)$, $\text{SO}(n)$. 

Dealing with $\text{SE}(n)$ and the Grassmannian of affine subspaces remains an open problem.
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