Homology, Cohomology, and Sheaf Cohomology for Algebraic Topology, Algebraic Geometry, and Differential Geometry

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Preface

The main topics of this book are cohomology, sheaves, and sheaf cohomology. Why? Mostly because for more than thirty years the senior author has been trying to learn algebraic geometry. To his dismay, he realized that since 1960, under the influence and vision of A. Grothendieck and his collaborators, in particular Serre, the foundations of algebraic geometry were built on sheaves and cohomology. But the invasion of these theories was not restricted to algebraic geometry. Cohomology was already a pillar of algebraic topology but sheaves and sheaf cohomology sneaked in too.

For a novice the situation seems hopeless. Even before one begins to discuss curves or surfaces, one has to spend years learning

(1) Some algebraic topology (especially homology and cohomology).

(2) Some basic homological algebra (chain complexes, cochain complexes, exact sequences, chain maps, *etc.*). Some commutative algebra (injective and projective modules, injective and projective resolutions).

(3) Some sheaf theory.

This book represents the result of an unfinished journey in attempting to accomplish the above. What we discovered on the way is that algebraic topology is a captivating and beautiful subject. We also believe that it is hard to appreciate sophisticated concepts such as sheaf cohomology without prior exposure to fundamentals of algebraic topology, simplicial homology, singular homology, and CW complexes, in particular.

With the above motivation in mind, this book consists of two parts. The first part consisting of the first seven chapters gives a crash-course on the homological and cohomological aspects of algebraic topology, with a bias in favor of cohomology. Unfortunately homotopy theory is omitted. Generally we do not provide proofs, with the exception of the homological tools needed later in the second part (such as the "zig-zag lemma"). Instead we try to provide intuitions and motivations, but we still provide rigorous definitions.

We conclude this overview of algebraic topology with a presentation of Poincaré duality, one of the jewels of algebraic topology. We follow Milnor and Stasheff's exposition [45] using the cap product, occasionally supplemented by Massey [41]. Contrary to the previous chapters we provide almost all proofs.

Hopefully this approach will not be frustrating to the reader. Our advice is to keep a copy of Hatcher [31] or Munkres [48] and Massey [41] at hand. Omitted details will be found in these references. Spanier [59] may also be helpful for some of the more advanced topics.

The second part is devoted to presheaves, sheaves, Čech cohomology, derived functors, sheaf cohomology, and spectral sequences.

Every book on algebraic geometry that goes beyond the classical material known before 1960 discusses sheaves and cohomology. The classic on the subject is Hartshorne [30]. The joke in certain circles is that most people are so exhausted after reading Chapters II and III that they never get to read the subsequent chapters on curves and surfaces.

It appears that after almost seventy five years it is not easy to find thorough expositions of sheaf cohomology designed for a "general" audience, with the exception of Rotman (second edition) [52]. Godement was already lamenting about this in the preface of his book [24] published in 1958. He says that ironically, someone with expertise in functional analysis (him) was compelled to give a complete exposition, that is, less incomplete than the other existing expositions of sheaf theory.

Godement writes in French in the Bourbaki style, which means that the exposition is terse, motivations are missing, and examples are few. This is very unfortunate because Godement's book contains some interesting material that is not easily found elsewhere, such as the spectral sequence of a differential sheaf and the spectral sequence of Čech cohomology. We discuss these topics in Chapter 15.

Our own experience is that the process of learning sheaves is facilitated by proceeding in stages. The first stage is to just define presheaves and sheaves and to give several examples. We do this in Chapter 8.

The second stage is to define the Čech cohomology of sheaves. Čech cohomology is combinatorial in nature and quite concrete so one can see how sheaves provide varying coefficients. This is the approach followed by Bott and Tu [4]. It is even possible without getting too technical to explain why De Rham cohomology is equivalent to Čech cohomology with coefficients in $\mathbb{R}$ by introducing the double complex known as the *Čech–de–Rham complex*. This material is discussed in Chapter 9.

The third stage is to explain the sheafification process, making a presheaf into a sheaf, and the approach to sheaves in terms of stalk spaces due to Lazard and Cartan. One would like to define the notion of exact sequence of sheaves, but unfortunately the obvious notion of image of a sheaf is not a sheaf in general, so the sheafification process can't be avoided. The right way to define the image of a sheaf is to define the notion of cokernel map of a sheaf and to define the image as the kernel of the cokernel map. These gymnastics are inspired by the notion of image of a map in an abelian categories, so we proceed with a basic presentation of the notions of categories, additive categories, and abelian categories. This way we can rightly claim that sheaves form an abelian category.

Personally, we find Serre's explanation of the sheafification process to be one of the clearest and we borrowed much from his famous paper FAC [55] (actually, his presentation of Čech cohomology of sheaves is also very precise and clear). The above material is presented in Chapter 10.

Having the machinery of sheaves at our disposal, the next step is to introduce sheaf cohomology. This can be done in two ways:

(1) In terms of resolutions by injectives.

(2) In terms of resolutions by flasque sheaves, a method invented by Godement [24].

In either case it is not possible to escape discussing the concept of resolution. We decided that we might as well go further and present some notions of homological algebra, namely projective and injective resolutions, as well as the notion of derived functor. Given a module A, a resolution is an exact sequence starting with A involving projective and injective modules. Projective and injective modules are modules satisfying certain extension properties. Given an additive functor T and an object A, it is possible to define uniquely some homology groups $L_nT(A)$ induced by projective resolutions of A and independent of such resolutions. It is also possible to define uniquely some cohomology groups $R^nT(A)$ induced by injective resolutions of A and independent of such resolutions; see Chapter 11. As special cases we obtain the Tor modules (associated with the tensor product) and the Ext modules (associated with the Hom functor). The modules Tor and Ext play a crucial role in the universal coefficient theorems; see Chapter 12. Our presentation of the homological algebra given in Chapters 11 and 12 is heavily inspired by Rotman's excellent exposition [52]. Although Mac Lane's presentation is more concise it is still a very reliable and elegantly written source which also contains historical sections [37].

Having gone that far, we also discuss Grothendieck's notion of δ-functors and universal δ-functors. The significance of this notion is that *the machinery of universal δ-functors can be used to prove that different kinds of cohomology theories yield isomorphic groups*. This technique will be used in Chapter 13 to prove that sheaf cohomology and Čech cohomology are isomorphic for paracompact spaces.

Grothendieck's legendary Tohoku paper [27] is written in French in a very terse style and many proof details are omitted (there are also quite a few typos). We are not aware of any source that gives detailed proofs of the main results about δ-functors (in particular, Proposition 2.2.1 on Page 141 of [27]). Lang [35] gives a fairly complete proof but omits the proof that the construction of the required morphism is unique. We fill in this step using an argument communicated to us by Steve Shatz; see Chapter 11.

Having the machinery of resolutions and derived functors at our disposal we are in the position to discuss sheaf cohomology quite thoroughly in Chapter 13. We show that the definition of sheaf cohomology in terms of derived functors is equivalent to the definition in terms of resolutions by flasque sheaves (due to Godement). We prove the equivalence of sheaf cohomology and Čech cohomology for paracompact spaces. We also discuss soft and fine sheaves, and prove that for a paracompact topological space, singular cohomology, Čech cohomology, Alexander–Spanier cohomology, and sheaf cohomology (for a suitable constant sheaf) are equivalent.

The purpose of Chapter 14 is to present various generalizations of Poincaré duality. These versions of duality involve taking direct limits of direct mapping families of singular cohomology groups which, in general, are not singular cohomology groups. However, such limits are isomorphic to Alexander–Spanier cohomology groups, and thus to Čech cohomology groups. These duality results also require relative versions of homology and cohomology.

The last chapter of our book (Chapter 15) is devoted to spectral sequences. A spectral sequence is a tool of homological algebra whose purpose is to approximate the cohomology (or homology) $H(M)$ of a module M endowed with a family $(F^pM)_{p\in\mathbb{Z}}$ of submodules called a filtration. The module M is also equipped with a linear map $d\colon M \to M$ called differential such that $d \circ d = 0$, so that it makes sense to define

$$H(M) = \mathrm{Ker}\, d/\mathrm{Im}\, d.$$

We say that (M, d) is a differential module. To be more precise, the filtration induces cohomology submodules $H(M)^p$ of $H(M)$, the images of $H(F^pM)$ in $H(M)$, and a spectral sequence is a sequence of modules E_r^p (equipped with a differential d_r^p), for $r \geq 1$, such that E_r^p approximates the "graded piece" $H(M)^p/H(M)^{p+1}$ of $H(M)$.

Actually, to be useful, the machinery of spectral sequences must be generalized to filtered cochain complexes. Technically this implies dealing with objects $E_r^{p,q}$ involving three indices, which makes its quite challenging to follow the exposition.

Many presentations jump immediately to the general case, but it seems pedagogically advantageous to begin with the simpler case of a single filtered differential module. This is the approach followed by Serre in his dissertation [56] (Pages 24–104, *Annals of Mathematics*, 54 (1951), 425–505), Godement [24], and Cartan and Eilenberg [10].

Spectral sequences were first introduced by Jean Leray in 1945 and 1946. Paraphrazing Jean Dieudoné [11], Leray's definitions were cryptic and proofs were incomplete. Koszul was the first to give a clear definition of spectral sequences in 1947. Independently, in his dissertation (1946), Lyndon introduced spectral sequences in the context of group extensions.

Detailed expositions of spectral sequences do not seem to have appeared until 1951, in lecture notes by Henri Cartan and in Serre's dissertation [56], which we highly recommend for its clarity (Serre defines homology spectral sequences, but the translation to cohomology is immediate). A concise but very clear description of spectral sequences appears in Dieudonné [11] (Chapter 4, Section 7, Parts D, E, F). More extensive presentations appeared in Cartan and Eilenberg [10] and Godement [24] around 1955.

There are several methods for defining spectral sequences, including the following three:

(1) Koszul's original approach as described by Serre [56] and Godement [24]. In our opinion it is the simplest method to understand what is going on.

(2) Cartan and Eilenberg's approach [10]. This is a somewhat faster and slicker method than the previous method.

(3) Exact couples of Massey (1952). This somewhat faster method for defining spectral sequences is adopted by Rotman [50, 52] and Bott and Tu [4]. Mac Lane [37], Weibel [63], and McCleary [44] also present it and show its equivalence with the first approach. It appears to be favored by algebraic topologists. This approach leads to spectral sequences in a quicker fashion and is more general because exact couples need not arise from a filtration, but our feeling is that it is even more mysterious to a novice than the first two approaches.

We will primarily follow Method (1) and present Method (2) and Method (3) in starred sections (Method (2) in Section 15.15 and Method (3) in Section 15.14). All three methods produce isomorphic sequences, and we will show their equivalence. We will also discuss the spectral sequences induced by double complexes and give as illustrations the spectral sequence of a differential sheaf and the spectral sequence of Čech cohomology. These spectral sequences are discussed in Godement [24].

We hope that the reader who read this book, especially the second part, will be well prepared to tackle Hartshorne [30] or comparable books on algebraic geometry. But we will be even happier if our readers found the topics of algebraic topology and homological algebra presented lovable (as Rotman hopes in his preface), and even beautiful.

In the second part of our book, except for a few exceptions we provide complete proofs. We did so to make this book self-contained, but also because we believe that no deep knowledge of this material can be acquired without working out some proofs. However, our advice is to skip some of the proofs upon first reading, especially if they are long and intricate.

The chapters or sections marked with the symbol ⊛ contain material that is typically more specialized or more advanced, and they can be omitted upon first (or second) reading.

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Contents

Chapter 1

Introduction

One of the main problems, if not "the" problem of topology, is to understand when two spaces X and Y are similar or dissimilar. A related problem is to understand the connectivity structure of a space in terms of its holes and "higher-order" holes. Of course, one has to specify what "similar" means. Intuitively, two topological spaces X and Y are similar if there is a "good" bijection $f\colon X \to Y$ between them. More precisely, "good" means that f is a continuous bijection whose inverse f^{-1} is also continuous; in other words, f is a *homeomorphism*. The notion of homeomorphism captures the notion proposed in the mid 1860s that X can be deformed into Y without tearing or overlapping. The problem then is to describe the equivalence classes of spaces under homeomorphism; it is a *classification problem*.

The classification problem for surfaces was investigated as early as the mid 1860s by Möbius and Jordan. These authors discovered that two (compact) surfaces are equivalent iff they have the same *genus* (the number of holes) and orientability type. Their "proof" could not be rigorous since they did not even have a precise definition of what a 2-manifold is! We have to wait until 1921 for a complete and rigorous proof of the classification theorem for compact surfaces; see Gallier and Xu [22] for a historical as well as technical account of this remarkable result.

What if X and Y do not have the nice structure of a surface or if they have higher-order dimension? In the words of Dieudonné, the problem is a "hopeless undertaking;" see Dieudonné's introduction [11].

The reaction to this fundamental difficulty was the creation of algebraic and differential topology, whose major goal is to associate "invariant" objects to various types of spaces, so that homeomorphic spaces have "isomorphic" invariants. If two spaces X and Y happen to have some distinct invariant objects, then for sure they are not homeomorphic.

Poincaré was one of the major pioneers of this approach. At first these invariant objects were integers (Betti numbers and torsion numbers), but it was soon realized that much more information could be extracted from invariant algebraic structures such as groups, ring, and modules.

Three types of invariants can be assigned to a topological space:

(1) Homotopy groups.

(2) Homology groups.

(3) Cohomology groups.

The above are listed in the chronological order of their discovery. It is interesting that the first homotopy group $\pi_1(X)$ of the space X, also called *fundamental group*, was invented by Poincaré (*Analysis Situs*, 1895), but homotopy basically did not evolve until the 1930s. One of the reasons is that the first homotopy group is generally nonabelian, so harder to study.

On the other hand, homology and cohomology groups (or rings, or modules) are abelian, so results about commutative algebraic structures can be leveraged. This is true in particular if the ring R is a PID, where the structure of the finitely generated R-modules is completely determined.

There are different kinds of homology groups. They usually correspond to some geometric intuition about decomposing a space into simple shapes such as triangles, tetrahedra, *etc.* Cohomology is more abstract because it usually deals with functions on a space. However, we will see that it yields more information than homology precisely because certain kinds of operations on functions can be defined (cup and cap products).

As often in mathematics, some machinery that is created to solve a specific problem, here a problem in topology, unexpectedly finds fruitful applications to other parts of mathematics and becomes a major component of the arsenal of mathematical tools, in the present case *homological algebra* and *category theory*. In fact, category theory, invented by Mac Lane and Eilenberg, permeates algebraic topology and is really put to good use, rather than being a fancy attire that dresses up and obscures some simple theory, as often is the case.

In view of the above discussion, it appears that algebraic topology might involve more algebra than topology. This is great if one is quite proficient in algebra, but not so good news for a novice who might be discouraged by the abstract and arcane nature of homological algebra. After all, what do the zig-zag lemma and the five lemma have to do with topology?

Unfortunately, it is true that a firm grasp of the basic concepts and results of homological algebra is essential to really understand what are the homology and the cohomology groups and what are their roles in topology.

One of our goals is to attempt to demystify homological algebra. For those of us fond of puns, keep this simple analogy in mind and all trepidation will (hopefully) fade. Homology groups describe what man does in his home; in French, l'homme au logis. Cohomology groups describe what co-man does in his home; in French, le co-homme au logis, that is, la femme au logis. Obviously this is not politically correct, so cohomology should be renamed. The big question is: what is a better name for cohomology?

In the following sections we give a brief description of the topics that we are going to discuss in this book, and we try to provide motivations for the introduction of the concepts and tools involved. These sections introduce topics in the same order in which they are presented in the book. All historical references are taken from Dieudonné [11]. This is a remarkable account of the history of algebraic and differential topology from 1900 to the 1960s which contains a wealth of information.

1.1 Exact Sequences, Chain Complexes, Homology and Cohomology

There are various kinds of homology groups (simplicial, singular, cellular, *etc.*), but they all arise the same way, namely from a (possibly infinite) sequence called a *chain complex*

$$0 \xleftarrow{d_0} C_0 \xleftarrow{d_1} C_1 \longleftarrow \cdots \xleftarrow{d_{p-1}} C_{p-1} \xleftarrow{d_p} C_p \xleftarrow{d_{p+1}} C_{p+1} \longleftarrow \cdots,$$

in which the C_p are vector spaces, or more generally abelian groups (typically freely generated), and the maps $d_p\colon C_p \to C_{p-1}$ are linear maps (homomorphisms of abelian groups) satisfying the condition

$$d_p \circ d_{p+1} = 0 \quad \text{for all } p \geq 0. \tag{$*_1$}$$

The elements of C_p are called *p-chains* and the maps d_p are called *boundary operators (or boundary maps)*. The intuition behind Condition $(*_1)$ is that elements of the form $d_p(c) \in C_{p-1}$ with $c \in C_p$ are *boundaries*, and "a boundary has no boundary." For example, in $\mathbb{R}^2$, the points on the boundary of a closed unit disk form the unit circle, and the points on the unit circle have no boundary.

Since $d_p \circ d_{p+1} = 0$, we have $B_p(C) = \operatorname{Im} d_{p+1} \subseteq \operatorname{Ker} d_p = Z_p(C)$ so the quotient $Z_p(C)/B_p(C) = \operatorname{Ker} d_p/\operatorname{Im} d_{p+1}$ makes sense. The quotient module

$$H_p(C) = Z_p(C)/B_p(C) = \operatorname{Ker} d_p/\operatorname{Im} d_{p+1}$$

is the *p-th homology module* of the chain complex C. Elements of Z_p are called *p-cycles* and elements of B_p are called *p-boundaries*; see Figure 1.1.

A condition stronger that Condition $(*_1)$ is that

$$\operatorname{Im} d_{p+1} = \operatorname{Ker} d_p \quad \text{for all } p \geq 0. \tag{$**_1$}$$

A sequence satisfying Condition $(**_1)$ is called an *exact sequence*. Thus, we can view the homology groups as a measure of the failure of a chain complex to be exact. Surprisingly, exact sequences show up in various areas of mathematics, especially abstract algebra.

In the case of many homology theories, chain complexes are constructed by "nicely" mapping simple geometric objects into a given topological space X. For singular homology

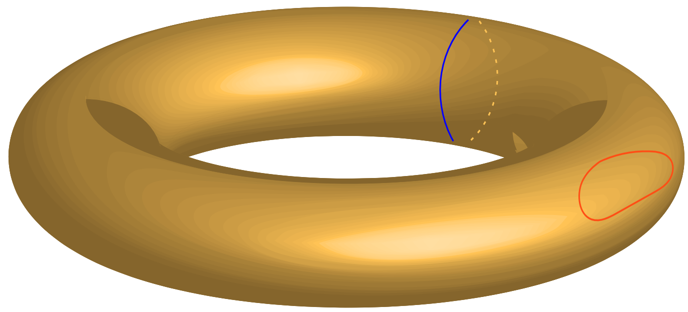


Figure 1.1: Let X be the surface of the torus. Elements of Z_1 are geometrically represented by curves which are homeomorphic to S^1. Thus both the red and blue curves are 1-cycles. The red curve is also a 1-boundary since it is the boundary of a region in X which is homeomorphic to the closed unit disk.

the C_p's are the abelian groups $C_p = S_p(X; \mathbb{Z})$ consisting of all (finite) linear combinations of the form $\sum n_i \sigma_i$, where $n_i \in \mathbb{Z}$ and each σ_i, a *singular p-simplex*, is a continuous function $\sigma_i \colon \Delta^p \to X$ from the p-simplex Δ^p to the space X. A 0-simplex is a single point, a 1-simplex is a line segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and a p-simplex is a higher-order generalization of a tetrahedron; see Figure 1.2.

A p-simplex Δ^p has $p+1$ *faces*, and the ith face is a $(p-1)$-singular simplex $\sigma \circ \phi_i^{p-1} \colon \Delta^{p-1} \to X$ defined in terms of a certain function $\phi_i^{p-1} \colon \Delta^{p-1} \to \Delta^p$; see Section 4.1. In the framework of singular homology, the boundary map d_p is denoted by ∂_p, and for any singular p-simplex σ, $\partial\sigma$ is the singular $(p-1)$-chain given by

$$\partial\sigma = \sigma \circ \phi_0^{p-1} - \sigma \circ \phi_1^{p-1} + \cdots + (-1)^p \sigma \circ \phi_p^{p-1}.$$

A simple calculation confirms that $\partial_p \circ \partial_{p+1} = 0$. Consequently the free abelian groups $S_p(X; \mathbb{Z})$ together with the boundary maps ∂_p form a chain complex denoted $S_*(X; \mathbb{Z})$ called the *simplicial chain complex* of X. Then the quotient module

$$H_p(X; \mathbb{Z}) = H_p(S_*(X; \mathbb{Z})) = \mathrm{Ker}\, \partial_p / \mathrm{Im}\, \partial_{p+1},$$

also denoted $H_p(X)$, is called the *p-th homology group* of X. Singular homology is discussed in Chapter 4, especially in Section 4.1.

Historically, singular homology did not come first. According to Dieudonné [11], singular homology emerged around 1925 in the work of Veblen, Alexander and Lefschetz (the "Princeton topologists," as Dieudonné calls them), and was defined rigorously and in complete generality by Eilenberg (1944). The definition of the homology modules $H_p(C)$ in terms

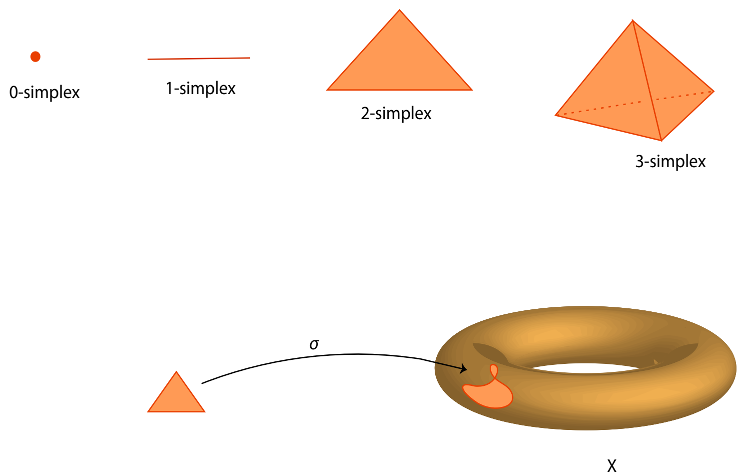


Figure 1.2: The top row illustrates lower order p-simplicies while the bottom figure illustrates a singular 2-simplex within the 2-dimensional torus.

of sequences of abelian groups C_p and boundary homomorphisms $d_p \colon C_p \to C_{p-1}$ satisfying the condition $d_p \circ d_{p+1} = 0$ as quotients $\mathrm{Ker}\, d_p/\mathrm{Im}\, d_{p+1}$ seems to have been suggested to H. Hopf by Emmy Noether while Hopf was visiting Göttingen in 1925. Hopf used this definition in 1928, and independently so did Vietoris in 1926, and then Mayer in 1929.

The first occurrence of a chain complex is found in Poincaré's papers of 1900, although he did not use the formalism of modules and homomorphisms as we do now, but matrices instead. Poincaré introduced the homology of *simplicial complexes*, which are combinatorial triangulated objects objects made up of simplices; see Figure 1.3.

Given a simplicial complex K, we have free abelian groups $C_p(K)$ consisting of $\mathbb{Z}$-linear combinations of oriented p-simplices, and the boundary maps $\partial_p \colon C_p(K) \to C_{p-1}(K)$ are defined by

$$\partial_p \sigma = \sum_{i=0}^{p} (-1)^i [\alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \alpha_p],$$

for any oriented p-simplex, $\sigma = [\alpha_0, \ldots, \alpha_p]$, where $[\alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \alpha_p]$ denotes the oriented $(p-1)$-simplex obtained by deleting vertex α_i. Then we have a *simplicial chain complex* $(C_p(K), \partial_p)$ denoted $C_*(K)$, and the corresponding homology groups $H_p(C_*(K))$ are denoted $H_p(K)$ and called the *simplicial homology groups* of K. Simplicial homology is discussed in Chapter 5. We discussed singular homology first because it subsumes simplicial homology, as shown in Section 5.2.

A simplicial complex K is a purely combinatorial object, thus it is not a space, but it has a *geometric realization* K_g, which is a (triangulated) topological space. This brings up the

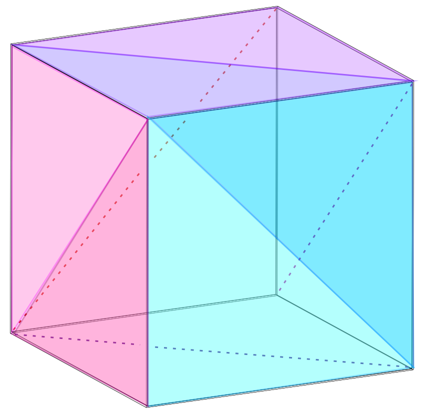


Figure 1.3: The surface of a cube as a simplicial complex consisting of 12 triangles (2-simplicies), 18 edges (1-simplicies), and 8 vertices (0-simplices).

following question: if K_1 and K_2 are two simplicial complexes whose geometric realizations $(K_1)_g$ and $(K_2)_g$ are homeomorphic, are the simplicial homology groups $H_p(K_1)$ and $H_p(K_2)$ isomorphic?

Poincaré conjectured that the answer was "yes," and this conjecture was first proved by Alexander. The proof is nontrivial, and we present a version of it in Section 5.2.

The above considerations suggest that it would be useful to understand the relationship between the homology groups of two spaces X and Y related by a continuous map $f\colon X \to Y$. For this, we define mappings between chain complexes called chain maps.

Given two chain complexes C and C', a *chain map* $f\colon C \to C'$ is a family $f = (f_p)_{p\geq 0}$ of homomorphisms $f_p\colon C_p \to C'_p$ such that all the squares of the following diagram commute:

$$\begin{array}{ccccccccccccc}
0 & \xleftarrow{d_0} & C_0 & \xleftarrow{d_1} & C_1 & \longleftarrow & \cdots & \xleftarrow{d_{p-1}} & C_{p-1} & \xleftarrow{d_p} & C_p & \xleftarrow{d_{p+1}} & C_{p+1} & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & & & \downarrow{\scriptstyle f_{p-1}} & & \downarrow{\scriptstyle f_p} & & \downarrow{\scriptstyle f_{p+1}} & & \\
0 & \xleftarrow[d'_0]{} & C'_0 & \xleftarrow[d'_1]{} & C'_1 & \longleftarrow & \cdots & \xleftarrow[d'_{p-1}]{} & C'_{p-1} & \xleftarrow[d'_p]{} & C'_p & \xleftarrow[d'_{p+1}]{} & C'_{p+1} & \longleftarrow & \cdots,
\end{array}$$

that is, $f_p \circ d_{p+1} = d'_{p+1} \circ f_{p+1}$, for all $p \geq 0$.

A chain map $f\colon C \to C'$ induces homomorphisms of homology

$$H_p(f)\colon H_p(C) \to H_p(C')$$

for all $p \geq 0$. Furthermore, given three chain complexes C, C', C'' and two chain maps $f\colon C \to C'$ and $g\colon C' \to C''$, we have

$$H_p(g \circ f) = H_p(g) \circ H_p(f) \quad \text{for all } p \geq 0$$

and

$$H_p(\mathrm{id}_C) = \mathrm{id}_{H_p(C)} \quad \text{for all } p \geq 0.$$

We say that the map $C \mapsto (H_p(C))_{p\geq 0}$ is *functorial* (to be more precise, it is a functor from the category of chain complexes and chain maps to the category of abelian groups and groups homomorphisms).

For example, in singular homology, a continuous function $f\colon X \to Y$ between two topological spaces X and Y induces a chain map $f_\sharp\colon S_*(X;\mathbb{Z}) \to S_*(Y;\mathbb{Z})$ between the two simplicial chain complexes $S_*(X;\mathbb{Z})$ and $S_*(Y;\mathbb{Z})$ associated with X and Y, which in turn yield homology homomorphisms usually denoted $f_{*,p}\colon H_p(X;\mathbb{Z}) \to H_p(Y;\mathbb{Z})$. Thus the map $X \mapsto (H_p(X;\mathbb{Z}))_{p\geq 0}$ is a functor from the category of topological spaces and continuous maps to the category of abelian groups and groups homomorphisms. Functoriality implies that if $f\colon X \to Y$ is a homeomorphism, then the maps $f_{*,p}\colon H_p(X;\mathbb{Z}) \to H_p(Y;\mathbb{Z})$ are *isomorphisms*. Thus, the singular homology groups are topological invariants. This is one of the advantages of singular homology; topological invariance is basically obvious.

This is not the case for simplicial homology where it takes a fair amount of work to prove that if K_1 and K_2 are two simplicial complexes whose geometric realizations $(K_1)_g$ and $(K_2)_g$ are homeomorphic, then the simplicial homology groups $H_p(K_1)$ and $H_p(K_2)$ isomorphic.

One might wonder what happens if we reverse the arrows in a chain complex? Abstractly, this is how cohomology is obtained, although this point of view was not considered until at least 1935.

A *cochain complex* is a sequence

$$0 \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \longrightarrow C^{p-1} \xrightarrow{d^{p-1}} C^p \xrightarrow{d^p} C^{p+1} \xrightarrow{d^{p+1}} C^{p+2} \longrightarrow \cdots,$$

in which the C^p are abelian groups, and the maps $d^p\colon C^p \to C^{p+1}$ are homomorphisms of abelian groups satisfying the condition

$$d^{p+1} \circ d^p = 0 \quad \text{for all } p \geq 0 \tag{$*_2$}$$

The elements of C^p are called *cochains* and the maps d^p are called *coboundary maps*. This time it is not clear how coboundary maps arise naturally. Since $d^{p+1} \circ d^p = 0$, we have $B^p = \mathrm{Im}\, d^p \subseteq \mathrm{Ker}\, d^{p+1} = Z^{p+1}$, so the quotient $Z^p/B^p = \mathrm{Ker}\, d^{p+1}/\mathrm{Im}\, d^p$ makes sense and the quotient module

$$H^p(C) = Z^p/B^p = \mathrm{Ker}\, d^{p+1}/\mathrm{Im}\, d^p$$

is the *pth cohomology module* of the cochain complex C. Elements of Z^p are called *p-cocycles* and elements of B^p are called *p-coboundaries*.

There seems to be an unwritten convention that when dealing with homology we use subscripts, and when dealing with cohomology we use with superscripts. Also, the “dual” of any “notion” is the “co-notion.”

As in the case of a chain complex, a condition stronger that Condition $(*_2)$ is that

$$\operatorname{Im} d^p = \operatorname{Ker} d^{p+1} \quad \text{for all } p \geq 0. \tag{$**_2$}$$

A sequence satisfying Condition $(**_2)$ is also called an *exact sequence*. Thus, we can view the cohomology groups as a measure of the failure of a cochain complex to be exact.

Given two cochain complexes C and C', a *(co)chain map* $f\colon C \to C'$ is a family $f = (f^p)_{p\geq 0}$ of homomorphisms $f^p\colon C^p \to C'^p$ such that all the squares of the following diagram commute:

$$\begin{array}{ccccccccccccccc}
0 & \xrightarrow{d^{-1}} & C^0 & \xrightarrow{d^0} & C^1 & \xrightarrow{d^1} & \cdots & \longrightarrow & C^{p-1} & \xrightarrow{d^{p-1}} & C^p & \xrightarrow{d^p} & C^{p+1} & \xrightarrow{d^{p+1}} & \cdots \\
 & & \Big\downarrow{\scriptstyle f^0} & & \Big\downarrow{\scriptstyle f^1} & & & & \Big\downarrow{\scriptstyle f^{p-1}} & & \Big\downarrow{\scriptstyle f^p} & & \Big\downarrow{\scriptstyle f^{p+1}} & & \\
0 & \xrightarrow[d'^{-1}]{} & C'^0 & \xrightarrow[d'^0]{} & C'^1 & \xrightarrow[d'^1]{} & \cdots & \longrightarrow & C'^{p-1} & \xrightarrow[d'^{p-1}]{} & C'^p & \xrightarrow[d'^p]{} & C'^{p+1} & \xrightarrow[d'^{p+1}]{} & \cdots,
\end{array}$$

that is, $f^{p+1} \circ d^p = d'^p \circ f^p$ for all $p \geq 0$. A chain map $f\colon C \to C'$ induces homomorphisms of cohomology

$$H^p(f)\colon H^p(C) \to H^p(C')$$

for all $p \geq 0$. Furthermore, this assignment is functorial (more precisely, it is a functor from the category of cochain complexes and chain maps to the category of abelian groups and their homomorphisms).

At first glance cohomology appears to be very abstract so it is natural to look for explicit examples. A way to obtain a cochain complex is to apply the operator (functor) $\operatorname{Hom}_{\mathbb{Z}}(-, G)$ to a chain complex C, where G is any abelian group. Given a fixed abelian group A, for any abelian group B we denote by $\operatorname{Hom}_{\mathbb{Z}}(B, A)$ the abelian group of all homomorphisms from B to A. Given any two abelian groups B and C, for any homomorphism $f\colon B \to C$, the homomorphism $\operatorname{Hom}_{\mathbb{Z}}(f, A)\colon \operatorname{Hom}_{\mathbb{Z}}(C, A) \to \operatorname{Hom}_{\mathbb{Z}}(B, A)$ is defined by

$$\operatorname{Hom}_{\mathbb{Z}}(f, A)(\varphi) = \varphi \circ f \quad \text{for all } \varphi \in \operatorname{Hom}_{\mathbb{Z}}(C, A);$$

see the commutative diagram below:

$$\begin{array}{ccc}
B & \xrightarrow{f} & C \\
 & {\scriptstyle \operatorname{Hom}_{\mathbb{Z}}(f,A)(\varphi)} \searrow & \Big\downarrow{\scriptstyle \varphi} \\
 & & A.
\end{array}$$

The map $\operatorname{Hom}_{\mathbb{Z}}(f, A)$ is also denoted by $\operatorname{Hom}_{\mathbb{Z}}(f, \mathrm{id}_A)$ or even $\operatorname{Hom}_{\mathbb{Z}}(f, \mathrm{id})$. Observe that the effect of $\operatorname{Hom}_{\mathbb{Z}}(f, \mathrm{id})$ on φ is to precompose φ with f.

If $f\colon B \to C$ and $g\colon C \to D$ are homomorphisms of abelian groups, a simple computation shows that

$$\operatorname{Hom}_R(g \circ f, \mathrm{id}) = \operatorname{Hom}_R(f, \mathrm{id}) \circ \operatorname{Hom}_R(g, \mathrm{id}).$$

Observe that $\mathrm{Hom}_{\mathbb{Z}}(f, \mathrm{id})$ and $\mathrm{Hom}_{\mathbb{Z}}(g, \mathrm{id})$ are composed in the reverse order of the composition of f and g. It is also immediately verified that

$$\mathrm{Hom}_{\mathbb{Z}}(\mathrm{id}_A, \mathrm{id}) = \mathrm{id}_{\mathrm{Hom}_{\mathbb{Z}}(A,G)}.$$

We say that $\mathrm{Hom}_{\mathbb{Z}}(-, \mathrm{id})$ is a *contravariant functor* (from the category of abelian groups and group homomorphisms to itself). Then given a chain complex

$$0 \xleftarrow{d_0} C_0 \xleftarrow{d_1} C_1 \longleftarrow \cdots \xleftarrow{d_{p-1}} C_{p-1} \xleftarrow{d_p} C_p \xleftarrow{d_{p+1}} C_{p+1} \longleftarrow \cdots,$$

we can form the cochain complex

$$0 \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(d_0,\mathrm{id})} \mathrm{Hom}_{\mathbb{Z}}(C_0, G) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(C_p, G) \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(d_{p+1},\mathrm{id})} \mathrm{Hom}_{\mathbb{Z}}(C_{p+1}, G) \longrightarrow \cdots$$

obtained by applying $\mathrm{Hom}_{\mathbb{Z}}(-, G)$, and denoted $\mathrm{Hom}_{\mathbb{Z}}(C, G)$. The coboundary map d^p is given by

$$d^p = \mathrm{Hom}_{\mathbb{Z}}(d_{p+1}, \mathrm{id}),$$

which means that for any $f \in \mathrm{Hom}_{\mathbb{Z}}(C_p, G)$, we have

$$d^p(f) = f \circ d_{p+1}.$$

Thus, for any $(p+1)$-chain $c \in C_{p+1}$ we have

$$(d^p(f))(c) = f(d_{p+1}(c)).$$

We obtain the cohomology groups $H^p(\mathrm{Hom}_{\mathbb{Z}}(C, G))$ associated with the cochain complex $\mathrm{Hom}_{\mathbb{Z}}(C, G)$. The cohomology groups $H^p(\mathrm{Hom}_{\mathbb{Z}}(C, G))$ are also denoted $H^p(C; G)$.

This process was applied to the simplicial chain complex $C_*(K)$ associated with a simplicial complex K by Alexander and Kolmogoroff to obtain the *simplicial cochain complex* $\mathrm{Hom}_{\mathbb{Z}}(C_*(K); G)$ denoted $C^*(K; G)$ and the *simplicial cohomology groups* $H^p(K; G)$ of the simplicial complex K; see Section 5.6. Soon after, this process was applied to the singular chain complex $S_*(X; \mathbb{Z})$ of a space X to obtain the *singular cochain complex* $\mathrm{Hom}_{\mathbb{Z}}(S_*(X; \mathbb{Z}); G)$ denoted $S^*(X; G)$ and the *singular cohomology groups* $H^p(X; G)$ of the space X; see Section 4.8.

Given a continuous map $f: X \to Y$, there is an induced chain map $f^\sharp: S^*(Y; G) \to S^*(X; G)$ between the singular cochain complexes $S^*(Y; G)$ and $S^*(X; G)$, and thus homomorphisms of cohomology $f^*: H^p(Y; G) \to H^p(X; G)$. Observe the reversal: f is a map from X to Y, but f^* maps $H^p(Y; G)$ to $H^p(X; G)$. We say that the map $X \mapsto (H^p(X; G))_{p \geq 0}$ is a contravariant functor from the category of topological spaces and continuous maps to the category of abelian groups and their homomorphisms.

So far our homology groups have coefficients in $\mathbb{Z}$, but the process of forming a cochain complex $\mathrm{Hom}_{\mathbb{Z}}(C, G)$ from a chain complex C allows the use of coefficients in any abelian group G, not just the integers. Actually, it is a trivial step to define chain complexes consisting of R-modules in any commutative ring R with a multiplicative identity element 1, and such complexes yield homology modules $H_p(C; R)$ with coefficients in R. This process immediately applies to the singular homology groups $H_p(X; R)$ and to the simplicial homology groups $H_p(K; R)$. Also, given a chain complex C where the C_p are R-modules, for any R-module G we can form the cochain complex $\mathrm{Hom}_R(C, G)$ and we obtain cohomology modules $H^p(C; G)$ with coefficients in any R-module G; see Section 4.8 and Chapter 12.

We can generalize homology with coefficients in a ring R to modules with coefficients in a R-module G by applying the operation (functor) $- \otimes_R G$ to a chain complex C, where the C_p's are R-modules, to get the chain complex denoted $C \otimes_R G$. The homology groups of this complex are denoted $H_p(C, G)$. We will discuss this construction in Section 4.7 and Chapter 12.

If the ring R is a PID, given a chain complex C where the C_p are R-modules, the homology groups $H_p(C; G)$ of the complex $C \otimes_R G$ are determined by the homology groups $H_{p-1}(C; R)$ and $H_p(C; R)$ *via* a formula called the *Universal Coefficient Theorem for Homology*; see Theorem 12.1. This formula involves a term $\mathrm{Tor}_1^R(H_{n-1}(C); G)$ that corresponds to the fact that the operation $- \otimes_R G$ on linear maps generally does not preserve injectivity ($- \otimes_R G$ is not left-exact). These matters are discussed in Chapter 11.

Similarly, if the ring R is a PID, given a chain complex C where the C_p are R-modules, the cohomology groups $H^p(C; G)$ of the complex $\mathrm{Hom}_R(C, G)$ are determined by the homology groups $H_{p-1}(C; R)$ and $H_p(C; R)$ *via* a formula called the *Universal Coefficient Theorem for Cohomology*; see Theorem 12.6. This formula involves a term $\mathrm{Ext}^1_R(H_{n-1}(C); G)$ that corresponds to the fact that if the linear map f is injective, then $\mathrm{Hom}_R(f, \mathrm{id})$ is not necessarily surjective ($\mathrm{Hom}_R(-, G)$ is not right-exact). These matters are discussed in Chapter 11.

One of the advantages of singular homology (and cohomology) is that it is defined for *all* topological spaces, but one of its disadvantages is that in practice it is very hard to compute. On the other hand, simplicial homology (and cohomology) only applies to triangulable spaces (geometric realizations of simplicial complexes), but in principle it is computable (for finite complexes). One of the practical problems is that the triangulations involved may have a large number of simplices. J.H.C. Whiteahead invented a class of spaces called *CW complexes* that are more general than triangulable spaces and for which the computation of the singular homology groups is often more tractable. Unlike a simplicial complex, a CW complex is obtained by gluing spherical cells as shown in Figure 1.4. CW complexes are discussed in Chapter 6.

There are at least four other ways of defining cohomology groups of a space X by directly forming a cochain complex without first forming a homology chain complex and then dualizing by applying $\mathrm{Hom}_{\mathbb{Z}}(-, G)$:

(1) If X is a smooth manifold, then there is the *de Rham complex* which uses the modules of smooth p-forms $\mathcal{A}^p(X)$ and the exterior derivatives $d^p\colon \mathcal{A}^p(X) \to \mathcal{A}^{p+1}(X)$. The

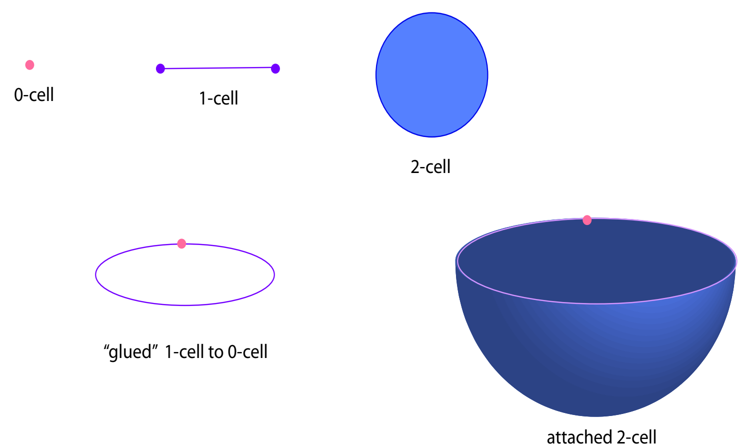


Figure 1.4: The spherical hemisphere is a CW complex consisting of a point (0-cell), a line segment (1-cell), and a closed unit disk (2-cell).

corresponding cohomology groups are the *de Rham cohomology groups* $H^p_{\mathrm{dR}}(X)$. These are actually real vector spaces. de Rham cohomology is discussed in Chapter 3.

(2) If X is any space and $\mathcal{U} = (U_i)_{i\in I}$ is any open cover of X, we can define the *Čech cohomology groups* $\check{H}^p(X, \mathcal{U})$ in a purely combinatorial fashion. Then we can define the notion of *refinement* of a cover and define the *Čech cohomology groups* $\check{H}^p(X, G)$ with values in an abelian group G using a limiting process known as a *direct limit* (see Section 8.3, Definition 8.10). Čech cohomology is discussed in Chapter 9.

(3) If X is any space, then there is the *Alexander–Spanier* cochain complex which yields the *Alexander–Spanier cohomology groups* $A^p_{\text{A-S}}(X; G)$. Alexander–Spanier cohomology is discussed in Section 13.8 and in Chapter 14.

(4) Sheaf cohomology, based on derived functors and injective resolutions. This is the most general kind of cohomology of a space X, where cohomology groups $H^p(X, \mathcal{F})$ with values in a sheaf $\mathcal{F}$ on the space X are defined. Intuitively, this means that the modules $\mathcal{F}(U)$ of "coefficients" in which these groups take values may vary with the open domain $U \subseteq X$. Sheaf cohomology is discussed in Chapter 13, and the algebraic machinery of derived functors is discussed in Chapter 11.

We will see that for topological manifolds, all these cohomology theories are equivalent; see Chapter 13. For paracompact spaces, Čech cohomology, Alexander–Spanier cohomology, and derived functor cohomology (for constant sheaves) are equivalent (see Chapter 13). In fact, Čech cohomology and Alexander–Spanier cohomology are equivalent for any space; see Chapter 14.

1.2 Relative Homology and Cohomology

In general, computing homology groups is quite difficult so it would be helpful if we had techniques that made this process easier. Relative homology and excision are two such tools that we discuss in this section.

Lefschetz (1928) introduced the *relative homology groups* $H_p(K, L; \mathbb{Z})$, where K is a simplicial complex and L is a subcomplex of K. The same idea immediately applies to singular homology and we can define the relative singular homology groups $H_p(X, A; R)$ where A is a subspace of X. The intuition is that the module of p-chains of a relative chain complex consists of chains of K modulo chains of L. For example, given a space X and a subspace $A \subseteq X$, the *singular chain complex* $S_*(X, A; R)$ *of the pair* (X, A) is the chain complex in which each R-module $S_p(X, A; R)$ is the quotient module

$$S_p(X, A; R) = S_p(X; R)/S_p(A; R).$$

It is easy to see that $S_p(X, A; R)$ is actually a free R-module; see Section 4.3.

Although this is not immediately apparent, the motivation is that the groups $H_p(A; R)$ and $H_p(X, A; R)$ are often "simpler" than the groups $H_p(X; R)$, and there is an exact sequence called the *long exact sequence of relative homology* that can often be used to come up with an inductive argument that allows the determination of $H_p(X; R)$ from $H_p(A; R)$ and $H_p(X, A; R)$. Indeed, we have the following exact sequence as shown in Section 4.3 (see Theorem 4.9):

$$\cdots \longrightarrow H_{p+2}(X, A; R) \xrightarrow{\partial_{*p+2}}$$
$$H_{p+1}(A; R) \xrightarrow{i_*} H_{p+1}(X; R) \xrightarrow{j_*} H_{p+1}(X, A; R) \xrightarrow{\partial_{*p+1}}$$
$$H_p(A; R) \xrightarrow{i_*} H_p(X; R) \xrightarrow{j_*} H_p(X, A; R) \xrightarrow{\partial_{*p}}$$
$$H_{p-1}(A; R) \longrightarrow \cdots$$

ending in

$$H_0(A; R) \longrightarrow H_0(X; R) \longrightarrow H_0(X, A; R) \longrightarrow 0.$$

Furthermore, if (X, A) is a "good pair," then there is an isomorphism

$$H_p(X, A; R) \cong H_p(X/A, \{\mathrm{pt}\}; R),$$

where the space X/A, *called a quotient space*, is obtained from X by identifying A with a single point, and where pt stands for any point in X.

The long exact sequence of relative homology is a corollary of one the staples of homology theory, the "zig-zag lemma." The zig-zag lemma says that for any short exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

of chain complexes X, Y, Z there is a long exact sequence of cohomology

$$\begin{array}{l} \cdots \longrightarrow H^{p-1}(Z) \xrightarrow{\delta^{p-1}} \\ H^p(X) \xrightarrow{f^*} H^p(Y) \xrightarrow{g^*} H^p(Z) \xrightarrow{\delta^{p}} \\ H^{p+1}(X) \xrightarrow{f^*} H^{p+1}(Y) \xrightarrow{g^*} H^{p+1}(Z) \xrightarrow{\delta^{p+1}} \\ H^{p+2}(X) \longrightarrow \cdots \end{array}$$

The zig-zag lemma is fully proven in Section 2.7; see Theorem 2.22. There is also a homology version of this theorem.

Another very important aspect of relative singular homology is that it satisfies the *excision axiom*, another useful tool to compute homology groups. This means that removing a subspace $Z \subseteq A \subseteq X$ which is clearly inside of A, in the sense that Z is contained in the interior of A, does not change the relative homology group $H_p(X, A; R)$. More precisely, there is an isomorphism

$$H_p(X - Z, A - Z; R) \cong H_p(X, A; R);$$

see Section 4.5 (Theorem 4.14). A good illustration of the use of excision and of the long exact sequence of relative homology is the computation of the homology of the sphere S^n; see Section 4.6. Relative singular homology also satisfies another important property: the *homotopy axiom*, which says that if two spaces are homotopy equivalent, then their homology is isomorphic; see Theorem 4.8.

Following the procedure for obtaining cohomology from homology described in Section 1.1, by applying $\mathrm{Hom}_R(-, G)$ to the chain complex $S_*(X, A; R)$ we obtain the cochain complex $S^*(X, A; G) = \mathrm{Hom}_R(S_*(X, A; R), G)$, and thus the *singular relative cohomology groups* $H^p(X, A; G)$; see Section 4.9. In this case we can think of the elements of $S^p(X, A; G)$ as linear maps (with values in G) on singular p-simplices in X that vanish on singular p-simplices in A.

Fortunately, since each $S_p(X, A; R)$ is a free R-module, it can be shown that there is a long exact sequence of relative cohomology (see Theorem 4.36):

$$\cdots \longrightarrow H^{p-1}(A;G) \xrightarrow{\delta^*_{p-1}} H^p(X,A;G) \xrightarrow{(j^\top)^*} H^p(X;G) \xrightarrow{(i^\top)^*} H^p(A;G) \xrightarrow{\delta^*_p} H^{p+1}(X,A;G) \xrightarrow{(j^\top)^*} H^{p+1}(X;G) \xrightarrow{(i^\top)^*} H^{p+1}(A;G) \xrightarrow{\delta^*_{p+1}} H^{p+2}(X,A;G) \longrightarrow \cdots$$

Relative singular cohomology also satisfies the excision axiom and the homotopy axioms (see Section 4.9).

1.3 Duality; Poincaré, Alexander, Lefschetz

Roughly speaking, duality is a kind of symmetry between the homology and the cohomology groups of a space. Historically, duality was formulated only for homology, but it was later found that more general formulations are obtained if both homology and cohomology are considered. We will discuss two duality theorems: Poincaré duality, and Alexander–Lefschetz duality. Original versions of these theorems were stated for homology and applied to special kinds of spaces. It took at least thirty years to obtain the versions that we will discuss.

The result that Poincaré considered as the climax of his work in algebraic topology was a *duality theorem* (even though the notion of duality was not very clear at the time). Since Poincaré was working with finite simplicial complexes, for him duality was a construction which, given a simplicial complex K of dimension n, produced a "dual" complex K^*; see Munkres [48] (Chapter 8, Section 64). If done the right way, the matrices of the boundary maps $\partial\colon C_p(K) \to C_{p-1}(K)$ are transposes of the matrices of the boundary maps $\partial^*\colon C_{n-p+1}(K^*) \to C_{n-p}(K^*)$. As a consequence, the homology groups $H_p(K)$ and $H_{n-p}(K^*)$ are isomorphic. Note that this type of duality relates homology groups, not homology and cohomology groups as it usually does nowadays, for the good reason that cohomology did not exist until about 1935.

Around 1930 de Rham gave a version of Poincaré duality for smooth orientable, compact manifolds. If M is a smooth, oriented, and compact n-manifold, then there are isomorphisms

$$H^p_{\mathrm{dR}}(M) \cong (H^{n-p}_{\mathrm{dR}}(M))^*,$$

where $(H^{n-p}(M))^*$ is the dual of the vector space $H^{n-p}(M)$. This duality is actually induced by a nondegenerate pairing

$$\langle -,-\rangle\colon H^p_{\mathrm{dR}}(M) \times H^{n-p}_{\mathrm{dR}}(M) \to \mathbb{R}$$

given by integration, namely

$$\langle [\omega], [\eta] \rangle = \int_M \omega \wedge \eta,$$

where ω is a differential p-form and η is a differential $(n-p)$-form. For details, see Chapter 3, Theorem 3.8. The proof uses several tools from the arsenal of homological algebra: the zig-zag lemma (in the form of Mayer–Vietoris sequences), the five lemma, and an induction on finite "good covers."

Around 1935, inspired by Pontrjagin's duality theorem and his introduction of the notion of nondegenerate pairing (see the end of this section), Alexander and Kolmogoroff independently started developing cohomology, and soon after this it was realized that because cohomology primarily deals with functions, it is possible to define various products. Among those, the *cup product* is particularly important because it induces a multiplication operation on what is called the *cohomology algebra* $H^*(X;R)$ of a space X, and the *cap product* yields a stronger version of Poincaré duality.

Recall that $S^*(X;R)$ is the R-module $\bigoplus_{p\geq 0} S^p(X;R)$, where the $S^p(X;R)$ are the singular cochain modules. For all $p, q \geq 0$, it possible to define a function

$$\smile \colon S^p(X;R) \times S^q(X;R) \to S^{p+q}(X;R),$$

called *cup product*. These functions induce a multiplication on $S^*(X;R)$ also called the cup product, which is bilinear, associative, and has an identity element. The cup product satisfies the following equation

$$\delta(c \smile d) = (\delta c) \smile d + (-1)^p c \smile (\delta d),$$

reminiscent of a property of the wedge product. (In the above equation δ is the coboundary map, i.e. $\delta^p \colon S^p(X;R) \to S^{p+1}(X;R)$.) This equation can be used to show that the cup product is a well defined on cohomology classes:

$$\smile \colon H^p(X;R) \times H^q(X;R) \to H^{p+q}(X;R).$$

These operations induce a multiplication operation on $H^*(X;R) = \bigoplus_{p\geq 0} H^p(X;R)$ which is bilinear and associative. Together with the cup product, $H^*(X;R)$ is called the *cohomology ring* of X. For details, see Section 4.10.

The cup product for simplicial cohomology was invented independently by Alexander and Kolmogoroff (in addition to simplicial cohomology) and presented at a conference held in Moscow in 1935. Alexander's original definition was not quite correct and he modified his definition following a suggestion of Čech (1936). This modified version was discovered independently by Whitney (1938), who introduced the notation $\smile$. Eilenberg extended the definition of the cup product to singular cohomology (1944).

The significance of the cohomology ring is that two spaces X and Y may have isomorphic cohomology modules but nonisomorphic cohomology rings. Therefore the cohomology ring is an invariant of a space X that is finer than its cohomology.

Another product related to the cup product is the cap product. The *cap product* combines cohomology and homology classes, it is an operation

$$\frown \colon H^p(X;R) \times H_n(X;R) \to H_{n-p}(X;R);$$

see Section 7.2.

The cap product was introduced by Čech (1936) and independently by Whitney (1938), who introduced the notation $\frown$ and the name *cap product*. Again Eilenberg generalized the cap product to singular homology and cohomology.

The cup product and the cap product are related by the following equation:

$$a(b \frown \sigma) = (a \smile b)(\sigma)$$

for all $a \in S^{n-p}(X;R)$, all $b \in S^p(X;R)$, and all $\sigma \in S_n(X;R)$, or equivalently using the bracket notation for evaluation as

$$\langle a, b \frown \sigma \rangle = \langle a \smile b, \sigma \rangle,$$

which shows that $\frown$ is the adjoint of $\smile$ with respect to the evaluation pairing $\langle -, - \rangle$.

The reason why the cap product is important is that it can be used to state a sharper version of Poincaré duality. But first we need to talk about orientability.

If M is a topological manifold of dimension n, it turns out that for every $x \in M$ the relative (singular) homology groups $H_p(M, M - \{x\}; \mathbb{Z})$ are either (0) if $p \neq n$, or equal to $\mathbb{Z}$ if $p = n$. An *orientation* of M is a choice of a generator $\mu_x \in H_n(M, M - \{x\}; \mathbb{Z}) \cong \mathbb{Z}$ for each $x \in M$ which varies "‘continuously" with x. A manifold that has an orientation is called *orientable*.

Technically, this means that for every $x \in M$, locally on some small open subset U of M containing x there is some homology class $\mu_U \in H_n(M, M - U; \mathbb{Z})$ such that all the chosen $\mu_x \in H_n(M, M - \{x\}; \mathbb{Z})$ for all $x \in U$ are obtained as images of μ_U; see Figure 1.5.

If such a μ_U can be found when $U = M$, we call it a *fundamental class* of M and denote it by μ_M; see Section 7.3. Readers familiar with differential geometry may think of the fundamental class as a discrete analog to the notion of volume form. The crucial result is that a compact manifold of dimension n is orientable iff it has a unique fundamental class μ_M; see Theorem 7.7.

The notion of orientability can be generalized to the notion of R-orientability. One of the advantages of this notion is that every manifold is $\mathbb{Z}/2\mathbb{Z}$ orientable. We are now in a position to state the Poincaré duality theorem in terms of the cap product.

If M is compact and orientable, then there is a fundamental class μ_M. In this case (if $0 \leq p \leq n$) we have a map

$$D_M \colon H^p(M;\mathbb{Z}) \to H_{n-p}(M;\mathbb{Z})$$

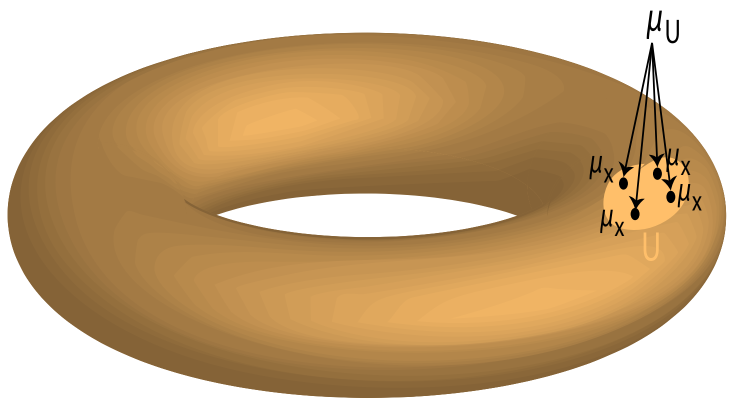


Figure 1.5: A schematic representation which shows μ_x as the image of μ_U.

given by

$$D_M(\omega) = \omega \frown \mu_M.$$

Poincaré duality asserts that the map

$$D_M\colon \omega \mapsto \omega \frown \mu_M$$

is an isomorphism between $H^p(M;\mathbb{Z})$ and $H_{n-p}(M;\mathbb{Z})$; see Theorem 7.16.

Poincaré duality can be generalized to compact R-orientable manifolds for any commutative ring R, and to coefficients in any R-module G. It can also be generalized to noncompact manifolds if we replace cohomology by cohomology with compact support (the modules $H_c^p(X;R)$); see Sections 7.3, 7.4, and 7.5. If $R = \mathbb{Z}/2\mathbb{Z}$, Poincaré duality holds for all manifolds, orientable or not.

Another kind of duality was introduced by Alexander in 1922. Alexander considered a compact proper subset A of the sphere S^n ($n \geq 2$) which is a curvilinear cell complex (A has some type of generalized triangulation). For the first time he defined the homology groups of the open subset $S^n - A$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$ (so that he did not have to bother with signs), and he proved that for $p \leq n-2$ there are isomorphisms

$$H_p(A;\mathbb{Z}/2\mathbb{Z}) \cong H_{n-p-1}(S^n - A;\mathbb{Z}/2\mathbb{Z});$$

see Figure 1.6. Since cohomology did not exist yet, the original version of Alexander duality was stated for homology.

Around 1928 Lefschetz started investigating homology with coefficients in $\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}$, or $\mathbb{Q}$, and defined relative homology. In his book published in 1930, using completely different methods from Alexander, Lefschetz proved a version of Alexander's duality in the case where

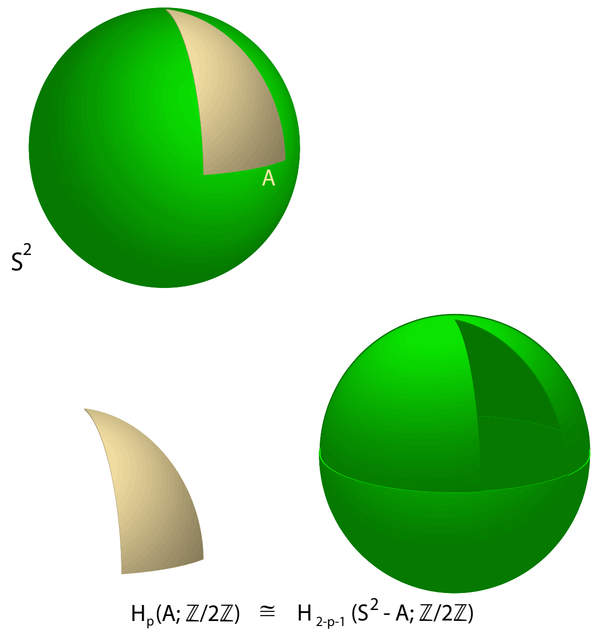


Figure 1.6: Let A be the peach spherical triangle in S^2. The original version of Alexander duality compares the homology of the peach spherical triangle with the homology of the surface consisting of $S^2 - A$.

A is a subcomplex of S^n. Soon after he obtained a homological version of what we call the Lefschetz duality theorem in Section 14.5 (Theorem 14.9):

$$H^p(M, L; \mathbb{Z}) \cong H_{n-p}(M - L; \mathbb{Z}),$$

where M and L are complexes and L is a subcomplex of M; see Figure 1.7.

Both Alexander and Lefschetz duality can be generalized to the situation where in Alexander duality A is an arbitrary closed subset of S^n, and in Lefschetz duality L is any compact subset of M and M is orientable, but new kinds of cohomology need to be introduced: *Čech cohomology and Alexander–Spanier cohomology*, which turn out to be equivalent. This is a nontrivial theorem due to Dowker [13]. Then a duality theorem generalizing both Poincaré duality and Alexander–Lefschetz duality can be proven. These matters are discussed in Chapter 9, Section 13.8, and Chapter 14.

Proving the general version of Alexander–Lefschetz duality takes a significant amount of work because it requires defining relative versions of Čech cohomology and Alexander–Spanier cohomology, and to prove their equivalence as well as their equivalence to another definition in terms of direct limits of singular cohomology groups (see Definition 14.13 and Proposition 14.7).

When discussing the notion of duality, we would be remiss if we did not mention the

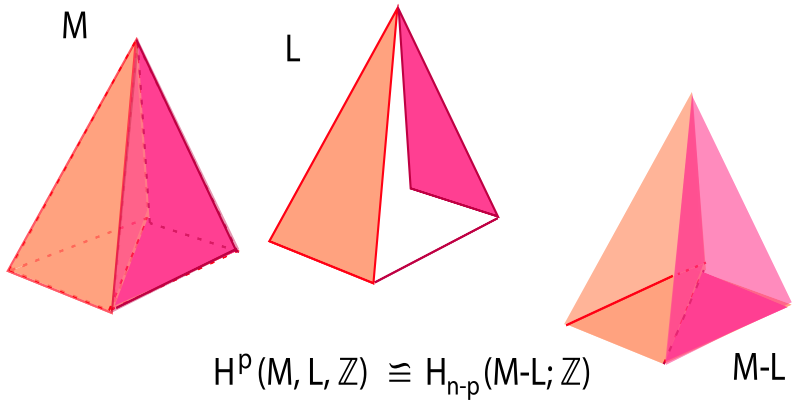


Figure 1.7: A representation of Lefschetz duality when M is the simplicial complex consisting of two solid tetrahedra while L is the subcomplex consisting of the front left peach face, the back right pink face, and the solid red edge.

important contributions of Pontrjagin. In a paper published in 1931 Pontrjagin investigates the duality between a closed subset A of $\mathbb{R}^n$ homeomorphic to a simplicial complex and $\mathbb{R}^n - A$. Pontrjagin introduces for the first time the notion of a *nondegenerate pairing* $\varphi\colon U \times V \to G$ between two finitely abelian groups U and V, where G is another abelian group (he uses $G = \mathbb{Z}$ or $G = \mathbb{Z}/m\mathbb{Z}$). This is a bilinear map $\varphi\colon U \times V \to G$ such that if $\varphi(u, v) = 0$ for all $v \in V$, then $u = 0$, and if $\varphi(u, v) = 0$ for all $u \in U$, then $v = 0$. Pontrjagin proves that U and V are isomorphic for his choice of G, and applies the notion of nondegenerate pairing to Poincaré duality and to a version of Alexander duality for certain subsets of $\mathbb{R}^n$. Pontrjagin also introduces the important notion of *direct limit* (see Section 8.3, Definition 8.10) which, among other things, plays a crucial role in the definition of Čech cohomology and in the construction of a sheaf from a presheaf (see Chapter 10).

In another paper published in 1934, Pontrjagin states and proves his famous duality theory between discrete and compact abelian topological groups. In this situation, U is a discrete group, $G = \mathbb{R}/\mathbb{Z} \cong S^1$, and $V = \widehat{U} = \mathrm{Hom}(U, \mathbb{R}/\mathbb{Z})$ (with the topology of simple convergence). Pontrjagin applies his duality theorem to a version of Alexander duality for compact subsets of $\mathbb{R}^n$ and for a version of Čech homology (cohomology had not been defined yet).

One notion that we still need to address, especially since it has appeared numerous times in our aforementioned discussions, is Čech cohomology. We will do so in the next section. It turns out that Čech cohomology accommodates very general types of coefficients, namely *presheaves and sheaves*. In Chapters 8, 9 and 10 we introduce these notions that play a major role in many area of mathematics, especially algebraic geometry and algebraic topology.

One can say that from a historical point of view, all the notions we presented so far are discussed in the landmark book by Eilenberg and Steenrod [15] (1952). This is a beautiful book well worth reading, but it is not for the beginner. The next landmark book is Spanier's [59] (1966). It is easier to read than Eilenberg and Steenrod but still quite demanding.

The next era of algebraic topology begins with the introduction of the notion of sheaf by Jean Leray around 1946.

1.4 Presheaves, Sheaves, and Čech Cohomology

The machinery of sheaves is applicable to problems designated by the vague notion of "passage from local to global properties." When some mathematical object attached to a topological space X can be "restricted" to any open subset U of X, and that restriction is known for sufficiently small U, what can be said about that "global" object? For example, consider the continuous functions defined over $\mathbb{R}^2$ and their restrictions to open subsets of $\mathbb{R}^2$.

Problems of this type had arisen since the 1880s in complex analysis in several variables and had been studied by Poincaré, Cousin, and later H. Cartan and Oka. Beginning in 1942, Leray considered a similar problem in cohomology. Given a space X, when the cohomology $H^*(U;G) = \bigoplus_{p\geq 0} H^p(U;G)$ is known for sufficiently small U, what can be said about $H^*(X;G) = \bigoplus_{p\geq 0} H^p(X;G)$?

Leray devised some machinery in 1946 that was refined and generalized by H. Cartan, M. Lazard, A. Borel, Koszul, Serre, Godement, and others to yield the notions of presheaves and sheaves.

Given a topological space X and a class $\mathbf{C}$ of structures (a category), say sets, vector spaces, R-modules, groups, commutative rings, *etc.*, a *presheaf on* X *with values in* $\mathbf{C}$ consists of an assignment of some object $\mathcal{F}(U)$ in $\mathbf{C}$ to every open subset U of X and of a map $\mathcal{F}(i)\colon \mathcal{F}(U) \to \mathcal{F}(V)$ of the class of structures in $\mathbf{C}$ to every inclusion $i\colon V \to U$ of open subsets $V \subseteq U \subseteq X$, such that

$$\begin{aligned}\mathcal{F}(i \circ j) &= \mathcal{F}(j) \circ \mathcal{F}(i)\\ \mathcal{F}(\mathrm{id}_U) &= \mathrm{id}_{\mathcal{F}(U)},\end{aligned}$$

for any two inclusions $i\colon V \to U$ and $j\colon W \to V$, with $W \subseteq V \subseteq U$; see Figure 1.8.

Note that the order of composition is switched in $\mathcal{F}(i \circ j) = \mathcal{F}(j) \circ \mathcal{F}(i)$.

Intuitively, the map $\mathcal{F}(i)\colon \mathcal{F}(U) \to \mathcal{F}(V)$ is a *restriction map* if we think of $\mathcal{F}(U)$ and $\mathcal{F}(V)$ as sets of functions (which is often the case). For this reason, the map $\mathcal{F}(i)\colon \mathcal{F}(U) \to \mathcal{F}(V)$ is also denoted by $\rho^U_V\colon \mathcal{F}(U) \to \mathcal{F}(V)$, and the first equation of the above definition is expressed by

$$\rho^U_W = \rho^V_W \circ \rho^U_V.$$

Presheaves, as defined above and in Section 8.1, are typically used to keep track of local information assigned to a global object (the space X). It is usually desirable to use consistent

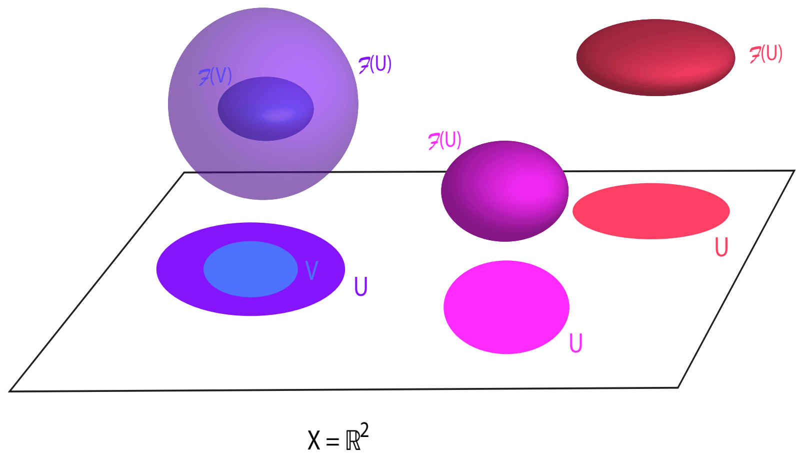


Figure 1.8: A schematic representation of the presheaf of continuous real valued function on $X = \mathbb{R}^2$. An open set U is a circle in the plane while $\mathcal{F}(U)$ is the "balloon" of functions floating above U.

local information to recover some global information, but this requires a sharper notion, that of a sheaf.

As stated at the beginning of Section 8.2, the motivation for the extra condition that a sheaf should satisfy is this. Suppose we consider the presheaf of continuous functions on a topological space X. If U is any open subset of X and if $(U_i)_{i\in I}$ is an open cover of U, for any family $(f_i)_{i\in I}$ of continuous functions $f_i\colon U_i \to \mathbb{R}$, if f_i and f_j agree on every overlap $U_i \cap U_j$, then the f_i patch to a unique continuous function $f\colon U \to \mathbb{R}$ whose restriction to U_i is f_i.

Given a topological space X and a class $\mathbf{C}$ of structures (a category), say sets, vector spaces, R-modules, groups, commutative rings, *etc.*, a *sheaf on* X *with values in* $\mathbf{C}$ is a presheaf $\mathcal{F}$ on X such that for any open subset U of X, for every open cover $(U_i)_{i\in I}$ of U (that is, $U = \bigcup_{i\in I} U_i$ for some open subsets $U_i \subseteq U$ of X), the following conditions hold:

(G) (Gluing condition) For every family $(f_i)_{i\in I}$ with $f_i \in \mathcal{F}(U_i)$, if the f_i are consistent, which means that
$$\rho^{U_i}_{U_i\cap U_j}(f_i) = \rho^{U_j}_{U_i\cap U_j}(f_j) \quad \text{for all } i, j \in I,$$
then there is some $f \in \mathcal{F}(U)$ such that $\rho^U_{U_i}(f) = f_i$ for all $i \in I$; see Figure 1.9.

(M) (Monopresheaf condition) For any two elements $f, g \in \mathcal{F}(U)$, if f and g agree on all the U_i, which means that
$$\rho^U_{U_i}(f) = \rho^U_{U_i}(g) \quad \text{for all } i \in I,$$
then $f = g$.

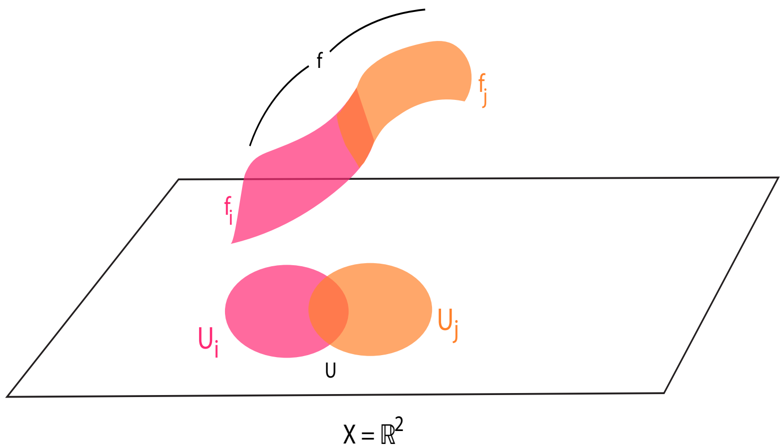


Figure 1.9: Let $\mathcal{F}$ be the sheaf of continuous real valued functions on $X = \mathbb{R}^2$. Let $U = U_1 \cup U_2$. The graph of the pink function f_1 and the peach function f_2 glue together over $U_1 \cap U_2$ to form the continuous function $f : U \to \mathbb{R}^2$.

Many (but not all) objects defined on a manifold are sheaves: the smooth functions $C^\infty(U)$, the smooth differential p-forms $\mathcal{A}^p(U)$, the smooth vector fields $\mathfrak{X}(U)$, where U is any open subset of M.

Given any commutative ring R and a fixed R-module G, the *constant presheaf* G_X is defined such that $G_X(U) = G$ for all nonempty open subsets U of X, and $G_X(\emptyset) = (0)$. The *constant sheaf* $\widetilde{G}_X$ is the sheaf given by $\widetilde{G}_X(U) =$ the set of locally constant functions on U (the functions $f : U \to G$ such that for every $x \in U$ there is some open subset V of U containing x such that f is constant on V), and $\widetilde{G}_X(\emptyset) = (0)$; see Figure 1.10.

In general a presheaf is not a sheaf. For example, the constant presheaf is not a sheaf. However, there is a procedure for converting a presheaf to a sheaf. We will return to this process in Section 1.5.

Čech cohomology with values in a presheaf of R-modules involves open covers of the topological space X; see Chapter 9.

Apparently, Čech himself did not introduce Čech cohomology, but he did introduce Čech homology using the notion of open cover (1932). Dowker, Eilenberg, and Steenrod introduced Čech cohomology in the early 1950s.

Given a topological space X, a family $\mathcal{U} = (U_j)_{j\in J}$ is an *open cover* of X if the U_j are open subsets of X and if $X = \bigcup_{j\in J} U_j$. Given any finite sequence $I = (i_0, \ldots, i_p)$ of elements of some index set J (where $p \geq 0$ and the i_j are not necessarily distinct), we let

$$U_I = U_{i_0\cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}.$$

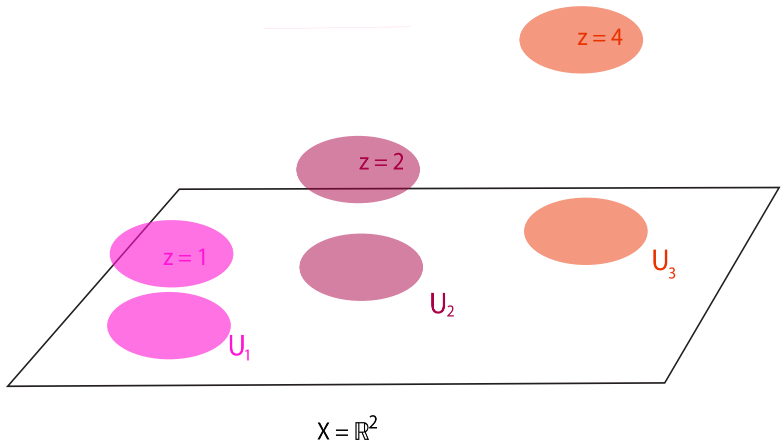


Figure 1.10: Let $\mathcal{F}$ be the sheaf of continuous real valued functions over $X = \mathbb{R}^2$, and let $U = U_1 \cup U_2 \cup U_3$, a disjoint union. The function f is locally constant over U since it takes a constant value over each U_i, where $1 \leq i \leq 3$.

Note that it may happen that $U_I = \emptyset$. We denote by $U_{i_0 \cdots \widehat{i_j} \cdots i_p}$ the intersection

$$U_{i_0 \cdots \widehat{i_j} \cdots i_p} = U_{i_0} \cap \cdots \cap \widehat{U_{i_j}} \cap \cdots \cap U_{i_p}$$

of the p subsets obtained by omitting U_{i_j} from $U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$ (the intersection of the $p+1$ subsets); see Figure 1.11.

Now given a presheaf $\mathcal{F}$ of R-modules, the R-module of *Čech p-cochains* $C^p(\mathcal{U}, \mathcal{F})$ is the set of all functions f with domain J^{p+1} such that $f(i_0, \ldots, i_p) \in \mathcal{F}(U_{i_0 \cdots i_p})$; in other words,

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{(i_0, \ldots, i_p) \in J^{p+1}} \mathcal{F}(U_{i_0 \cdots i_p}),$$

the set of all J^{p+1}-indexed families $(f_{i_0, \ldots, i_p})_{(i_0, \ldots, i_p) \in J^{p+1}}$ with $f_{i_0, \ldots, i_p} \in \mathcal{F}(U_{i_0 \cdots i_p})$. Observe that the coefficients (the modules $\mathcal{F}(U_{i_0 \cdots i_p})$) can "vary" from open subset to open subset.

We have $p+1$ inclusion maps

$$\delta^p_j \colon U_{i_0 \cdots i_p} \longrightarrow U_{i_0 \cdots \widehat{i_j} \cdots i_p}, \quad 0 \leq j \leq p.$$

Each inclusion map $\delta^p_j \colon U_{i_0 \cdots i_p} \longrightarrow U_{i_0 \cdots \widehat{i_j} \cdots i_p}$ induces a map

$$\mathcal{F}(\delta^p_j) \colon \mathcal{F}(U_{i_0 \cdots \widehat{i_j} \cdots i_p}) \longrightarrow \mathcal{F}(U_{i_0 \cdots i_p})$$

which is none other that the restriction map $\rho^{U_{i_0 \cdots \widehat{i_j} \cdots i_p}}_{U_{i_0 \cdots i_p}}$ which, for the sake of notational simplicity, we also denote by $\rho^j_{i_0 \cdots i_p}$.

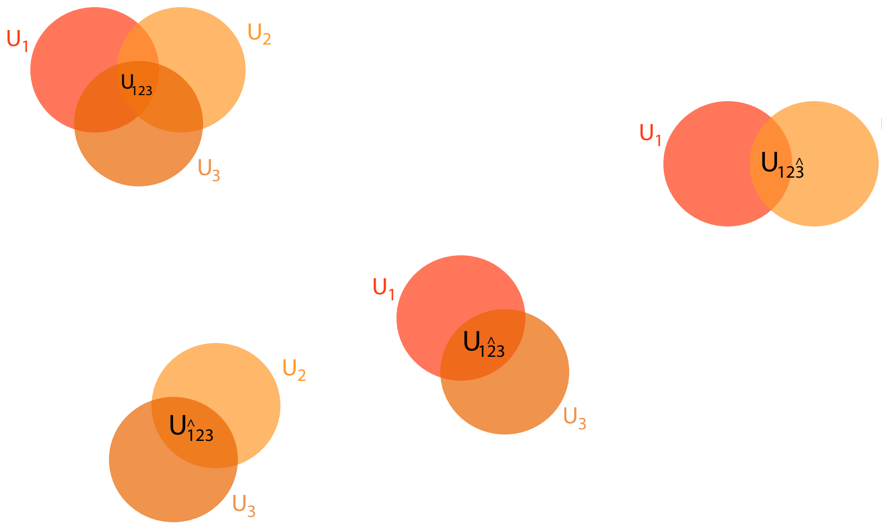


Figure 1.11: An illustration of the notation $U_{123} = U_1 \cap U_2 \cap U_3$ and the three cases of $U_{i_0 \cdots \widehat{i_j} \cdots i_p} = U_{i_0} \cap \cdots \cap \widehat{U_{i_j}} \cap \cdots \cap U_{i_p}$, where $i_0 = 1$ and $i_p = 3$.

Given a topological space X, an open cover $\mathcal{U} = (U_j)_{j \in J}$ of X, and a presheaf of R-modules $\mathcal{F}$ on X, the *coboundary maps* $\delta^p_{\mathcal{F}} \colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ are given by

$$\delta^p_{\mathcal{F}} = \sum_{j=0}^{p+1} (-1)^j \mathcal{F}(\delta^{p+1}_j), \quad p \geq 0.$$

More explicitly, for any p-cochain $f \in C^p(\mathcal{U}, \mathcal{F})$, for any sequence $(i_0, \ldots, i_{p+1}) \in J^{p+2}$, we have

$$(\delta^p_{\mathcal{F}} f)_{i_0, \ldots, i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \rho^j_{i_0 \cdots i_{p+1}} (f_{i_0, \ldots, \widehat{i_j}, \ldots, i_{p+1}}).$$

Unravelling the above definition for $p = 0$ we have

$$(\delta^0_{\mathcal{F}} f)_{i,j} = \rho^0_{ij}(f_j) - \rho^1_{ij}(f_i),$$

and for $p = 1$ we have

$$(\delta^1_{\mathcal{F}} f)_{i,j,k} = \rho^0_{ijk}(f_{j,k}) - \rho^1_{ijk}(f_{i,k}) + \rho^2_{ijk}(f_{i,j}).$$

It is easy to check that $\delta^{p+1}_{\mathcal{F}} \circ \delta^p_{\mathcal{F}} = 0$ for all $p \geq 0$, so we have a chain complex of cohomology

$$0 \xrightarrow{\delta^{-1}_{\mathcal{F}}} C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta^0_{\mathcal{F}}} C^1(\mathcal{U}, \mathcal{F}) \longrightarrow \cdots \xrightarrow{\delta^{p-1}_{\mathcal{F}}} C^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta^p_{\mathcal{F}}} C^{p+1}(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta^{p+1}_{\mathcal{F}}} \cdots$$

and we can define the Čech cohomology groups as follows.

Given a topological space X, an open cover $\mathcal{U} = (U_j)_{j \in J}$ of X, and a presheaf of R-modules $\mathcal{F}$ on X, the *Čech cohomology groups $\check{H}^p(\mathcal{U}, \mathcal{F})$ of the cover $\mathcal{U}$ with values in $\mathcal{F}$* are defined by

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = \mathrm{Ker}\, \delta^p_{\mathcal{F}} / \mathrm{Im}\, \delta^{p-1}_{\mathcal{F}}, \quad p \geq 0.$$

The *classical Čech cohomology groups $\check{H}^p(\mathcal{U}; G)$ of the cover $\mathcal{U}$ with coefficients in the R-module G* are the groups $\check{H}^p(\mathcal{U}, G_X)$, where G_X is the constant sheaf on X with values in G.

The next step is to define Čech cohomology groups that do not depend on the open cover $\mathcal{U}$. This is achieved by defining a notion of refinement on covers and by taking *direct limits* (see Section 8.3, Definition 8.10). Čech had used such a method in defining his Čech homology groups, by introducing the notion of *inverse limit* (which, curiously, was missed by Pontrjagin whose introduced direct limits!).

Without going into details, given two covers $\mathcal{U} = (U_i)_{i \in I}$ and $\mathcal{V} = (V_j)_{j \in J}$ of a space X, we say that *$\mathcal{V}$ is a refinement of $\mathcal{U}$*, denoted $\mathcal{U} \prec \mathcal{V}$, if there is a function $\tau \colon J \to I$ such that

$$V_j \subseteq U_{\tau(j)} \quad \text{for all } j \in J.$$

Under this notion of refinement, the open covers of X form a directed preorder, and the family $(\check{H}^p(\mathcal{U}, \mathcal{F}))_{\mathcal{U}}$ is what is called a direct mapping family so its direct limit

$$\varinjlim_{\mathcal{U}} \check{H}^p(\mathcal{U}, \mathcal{F})$$

makes sense. We define the *Čech cohomology groups $\check{H}^p(X, \mathcal{F})$ with values in $\mathcal{F}$* by

$$\check{H}^p(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^p(\mathcal{U}, \mathcal{F}).$$

The *classical Čech cohomology groups $\check{H}^p(X; G)$ with coefficients in the R-module G* are the groups $\check{H}^p(X, G_X)$ where G_X is the constant presheaf with value G. All this is presented in Chapter 9.

A natural question to ask is how does the classical Čech cohomology of a space compare with other types of cohomology, in particular singular cohomology. In general Čech cohomology can differ from singular cohomology, but for manifolds it agrees. Classical Čech cohomology also agrees with de Rham cohomology of the constant presheaf $\mathbb{R}_X$. These results are hard to prove; see Chapter 13.

1.5 Sheafification and Stalk Spaces

One of the major goals of this book is to introduce sheaf cohomology. This means we need to develop a deeper understanding of mappings between sheaves. A *map (or morphism)* $\varphi \colon \mathcal{F} \to$

$\mathcal{G}$ of presheaves (or sheaves) $\mathcal{F}$ and $\mathcal{G}$ on X consists of a family of maps $\varphi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ of the class of structures in $\mathbf{C}$, for any open subset U of X, such that

$$\varphi_V \circ (\rho_{\mathcal{F}})^U_V = (\rho_{\mathcal{G}})^U_V \circ \varphi_U$$

for every pair of open subsets U, V such that $V \subseteq U \subseteq X$. Equivalently, the following diagrams commute for every pair of open subsets U, V such that $V \subseteq U \subseteq X$

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ {\scriptstyle (\rho_{\mathcal{F}})^U_V}\downarrow & & \downarrow{\scriptstyle (\rho_{\mathcal{G}})^U_V} \\ \mathcal{F}(V) & \xrightarrow[\varphi_V]{} & \mathcal{G}(V). \end{array}$$

The notion of kernel $\operatorname{Ker}\varphi$ and image $\operatorname{Im}\varphi$ of a presheaf or sheaf map $\varphi \colon \mathcal{F} \to \mathcal{G}$ is easily defined. The presheaf $\operatorname{Ker}\varphi$ is defined by $(\operatorname{Ker}\varphi)(U) = \operatorname{Ker}\varphi_U$, and the presheaf $\operatorname{Im}\varphi$ is defined by $(\operatorname{Im}\varphi)(U) = \operatorname{Im}\varphi_U$. In the case of presheaves, they are also presheaves, but in the case of sheaves, the kernel $\operatorname{Ker}\varphi$ is indeed a sheaf, but the image $\operatorname{Im}\varphi$ is **not** a sheaf in general.

This failure of the image of a sheaf map to be a sheaf is a problem that causes significant technical complications. In particular, it is not clear what it means for a sheaf map to be surjective, and a "good" definition of the notion of an exact sequence of sheaves is also unclear.

Fortunately, there is a procedure for converting a presheaf $\mathcal{F}$ into a sheaf $\widetilde{\mathcal{F}}$ which is reasonably well-behaved. This procedure is called *sheafification*. There is a sheaf map $\eta \colon \mathcal{F} \to \widetilde{F}$ which is generally not injective.

The *sheafification* process is universal in the sense that given any presheaf $\mathcal{F}$ and any sheaf $\mathcal{G}$, for any presheaf map $\varphi \colon \mathcal{F} \to \mathcal{G}$, there is a unique sheaf map $\widehat{\varphi} \colon \widetilde{\mathcal{F}} \to \mathcal{G}$ such that

$$\varphi = \widehat{\varphi} \circ \eta_{\mathcal{F}}$$

as illustrated by the following commutative diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\eta_{\mathcal{F}}} & \widetilde{\mathcal{F}} \\ & {\scriptstyle \varphi}\searrow & \downarrow{\scriptstyle \widehat{\varphi}} \\ & & \mathcal{G}; \end{array}$$

see Theorem 10.12.

The *sheafification* process involves constructing a topological space $S\mathcal{F}$ from the presheaf $\mathcal{F}$ that we call the *stalk space* of $\mathcal{F}$; see Figure 1.12. Godement calls it the *espace étalé*. The stalk space is the disjoint union of sets (modules) $\mathcal{F}_x$ called *stalks*. Each stalk $\mathcal{F}_x$ is

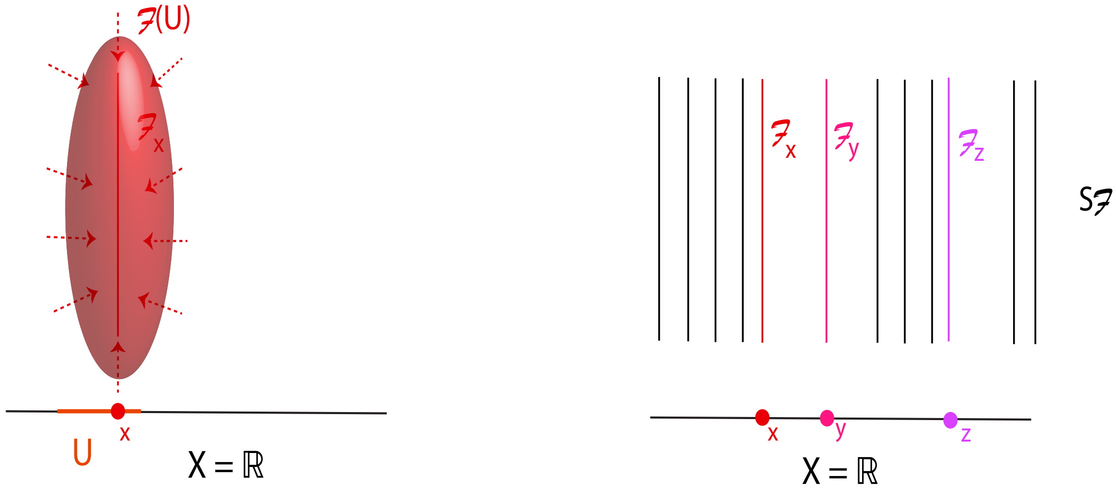


Figure 1.12: Let $X = \mathbb{R}$ and $\mathcal{F}$ be the sheaf of real valued continuous functions. An element $\mathcal{F}(U)$ is represented by the floating balloon. By "collapsing" the balloon (via the direct limiting process), we form the stalk $\mathcal{F}_x$, which is represented as a vertical line.

the direct limit $\varinjlim(\mathcal{F}(U))_{U \ni x}$ of the family of modules $\mathcal{F}(U)$ for all "small" open sets U containing x (see Definition 10.1).

There is a surjective map $p \colon S\mathcal{F} \to X$ which, under the topology given to $S\mathcal{F}$, is a local homeomorphism, which means that for every $y \in S\mathcal{F}$, there is some open subset V of $S\mathcal{F}$ containing y such that the restriction of p to V is a homeomorphism. The sheaf $\widetilde{\mathcal{F}}$ consists of the continuous sections of p, that is, the continuous functions $s \colon U \to S\mathcal{F}$ such that $p \circ s = \mathrm{id}_U$, for any open subset U of X. This construction is presented in detail in Sections 10.1, 10.2, and 10.4.

The construction of the pair $(S\mathcal{F}, p)$ from a presheaf $\mathcal{F}$ suggests another definition of a sheaf as a pair (E, p), where E is a topological space and $p \colon E \to X$ is a surjective local homeomorphism onto another space X. Such a pair (E, p) is often called a *sheaf space*, but we prefer to call it a *stalk space*. This is the definition that was given by H. Cartan and M. Lazard around 1950. The sheaf ΓE associated with the stalk space (E, p) is defined as follows: for any open subset U or X, the *sections* of ΓE are the continuous sections $s \colon U \to E$, that is, the continuous functions such that $p \circ s = \mathrm{id}$; see Figure 1.13. We can also define a notion of map between two stalk spaces. Stalk spaces are discussed in Section 10.3.

As this stage, given a topological space X we have three categories (classes of objects):

(1) The category $\mathbf{Psh}(X)$ of presheaves and their morphisms.

(2) The category $\mathbf{Sh}(X)$ of sheaves and their morphisms.

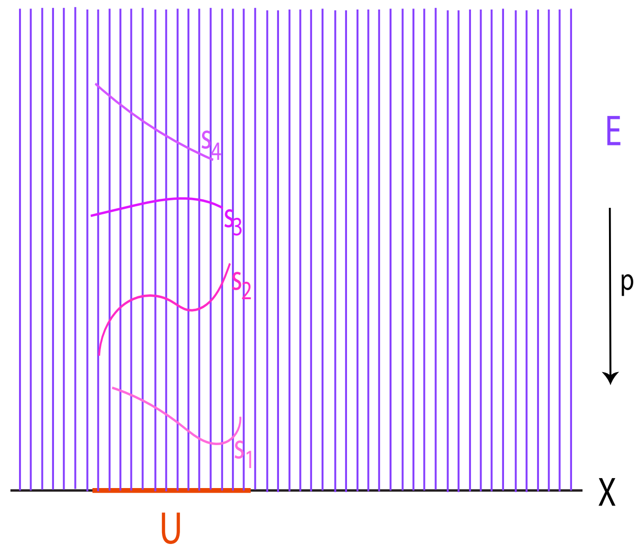


Figure 1.13: A schematic representation of a stalk space (E, p). We drew four sections over U, where each section is a colored curve such that $p \circ s = \mathrm{id}$.

(3) The category $\mathbf{StalkS}(X)$ of stalk spaces and their morphisms.

There is also a functor

$$S\colon \mathbf{PSh}(X) \to \mathbf{StalkS}(X)$$

from the category $\mathbf{PSh}(X)$ to the category $\mathbf{StalkS}(X)$ given by the construction of a stalk space $S\mathcal{F}$ from a presheaf $\mathcal{F}$, ($S(\mathcal{F}) = S\mathcal{F}$), and a functor

$$\Gamma\colon \mathbf{StalkS}(X) \to \mathbf{Sh}(X)$$

from the category $\mathbf{StalkS}(X)$ to the category $\mathbf{Sh}(X)$, given by the sheaf ΓE of continuous sections of E. Here we are using the term functor in an informal way. A more precise definition is given in Sections 1.7 and 10.10.

Note that every sheaf $\mathcal{F}$ is also a presheaf, and that every map $\varphi\colon \mathcal{F} \to \mathcal{G}$ of sheaves is also a map of presheaves. Therefore, we have an inclusion map

$$i\colon \mathbf{Sh}(X) \to \mathbf{PSh}(X),$$

which is a functor. As a consequence, S restricts to an operation (functor)

$$S\colon \mathbf{Sh}(X) \to \mathbf{StalkS}(X).$$

There is also a map η which maps a presheaf $\mathcal{F}$ to the sheaf $\Gamma S(\mathcal{F}) = \widetilde{\mathcal{F}}$. This map η is a natural isomorphism between the functors id (the identity functor) and ΓS from $\mathbf{Sh}(X)$

to itself. In other words, if we take $\mathcal{F}$, form the stalk space $S\mathcal{F}$, then turn this stalk space into the sheaf of continuous sections $\Gamma S\mathcal{F}$, this new sheaf is *isomorphic* to $\mathcal{F}$.

We can also define a map ϵ which takes a stalk space (E, p) and makes the stalk space $S\Gamma E$. The map ϵ is a natural isomorphism between the functors id (the identity functor) and $S\Gamma$ from $\mathbf{StalkS}(X)$ to itself. In other words, if we take the stalk space (E, p), form the sheaf of continuous sections ΓE, then form the stalk space of ΓE, namely $S\Gamma E$, this new stalk space is *isomorphic* to (E, p).

Then we see that the two operations (functors)

$$S\colon \mathbf{Sh}(X) \to \mathbf{StalkS}(X) \quad \text{and} \quad \Gamma\colon \mathbf{StalkS}(X) \to \mathbf{Sh}(X)$$

are almost mutual inverses, in the sense that there is a natural isomorphism η between ΓS and id and a natural isomorphism ϵ between $S\Gamma$ and id. In such a situation, we say that the classes (categories) $\mathbf{Sh}(X)$ and $\mathbf{StalkS}(X)$ are *equivalent*. The upshot is that it is basically a matter of taste (or convenience) whether we decide to work with sheaves or stalk spaces. In fact, for the aspects of sheaf cohomology that deal with soft and fine sheaves (Sections 13.5 and 13.6), it is best to use the stalk space construction of a sheaf.

We also have the operator (functor)

$$\Gamma S\colon \mathbf{PSh}(X) \to \mathbf{Sh}(X)$$

which "sheafifies" a presheaf $\mathcal{F}$ into the sheaf $\widetilde{\mathcal{F}}$. Theorem 10.12 can be restated as saying that there is an isomorphism

$$\mathrm{Hom}_{\mathbf{PSh}(X)}(\mathcal{F}, i(\mathcal{G})) \cong \mathrm{Hom}_{\mathbf{Sh}(X)}(\widetilde{\mathcal{F}}, \mathcal{G}),$$

between the set (category) of maps between the presheaves $\mathcal{F}$ and $i(\mathcal{G})$ and the set (category) of maps between the sheaves $\widetilde{\mathcal{F}}$ and $\mathcal{G}$. In fact, such an isomorphism is natural, so in categorical terms, i and $\widetilde{\ } = \Gamma S$ are *adjoint functors*.

All this is explained in Sections 10.3 and 10.4.

1.6 Cokernels and Images of Sheaf Maps

We still need to define the image of a sheaf map in such a way that the notion of exact sequence of sheaves makes sense. Recall that if $f\colon A \to B$ is a homomorphism of modules, the *cokernel* $\mathrm{Coker}\, f$ of f is defined by $B/\mathrm{Im}\, f$. It is a measure of the surjectivity of f. We also have the projection map $\mathrm{coker}(f)\colon B \to \mathrm{Coker}\, f$, and observe that

$$\mathrm{Im}\, f = \mathrm{Ker}\, \mathrm{coker}(f).$$

The above suggests defining notions of cokernels of presheaf maps and sheaf maps. For a presheaf map $\varphi\colon \mathcal{F} \to \mathcal{G}$ this is easy, and we can define the *presheaf cokernel* $\mathrm{PCoker}(\varphi)$. It comes with a presheaf map $\mathrm{pcoker}(\varphi)\colon \mathcal{G} \to \mathrm{PCoker}(\varphi)$.

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, we define the *sheaf cokernel* $\mathrm{SCoker}(\varphi)$ as the sheafification of $\mathrm{PCoker}(\varphi)$. It also comes with a presheaf map $\mathrm{scoker}(\varphi)\colon \mathcal{G} \to \mathrm{SCoker}(\varphi)$.

Then it can be shown that if $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a sheaf map, $\mathrm{SCoker}(\varphi) = (0)$ iff the stalk maps $\varphi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ are surjective for all $x \in X$; see Proposition 10.19.

It follows that the "correct" definition for the image $\mathrm{SIm}\,\varphi$ of a sheaf map $\varphi\colon \mathcal{F} \to \mathcal{G}$ is

$$\mathrm{SIm}\,\varphi = \mathrm{Ker}\,\mathrm{scoker}(\varphi).$$

With this definition, a sequence of sheaves

$$\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$$

is said to be exact if $\mathrm{SIm}\,\varphi = \mathrm{Ker}\,\psi$. Then it can be shown that

$$\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$$

is an exact sequence of sheaves iff the sequence

$$\mathcal{F}_x \xrightarrow{\varphi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x$$

is an exact sequence of R-modules (or rings) for all $x \in X$; see Proposition 10.24. This second characterization of exactness (for sheaves) is usually much more convenient than the first condition.

The definitions of cokernels and images of presheaves and sheaves as well as the notion of exact sequences of presheaves and sheaves are discussed in Sections 10.6, 10.7, 10.8, 10.9, and 10.10.

1.7 Injective and Projective Resolutions; Derived Functors

In order to define, even informally, the concept of derived functor, we need to describe what are functors and exact functors.

Suppose we have two types of structures (categories) **C** and **D** (for concreteness, think of **C** as the class of R-modules over some commutative ring R with an identity element 1 and of **D** as the class of abelian groups), and we have a transformation T (a functor) which works as follows:

(i) Each object A of **C** is mapped to some object $T(A)$ of **D**.

(ii) Each map $A \xrightarrow{f} B$ between two objects A and B in **C** (of example, an R-linear map) is mapped to some map $T(A) \xrightarrow{T(f)} T(B)$ between the objects $T(A)$ and $T(B)$ in **D** (for example, a homomorphism of abelian groups) in such a way that the following properties hold:

(a) Given any two maps $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ between objects A, B, C in $\mathbf{C}$ such that the composition $A \xrightarrow{g\circ f} C = A \xrightarrow{f} B \xrightarrow{g} C$ makes sense, the composition $T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C)$ makes sense in $\mathbf{D}$, and

$$T(g \circ f) = T(g) \circ T(f).$$

(b) If $A \xrightarrow{\mathrm{id}_A} A$ is the identity map of the object A in $\mathbf{C}$, then $T(A) \xrightarrow{T(\mathrm{id}_A)} T(A)$ is the identity map of $T(A)$ in $\mathbf{D}$; that is,

$$T(\mathrm{id}_A) = \mathrm{id}_{T(A)}.$$

Whenever a transformation $T\colon \mathbf{C} \to \mathbf{D}$ satisfies the Properties (i), (ii), (a), (b), we call it a *(covariant) functor* from $\mathbf{C}$ to $\mathbf{D}$.

If $T\colon \mathbf{C} \to \mathbf{D}$ satisfies Properties (i), (b), and if Properties (ii) and (a) are replaced by the Properties (ii') and (a') below

(ii') Each map $A \xrightarrow{f} B$ between two objects A and B in $\mathbf{C}$ is mapped to some map $T(B) \xrightarrow{T(f)} T(A)$ between the objects $T(B)$ and $T(A)$ in $\mathbf{D}$ in such a way that the following properties hold:

(a') Given any two maps $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ between objects A, B, C in $\mathbf{C}$ such that the composition $A \xrightarrow{g\circ f} C = A \xrightarrow{f} B \xrightarrow{g} C$ makes sense, the composition $T(C) \xrightarrow{T(g)} T(B) \xrightarrow{T(f)} T(A)$ makes sense in $\mathbf{D}$, and

$$T(g \circ f) = T(f) \circ T(g),$$

then T is called a *contravariant functor* from $\mathbf{C}$ to $\mathbf{D}$.

An example of a (covariant) functor is the functor $\mathrm{Hom}(A, -)$ (for a fixed R-module A) from R-modules to R-modules which maps a module B to the module $\mathrm{Hom}(A, B)$ and a module homomorphism $f\colon B \to C$ to the module homomorphism $\mathrm{Hom}(A, f)$ from $\mathrm{Hom}(A, B)$ to $\mathrm{Hom}(A, C)$ given by

$$\mathrm{Hom}(A, f)(\varphi) = f \circ \varphi \quad \text{for all } \varphi \in \mathrm{Hom}(A, B).$$

Another example is the functor T from R-modules to R-modules such that $T(A) = A \otimes_R M$ for any R-module A, and $T(f) = f \otimes_R \mathrm{id}_M$ for any R-linear map $f\colon A \to B$.

An example of a contravariant functor is the functor $\mathrm{Hom}(-, A)$ (for a fixed R-module A) from R-modules to R-modules which maps a module B to the module $\mathrm{Hom}(B, A)$ and a module homomorphism $f\colon B \to C$ to the module homomorphism $\mathrm{Hom}(f, A)$ from $\mathrm{Hom}(C, A)$ to $\mathrm{Hom}(B, A)$ given by

$$\mathrm{Hom}(f, A)(\varphi) = \varphi \circ f \quad \text{for all } \varphi \in \mathrm{Hom}(C, A).$$

Categories and functors were introduced by Eilenberg and Mac Lane, first in a paper published in 1942, and then in a more complete paper published in 1945.

Given a type of structures (category) $\mathbf{C}$, let us denote the set of all maps from an object A to an object B by $\mathrm{Hom}_{\mathbf{C}}(A, B)$. For all the types of structures $\mathbf{C}$ that we will dealing with, each set $\mathrm{Hom}_{\mathbf{C}}(A, B)$ has some additional structure; namely it is an abelian group.

Intuitively speaking an *abelian category* is a category in which the notion of kernel and cokernel of a map makes sense. Then we can define the notion of image of a map f as the kernel of the cokernel of f, so the notion of exact sequence makes sense, as we did in Section 1.6. The categories of R-modules and the categories of sheaves (or presheaves) are abelian categories. For more details, see Sections 10.10 and 10.11.

A sequence of R-modules and R-linear maps (more generally objects and maps between objects in an abelian category)

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{$*$}$$

is a *short exact sequence* if

(1) f is injective.

(2) $\mathrm{Im}\ f = \mathrm{Ker}\, g$.

(3) g is surjective.

According to Dieudonné [11], the notion of exact sequence first appeared in a paper of Hurewicz (1941), and then in a paper of Eilenberg and Steenrod and a paper of H. Cartan, both published in 1945. In 1947, Kelly and Pitcher generalized the notion of exact sequence to chain complexes, and apparently introduced the terminology *exact sequence*. In their 1952 treatise [15], Eilenberg and Steenrod took the final step of allowing a chain complex to be indexed by $\mathbb{Z}$ (as we do in Section 2.5).

Given two types of structures (categories) $\mathbf{C}$ and $\mathbf{D}$ in each of which the concept of exactness is defined (abelian categories), given an additive functor $T\colon \mathbf{C} \to \mathbf{D}$, by applying T to the short exact sequence $(*)$ we obtain the sequence

$$0 \longrightarrow T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C) \longrightarrow 0, \tag{$**$}$$

which is a chain complex (since $T(g) \circ T(f) = 0$). Then the following question arises:

Is the sequence $(**)$ also exact?

In general, the answer is **no**, but weaker forms of preservation of exactness suggest themselves.

A functor $T\colon \mathbf{C} \to \mathbf{D}$, is said to be *exact* if whenever the sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact in **C**, then the sequence

$$0 \longrightarrow T(A) \longrightarrow T(B) \longrightarrow T(C) \longrightarrow 0$$

is exact in **D**; *left exact* if whenever the sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C$$

is exact in **C**, then the sequence

$$0 \longrightarrow T(A) \longrightarrow T(B) \longrightarrow T(C)$$

is exact; and *right exact* if whenever the sequence

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact in **C**, then the sequence

$$T(A) \longrightarrow T(B) \longrightarrow T(C) \longrightarrow 0$$

is exact.

If $T\colon \mathbf{C} \to \mathbf{D}$ is a contravariant functor, then T is said to be *exact* if whenever the sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact in **C**, then the sequence

$$0 \longrightarrow T(C) \longrightarrow T(B) \longrightarrow T(A) \longrightarrow 0$$

is exact in **D**; *left exact* if whenever the sequence

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact in **C**, then the sequence

$$0 \longrightarrow T(C) \longrightarrow T(B) \longrightarrow T(A)$$

is exact; and *right exact* if whenever the sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C$$

is exact in **C**, then the sequence

$$T(C) \longrightarrow T(B) \longrightarrow T(A) \longrightarrow 0$$

is exact.

For example, the functor $\mathrm{Hom}(-, A)$ is (contravariant) *left-exact* but not exact in general (see Section 2.1). Similarly, the functor $\mathrm{Hom}(A, -)$ is *left-exact* but not exact in general (see Section 2.4).

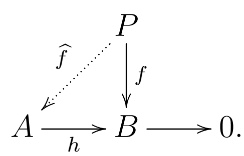
Modules for which the functor $\mathrm{Hom}(A, -)$ is exact play an important role. They are called *projective modules*. Similarly, modules for which the functor $\mathrm{Hom}(-, A)$ is exact are called *injective modules*.

The functor $- \otimes_R M$ is *right-exact* but not exact in general (see Section 2.4). Modules M for which the functor $- \otimes_R M$ is exact are called *flat*.

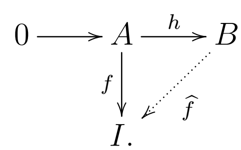
A good deal of homological algebra has to do with understanding how much a module fails to be projective or injective (or flat).

Injective and projective modules are also characterized by extension properties. As we will see later, these extension characterizations can be used to define injective and projective objects in an abelian category.

(1) A module P is projective iff for any surjective linear map $h\colon A \to B$ and any linear map $f\colon P \to B$, there is some linear map $\widehat{f}\colon P \to A$ lifting $f\colon P \to B$ in the sense that $f = h \circ \widehat{f}$, as in the following commutative diagram:



(2) A module I is injective iff for any injective linear map $h\colon A \to B$ and any linear map $f\colon A \to I$, there is some linear map $\widehat{f}\colon B \to I$ extending $f\colon A \to I$ in the sense that $f = \widehat{f} \circ h$, as in the following commutative diagram:



See Section 11.1.

Injective modules were introduced by Baer in 1940 and projective modules by Cartan and Eilenberg in the early 1950s. Every free module is projective. Injective modules are more elusive. If the ring R is a PID, an R-module M is injective iff it is divisible (which means that for every nonzero $\lambda \in R$, the map given by $u \mapsto \lambda u$ for $u \in M$ is surjective).

One of the most useful properties of projective modules is that every module M is the image of some projective (even free) module P, which means that there is a surjective homomorphism $\rho\colon P \to M$. Similarly, every module M can be embedded in an injective module I, which means that there is an injective homomorphism $i\colon M \to I$. This second fact is harder to prove (see Baer's embedding theorem, Theorem 11.6).

The above properties can be used to construct inductively projective and injective resolutions of a module M, a process that turns out to be remarkably useful. Intuitively, projective resolutions measure how much a module deviates from being projective, and injective resolutions measure how much a module deviates from being injective.

Hopf introduced free resolutions in 1945. A few years later Cartan and Eilenberg defined projective and injective resolutions.

Given any R-module A, a *projective resolution* of A is any exact sequence

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{p_0} A \longrightarrow 0 \qquad (*_1)$$

in which every P_n is a projective module. The exact sequence

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0$$

obtained by truncating the projective resolution of A after P_0 is denoted by $\mathbf{P}^A$, and the projective resolution $(*_1)$ is denoted by

$$\mathbf{P}^A \xrightarrow{p_0} A \longrightarrow 0.$$

Given any R-module A, an *injective resolution* of A is any exact sequence

$$0 \longrightarrow A \xrightarrow{i_0} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots \longrightarrow I^n \xrightarrow{d^n} I^{n+1} \longrightarrow \cdots \qquad (**_1)$$

in which every I^n is an injective module. The exact sequence

$$I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots \longrightarrow I^n \xrightarrow{d^n} I^{n+1} \longrightarrow \cdots$$

obtained by truncating the injective resolution of A before I^0 is denoted by $\mathbf{I}_A$, and the injective resolution $(**_1)$ is denoted by

$$0 \longrightarrow A \xrightarrow{i_0} \mathbf{I}_A.$$

Now suppose that we have a functor $T\colon \mathbf{C} \to \mathbf{D}$, where $\mathbf{C}$ is the category of R-modules and $\mathbf{D}$ is the category of abelian groups. If we apply T to $\mathbf{P}^A$ we obtain the chain complex

$$0 \longleftarrow T(P_0) \xleftarrow{T(d_1)} T(P_1) \xleftarrow{T(d_2)} \cdots \longleftarrow T(P_{n-1}) \xleftarrow{T(d_n)} T(P_n) \longleftarrow \cdots, \qquad \text{(Lp)}$$

denoted $T(\mathbf{P}^A)$. The above is no longer exact in general but it defines homology groups $H_p(T(\mathbf{P}^A))$.

Similarly If we apply T to $\mathbf{I}_A$ we obtain the cochain complex

$$0 \longrightarrow T(I^0) \xrightarrow{T(d^0)} T(I^1) \xrightarrow{T(d^1)} \cdots \longrightarrow T(I^n) \xrightarrow{T(d^n)} T(I^{n+1}) \longrightarrow \cdots, \tag{Ri}$$

denoted $T(\mathbf{I}_A)$. The above is no longer exact in general but it defines cohomology groups $H^p(T(\mathbf{I}_A))$.

The reason why projective resolutions are so special is that even though the homology groups $H_p(T(\mathbf{P}^A))$ appear to depend on the projective resolution $\mathbf{P}^A$, in fact they don't; *the groups* $H_p(T(\mathbf{P}^A))$ *only depend on* A *and* T. This is proven in Theorem 11.28.

Similarly, the reason why injective resolutions are so special is that even though the cohomology groups $H^p(T(\mathbf{I}_A))$ appear to depend on the injective resolution $\mathbf{I}_A$, in fact they don't; *the groups* $H^p(T(\mathbf{I}_A))$ *only depend on* A *and* T. This is proven in Theorem 11.27.

Proving the above facts takes some work; we make use of the *comparison theorems*; see Section 11.2, Theorem 11.17 and Theorem 11.21. In view of the above results, given a functor T as above, Cartan and Eilenberg were led to define the *left derived functors* L_nT of T by

$$L_nT(A) = H_n(T(\mathbf{P}^A)),$$

for any projective resolution $\mathbf{P}^A$ of A, and the *right derived functors* R^nT of T by

$$R^nT(A) = H^n(T(\mathbf{I}_A)),$$

for any injective resolution $\mathbf{I}_A$ of A. The functors L_nT and R^nT can also be defined on maps. If T is right-exact, then L_0T is isomorphic to T (as a functor), and if T is left-exact, then R^0T is isomorphic to T (as a functor).

For example, the left derived functors of the right-exact functor $T_B(A) = A \otimes B$ (with B fixed) are the "Tor" functors. We have $\mathrm{Tor}_0^R(A, B) \cong A \otimes B$, and the functor $\mathrm{Tor}_1^R(-, G)$ plays an important role in comparing the homology of a chain complex C and the homology of the complex $C \otimes_R G$; see Chapter 12. Čech introduced the functor $\mathrm{Tor}_1^R(-, G)$ in 1935 in terms of generators and relations. It is only after Whitney defined tensor products of arbitrary $\mathbb{Z}$-modules in 1938 that the definition of Tor was expressed in the intrinsic form that we are now familar with.

There are also versions of left and right derived functors for contravariant functors. For example, the right derived functors of the contravariant left-exact functor $T_B(A) = \mathrm{Hom}_R(A, B)$ (with B fixed) are the "Ext" functors. We have $\mathrm{Ext}_R^0(A, B) \cong \mathrm{Hom}_R(A, B)$, and the functor $\mathrm{Ext}_R^1(-, G)$ plays an important role in comparing the homology of a chain complex C and the cohomology of the complex $\mathrm{Hom}_R(C, G)$; see Chapter 12. The Ext functors were introduced in the context of algebraic topology by Eilenberg and Mac Lane (1942).

Everything we discussed so far is presented in Cartan and Eilenberg's groundbreaking book, Cartan–Eilenberg [10], published in 1956. It is in this book that the name *homological* *algebra* is introduced. MacLane [37] (1975) and Rotman [50, 52] give more "gentle" presentations (see also Weibel [63] and Eisenbud [16]). A more sophisticated presentation of homological algebra is found in Gelfand and Manin [23].

Derived functors can be defined for functors $T\colon \mathbf{C} \to \mathbf{D}$, where $\mathbf{C}$ or $\mathbf{D}$ is a more general category than the category of R-modules or the category of abelian groups. For example, in sheaf cohomology, the category $\mathbf{C}$ is the category of sheaves of rings. In general, it suffices that $\mathbf{C}$ and $\mathbf{D}$ are abelian categories.

We say that $\mathbf{C}$ *has enough projectives* if every object in $\mathbf{C}$ is the image of some projective object in $\mathbf{C}$, and that $\mathbf{C}$ *has enough injectives* if every object in $\mathbf{C}$ can be embedded (injectively) into some injective object in $\mathbf{C}$.

There are situations (for example, when dealing with sheaves) where it is useful to know that right derived functors can be computed by resolutions involving objects that are not necessarily injective, but T-acyclic, as defined below.

Given a left-exact functor $T\colon \mathbf{C} \to \mathbf{D}$, an object $J \in \mathbf{C}$ is T*-acyclic* if $R^nT(J) = (0)$ for all $n \geq 1$.

The following proposition shows that right derived functors can be computed using T-acyclic resolutions.

Proposition *Given an additive left-exact functor $T\colon \mathbf{C} \to \mathbf{D}$, for any $A \in \mathbf{C}$ suppose there is an exact sequence*

$$0 \longrightarrow A \xrightarrow{\epsilon} J^0 \xrightarrow{d^0} J^1 \xrightarrow{d^1} J^2 \xrightarrow{d^2} \cdots \qquad (\dagger)$$

in which every J^n is T-acyclic (a right T-acyclic resolution $\mathbf{J}^A$ formed by truncating $(\dagger)$ before J^0). Then for every $n \geq 0$ we have an isomorphism between $R^nT(A)$ and $H^n(T(\mathbf{J}_A))$.

The above proposition is used several times in Chapter 13.

1.8 Universal δ-Functors

The most important property of derived functors is that short exact sequences yield long exact sequences of homology or cohomology. This property was proven by Cartan and Eilenberg, but Grothendieck realized how crucial it was and this led him to the fundamental concept of *universal δ-functor*. Since we will be using right derived functors much more than left derived functors we state the existence of the long exact sequences of cohomology for right derived functors.

Theorem *Assume the abelian category $\mathbf{C}$ has enough injectives, let $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ be an exact sequence in $\mathbf{C}$, and let $T\colon \mathbf{C} \to \mathbf{D}$ be a left-exact (additive) functor.*

(1) Then for every $n \geq 0$, there is a map

$$(R^nT)(A'') \xrightarrow{\delta^n} (R^{n+1}T)(A'),$$

and the sequence

$$
\begin{array}{l}
0 \longrightarrow T(A') \longrightarrow T(A) \longrightarrow T(A'') \xrightarrow{\delta^0} \\
\longrightarrow (R^1T)(A') \longrightarrow \cdots \longrightarrow \cdots \longrightarrow \\
\longrightarrow (R^nT)(A') \longrightarrow (R^nT)(A) \longrightarrow (R^nT)(A'') \xrightarrow{\delta^n} \\
\longrightarrow (R^{n+1}T)(A') \longrightarrow \cdots \longrightarrow \cdots \longrightarrow \cdots
\end{array}
$$

is exact. This property is similar to the property of the zig-zag lemma from Section 1.2.

(2) If $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ *is another exact sequence in* $\mathbf{C}$*, and if there is a commutative diagram*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0,
\end{array}
$$

then the induced diagram beginning with

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T(A') & \longrightarrow & T(A) & \longrightarrow & T(A'') & \xrightarrow{\delta^0_A} & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T(B') & \longrightarrow & T(B) & \longrightarrow & T(B'') & \xrightarrow[\delta^0_B]{} &
\end{array}
$$

and continuing with

$$
\begin{array}{ccccccccccc}
\cdots & \longrightarrow & R^nT(A') & \longrightarrow & R^nT(A) & \longrightarrow & R^nT(A'') & \xrightarrow{\delta^n_A} & (R^{n+1}T)(A') & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & R^nT(B') & \longrightarrow & R^nT(B) & \longrightarrow & R^nT(B'') & \xrightarrow[\delta^n_B]{} & (R^{n+1}T)(B') & \longrightarrow & \cdots
\end{array}
$$

is also commutative.

The proof of this result (Theorem 11.31) is fairly involved and makes use of the horseshoe lemma (Theorem 11.25).

The previous theorem suggests the definition of families of functors originally proposed by Cartan and Eilenberg [10] and then investigated by Grothendieck in his legendary “Tohoku” paper [27] (1957).

A *δ-functors* consists of a countable family $T = (T^n)_{n\geq 0}$ of functors $T^n\colon \mathbf{C} \to \mathbf{D}$ that satisfy the two conditions of the previous theorem. There is a notion of map, also called morphism, between δ-functors.

Given two δ-functors $S = (S^n)_{n\geq 0}$ and $T = (T^n)_{n\geq 0}$, a *morphism* $\eta\colon S \to T$ between S and T is a family $\eta = (\eta^n)_{n\geq 0}$ of natural transformations $\eta^n\colon S^n \to T^n$ such that a certain diagram commutes; see Definition 11.21.

Grothendieck also introduced the key notion of universal δ-functor; see Grothendieck [27] (Chapter II, Section 2.2).

A δ-functor $T = (T^n)_{n\geq 0}$ is *universal* if for every δ-functor $S = (S^n)_{n\geq 0}$ and every natural transformation $\varphi\colon T^0 \to S^0$, there is a *unique* morphism $\eta\colon T \to S$ such that $\eta^0 = \varphi$; we say that η *lifts* φ.

The reason why universal δ-functors are important is the following kind of uniqueness property that shows that a universal δ-functor *is completely determined by the component* T^0; see Proposition 11.38.

Proposition *Suppose $S = (S^n)_{n\geq 0}$ and $T = (T^n)_{n\geq 0}$ are both universal δ-functors and there is an isomorphism $\varphi\colon S^0 \to T^0$ (a natural transformation φ which is an isomorphism). Then there is a unique isomorphism $\eta\colon S \to T$ lifting φ.*

One might wonder whether (universal) δ-functors exist. Indeed there are plenty of them; see Theorem 11.39.

Theorem *Assume the abelian category $\mathbf{C}$ has enough injectives. For every additive left-exact functor $T\colon \mathbf{C} \to \mathbf{D}$, the family $(R^nT)_{n\geq 0}$ of right derived functors of T is a δ-functor. Furthermore T is isomorphic to R^0T.*

In fact, the δ-functors $(R^nT)_{n\geq 0}$ are universal.

Grothendieck came up with an ingenious sufficient condition for a δ-functor to be universal: the notion of an *erasable* functor. Since Grothendieck's paper is written in French, this notion defined in Section 2.2 (Page 141) of [27] is called *effaçable*, and many books and paper use it. Since the English translation of "effaçable" is "erasable," as advocated by Lang, we will use the the English word.

A functor $T\colon \mathbf{C} \to \mathbf{D}$ is *erasable* (or *effaçable*) if for every object $A \in \mathbf{C}$ there is some object M_A and an injection $u\colon A \to M_A$ such that $T(u) = 0$. In particular this will be the case if $T(M_A)$ is the zero object of $\mathbf{D}$. If the category $\mathbf{C}$ has enough injectives, it can be shown that T is erasable iff $T(I) = (0)$ for all injectives I.

Our favorite functors, namely the right derived functors R^nT, are erasable by injectives for all $n \geq 1$. The following result due to Grothendieck is crucial:

Theorem *Assume the abelian category $\mathbf{C}$ has enough injectives. Let $T = (T^n)_{n\geq 0}$ be a δ-functor between two abelian categories $\mathbf{C}$ and $\mathbf{D}$. If $T^n(I) = (0)$ for every injective I, for all $n \geq 1$, then T is a universal δ-functor.*

Finally, by combining the previous results, we obtain the most important theorem about universal δ-functors:

Theorem *Assume the abelian category* $\mathbf{C}$ *has enough injectives. For every left-exact functor* $T\colon \mathbf{C} \to \mathbf{D}$*, the right derived functors* $(R^nT)_{n\geq 0}$ *form a universal* δ*-functor such that* T *is isomorphic to* R^0T*. Conversely, every universal* δ*-functor* $T = (T^n)_{n\geq 0}$ *is isomorphic to the right derived* δ*-functor* $(R^nT^0)_{n\geq 0}$*.*

After all, the mysterious universal δ-functors are just the right derived functors of left-exact functors. As an example, the functors $\mathrm{Ext}^n_R(A, -)$ constitute a universal δ-functor (for any fixed R-module A).

The machinery of universal δ*-functors can be used to prove that different kinds of cohomology theories yield isomorphic groups.* If two cohomology theories $(H^n_S(-))_{n\geq 0}$ and $(H^n_T(-))_{n\geq 0}$ defined for objects in a category $\mathbf{C}$ (say, topological spaces) are given by universal δ-functors S and T in the sense that the cohomology groups $H^n_S(A)$ and $H^n_T(A)$ are given by $H^n_S(A) = S^n(A)$ and $H^n_T(A) = T^n(A)$ for all objects $A \in \mathbf{C}$, and if $H^0_S(A)$ and $H^0_T(A)$ are isomorphic, then $H^n_S(A)$ and $H^n_T(A)$ are isomorphic for all $n \geq 0$. This technique will be used in Chapter 13 to prove that sheaf cohomology and Čech cohomology are isomorphic for paracompact spaces.

In Section 1.10 we will further see how the machinery of right derived functors can be used to define sheaf cohomology (where the category $\mathbf{C}$ is the category of sheaves of R-modules, the category $\mathbf{D}$ is the category of abelian groups, and T is the left exact "global section functor").

1.9 Universal Coefficient Theorems

Suppose we have a homology chain complex

$$0 \xleftarrow{d_0} C_0 \xleftarrow{d_1} C_1 \longleftarrow \cdots \xleftarrow{d_{p-1}} C_{p-1} \xleftarrow{d_p} C_p \xleftarrow{d_{p+1}} C_{p+1} \longleftarrow \cdots,$$

where the C_i are R-modules over some commutative ring R with a multiplicative identity element (recall that $d_i \circ d_{i+1} = 0$ for all $i \geq 0$). Given another R-module G we can form the homology complex

$$0 \xleftarrow{d_0\otimes \mathrm{id}} C_0 \otimes_R G \xleftarrow{d_1\otimes \mathrm{id}} C_1 \otimes_R G \longleftarrow \cdots \xleftarrow{d_p\otimes \mathrm{id}} C_p \otimes_R G \longleftarrow \cdots,$$

obtained by tensoring with G, denoted $C \otimes_R G$, and the cohomology complex

$$0 \xrightarrow{\mathrm{Hom}_R(d_0,G)} \mathrm{Hom}_R(C_0, G) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_R(C_p, G) \xrightarrow{\mathrm{Hom}_R(d_{p+1},G)} \mathrm{Hom}_R(C_{p+1}, G) \longrightarrow \cdots$$

obtained by applying $\mathrm{Hom}_R(-, G)$, and denoted $\mathrm{Hom}_R(C, G)$.

The question is: what is the relationship between the homology groups $H_p(C \otimes_R G)$ and the original homology groups $H_p(C)$ in the first case, and what is the relationship between the cohomology groups $H^p(\mathrm{Hom}_R(C, G))$ and the original homology groups $H_p(C)$ in the second case?

The ideal situation would be that

$$H_p(C \otimes_R G) \cong H_p(C) \otimes_R G \quad \text{and} \quad H^p(\mathrm{Hom}_R(C, G)) \cong \mathrm{Hom}_R(H_p(C), G),$$

but this is generally not the case. If the ring R is nice enough and if the modules C_p are nice enough, then $H_p(C \otimes_R G)$ can be expressed in terms of $H_p(C) \otimes_R G$ and $\mathrm{Tor}_1^R(H_{p-1}(C), G)$, where $\mathrm{Tor}_1^R(-, G)$ is a one of the left-derived functors of $- \otimes_R G$, and $H^p(\mathrm{Hom}_R(C, G))$ can be expressed in terms of $\mathrm{Hom}_R(H_p(C), G))$ and $\mathrm{Ext}_R^1(H_{p-1}(C), G)$, where $\mathrm{Ext}_R^1(-, G)$ is one of the right-derived functors of $\mathrm{Hom}_R(-, G)$; both derived functors are defined in Section 11.2 and further discussed in Example 11.1. These formulae known as universal coefficient theorems are discussed in Chapter 12.

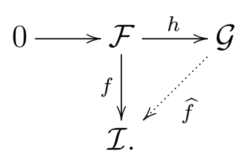
1.10 Sheaf Cohomology

Given a topological space X, we define the *global section functor* $\Gamma(X, -)$ such that for every sheaf of R-modules $\mathcal{F}$,

$$\Gamma(X, \mathcal{F}) = \mathcal{F}(X).$$

This is a functor from the category $\mathbf{Sh}(X)$ of sheaves of R-modules over X to the category of abelian groups.

A sheaf $\mathcal{I}$ is *injective* if for any injective sheaf map $h\colon \mathcal{F} \to \mathcal{G}$ and any sheaf map $f\colon \mathcal{F} \to \mathcal{I}$, there is some sheaf map $\widehat{f}\colon \mathcal{G} \to \mathcal{I}$ extending $f\colon \mathcal{F} \to \mathcal{I}$ in the sense that $f = \widehat{f} \circ h$, as in the following commutative diagram:



This is the same diagram that we used to define injective modules in Section 1.7, but here, the category involved is the category of sheaves.

A nice feature of the category of sheaves of R-modules is that its has enough injectives.

Proposition *For any sheaf $\mathcal{F}$ of R-modules, there is an injective sheaf $\mathcal{I}$ and an injective sheaf homomorphism $\varphi\colon \mathcal{F} \to \mathcal{I}$.*

As in the case of modules, the fact that the category of sheaves has enough injectives implies that any sheaf has an injective resolution.

On the other hand, the category of sheaves does not have enough projectives. This is the reason why projective resolutions of sheaves are of little interest.

Another good property is that the global section functor is left-exact. Then as in the case of modules in Section 1.7, the cohomology groups induced by the right derived functors $R^p\Gamma(X, -)$ are well defined.

The *cohomology groups* of the sheaf $\mathcal{F}$ (or the *cohomology groups of* X *with values in* $\mathcal{F}$), denoted by $H^p(X, \mathcal{F})$, are the groups $R^p\Gamma(X, -)(\mathcal{F})$ induced by the right derived functor $R^p\Gamma(X, -)$ (with $p \geq 0$).

To compute the sheaf cohomology groups $H^p(X, \mathcal{F})$, pick *any* resolution of $\mathcal{F}$

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \xrightarrow{d^0} \mathcal{I}^1 \xrightarrow{d^1} \mathcal{I}^2 \xrightarrow{d^2} \cdots$$

by injective sheaves $\mathcal{I}^n$, apply the global section functor $\Gamma(X, -)$ to obtain the complex of R-modules

$$0 \xrightarrow{\delta^{-1}} \mathcal{I}^0(X) \xrightarrow{\delta^0} \mathcal{I}^1(X) \xrightarrow{\delta^1} \mathcal{I}^2(X) \xrightarrow{\delta^2} \cdots,$$

and then

$$H^p(X, \mathcal{F}) = \mathrm{Ker}\, \delta^p / \mathrm{Im}\, \delta^{p-1}.$$

By Theorem 11.47 (stated in the previous section) the right derived functors $R^p\Gamma(X, -)$ constitute a universal δ-functor, so all the properties of δ-functors apply.

In principle, computing the cohomology groups $H^p(X, \mathcal{F})$ requires finding injective resolutions of sheaves. However injective sheaves are very big and hard to deal with. Fortunately, there is a class of sheaves known as *flasque* sheaves (due to Godement) which are $\Gamma(X, -)$-acyclic, and every sheaf has a resolution by flasque sheaves. Therefore, by Proposition 11.34 (stated in the previous section), the cohomology groups $H^p(X, \mathcal{F})$ can be computed using flasque resolutions.

Then we compare sheaf cohomology (defined by derived functors) to the other kinds of cohomology defined so far: de Rham, singular, Čech (for the constant sheaf $\widetilde{G}_X$).

If the space X is paracompact, then it turns out that for any sheaf $\mathcal{F}$, the Čech cohomology groups $\check{H}^p(X, \mathcal{F})$ are isomorphic to the cohomology groups $H^p(X, \mathcal{F})$. Furthermore, if $\mathcal{F}$ is a presheaf, then the Čech cohomology groups $\check{H}^p(X, \mathcal{F})$ and $\check{H}^p(X, \widetilde{\mathcal{F}})$ are isomorphic, where $\widetilde{\mathcal{F}}$ is the sheafification of $\mathcal{F}$. Several other results (due to Leray and Henri Cartan) about the relationship between Čech cohomology and sheaf cohomology will be stated.

When X is a topological manifold (thus paracompact), for every R-module G, we will show that the singular cohomology groups $H^p(X; G)$ are isomorphic to the cohomology groups $H^p(X, \widetilde{G}_X)$ of the constant sheaf $\widetilde{G}_X$. Technically, we will need to define *soft* and *fine* sheaves.

We will also define Alexander–Spanier cohomology and prove that it is equivalent to sheaf cohomology (and Čech cohomology) for paracompact spaces and for the constant sheaf $\widetilde{G}_X$.

In summary, for manifolds, singular cohomology, Čech cohomology, Alexander–Spanier cohomology, and sheaf cohomology all agree (for the constant sheaf $\widetilde{G}_X$). For smooth manifolds, we can add de Rham cohomology to the above list of equivalent cohomology theories, for the constant sheaf $\widetilde{\mathbb{R}}_X$. All these results are presented in Chapter 13.

1.11 Alexander and Alexander–Lefschetz Duality

The goal of Chapter 14 is to present various generalizations of Poincaré duality. These versions of duality involve taking direct limits of direct mapping families of singular cohomology groups which, in general, are not singular cohomology groups. However, such limits are isomorphic to Alexander–Spanier cohomology groups, and thus to Čech cohomology groups. These duality results also require relative versions of homology and cohomology.

1.12 Spectral Sequences

A spectral sequence is a tool of homological algebra whose purpose is to approximate the cohomology (or homology) $H(M)$ of a module M endowed with a family $(F^pM)_{p\in\mathbb{Z}}$ of submodules such that $F^{p+1}M \subseteq F^pM$ for all p and

$$M = \bigcup_{p\in\mathbb{Z}} F^pM,$$

called a filtration. The module M is also equipped with a linear map $d\colon M \to M$ called differential such that $d \circ d = 0$, so that it makes sense to define

$$H(M) = \mathrm{Ker}\, d/\mathrm{Im}\, d.$$

We say that (M, d) is a differential module. To be more precise, the filtration induces cohomology submodules $H(M)^p$ of $H(M)$, the images of $H(F^pM)$ in $H(M)$, and a spectral sequence is a sequence of modules E_r^p (equipped with a differential d_r^p), for $r \geq 1$, such that E_r^p approximates the "graded piece" $H(M)^p/H(M)^{p+1}$ of $H(M)$.

Actually, to be useful, the machinery of spectral sequences must be generalized to filtered cochain complexes. Technically this implies dealing with objects $E_r^{p,q}$ involving three indices, which makes its quite challenging to follow the exposition.

Many presentations jump immediately to the general case, but it seems pedagogically advantageous to begin with the simpler case of a single filtered differential module. This the approach followed by Serre in his dissertation [56] (Pages 24–104, *Annals of Mathematics*, 54 (1951), 425–505), Godement [24], and Cartan and Eilenberg [10]. Spectral sequences are discussed in great detail in Chapter 15.

There are several methods for defining spectral sequences, including the following three:

(1) Koszul's original approach as described by Serre [56] and Godement [24]. In our opinion it is the simplest method to understand what is going on.

(2) Cartan and Eilenberg's approach [10]. This is a somewhat faster and slicker method than the previous method.

(3) Exact couples of Massey (1952). This somewhat faster method for defining spectral sequences is adopted by Rotman [50, 52] and Bott and Tu [4]. Mac Lane [37], Weibel [63], and McCleary [44] also present it and show its equivalence with the first approach. It appears to be favored by algebraic topologists. This approach leads to spectral sequences in a quicker fashion and is more general because exact couples need not arise from a filtration, but our feeling is that it is even more mysterious to a novice than the first two approaches.

We will primarily follow Method (1) and present Method (2) and Method (3) in starred sections (Method (2) in Section 15.15 and Method (3) in Section 15.14). All three methods produce isomorphic sequences, and we will show their equivalence.

1.13 Suggestions On How to Use This Book

This book basically consists of two parts. The first part covers fairly basic material presented in the first seven chapters. The second part deals with more sophisticated material including sheaves, derived functors, sheaf cohomology, and spectral sequences.

Chapter 3 on de Rham cohomology, Chapter 5 on simplicial homology and cohomology, and Chapter 6 on CW-complexes, are written in such a way that they are pretty much independent of each other and of the rest of book, and thus can be safely skipped. Readers who have never heard about differential forms can skip Chapter 3, although of course they will miss a nice facet of the global picture. Chapter 5 on simplicial homology and cohomology was included mostly for historical sake, and because they have a strong combinatorial and computational flavor. Chapter 6 on CW-complexes was included to show that there are tools for computing homology goups and to compensate for the lack of computational flavor of singular homology. However, CW-complexes can't really be understood without a good knowledge of singular homology.

Our feeling is that singular homology is simpler to define than the other homology theories, and since it is also more general, we decided to choose it as our first presentation of homology.

Our main goal is really to discuss cohomology, but except for de Rham cohomology, we feel that a two step process where we first present singular homology, and then singular cohomology as the result of applying the functor $\mathrm{Hom}(-, G)$, is less abrupt than discussing Čech cohomology (or Alexander–Spanier cohomology) first. If the reader prefers, he/she may to go directly to Chapter 9.

In any case, we highly recommend first reading the first four sections of Chapter 2. Sections 2.7 and 2.2 can be skipped upon first reading. Next, either proceed with Chapter 3, or skip it, but read Chapter 4 entirely.

After this, we recommend reading Chapter 7 on Poincaré duality, since this is one of the jewels of algebraic topology.

Knowledge about manifolds is not necessary to read this book but definitely useful since manifolds form a large class of spaces for which all the main cohomology theories are equivalent. Among the many books that cover manifolds, we suggest (in alphabetic order) Lee [36], Morita [46], Tu [61], and Warner [62]. A detailed presentation, first at a basic level and then at a more advanced level is also provided in Gallier and Quaintance [20]. Chapter 3 requires knowledge of differential forms on smooth manifolds. Differential forms are discussed in Tu [61], Morita [46], Madsen and Tornehave [39], and Bott and Tu [4]. A detailed exposition, including an extensive review of tensor algebra, is also provided in Gallier and Quaintance [21]. A firm grasp of linear algebra and of some commutative algebra, at the level discussed in texts such as Artin [2] and Dummit and Foote [14], is required.

The second part, starting with presheaves and sheaves in Chapter 8, relies on more algebra, especially Chapter 11 on derived functors and Chapter 15 on spectral sequences. However, this is some of the most beautiful material, so do not be discouraged if the going is tough. Skip proofs upon first reading and try to plow through as much as possible. Stop to take a break, and go back!

One of our goals is to fully prepare the reader to read books like Hartshorne [30] (Chapter III). Others have expressed the same goal, and we hope to be more successful.

We have borrowed some proofs of Steve Shatz from Shatz and Gallier [58], and many proofs in Chapter 11 are borrowed from Rotman [50, 52]. Generally, we relied heavily on Bott and Tu [4], Bredon [7], Godement [24], Hatcher [31], Milnor and Stasheff [45], Munkres [48], Serre [55], Spanier [59], Tennison [60], and Warner [62]. These are wonderful books, and we hope that reading our book will prepare the reader to study them. We express our gratitude to these authors, and to all the others that have inspired us (including, of course, Dieudonné).

Since we made the decision not to include all proofs (this would have doubled if not tripled the size of the book!), we tried very hard to provide precise pointers to all omitted proofs. This may be irritating to the expert, but we believe that a reader with less knowledge will appreciate this. The reason for including a proof is that we feel that it presents a type of argument that the reader should be exposed to, but this often subjective and a reflection of our personal taste. When we omitted a proof, we tried to give an idea of what it would be, except when it was a really difficult proof. This should be an incentive for the reader to dig into these references.