What is a Proof?

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CIS, Upenn and
Hospices de Beaune

Reillag’s office

Another office

After a bad proof!

Finally, some peace!

Quick History

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- Modus Ponens: If (P implies Q) holds and P holds, then Q holds.

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- Proof by metaproof, etc.

Proof by intimidation!

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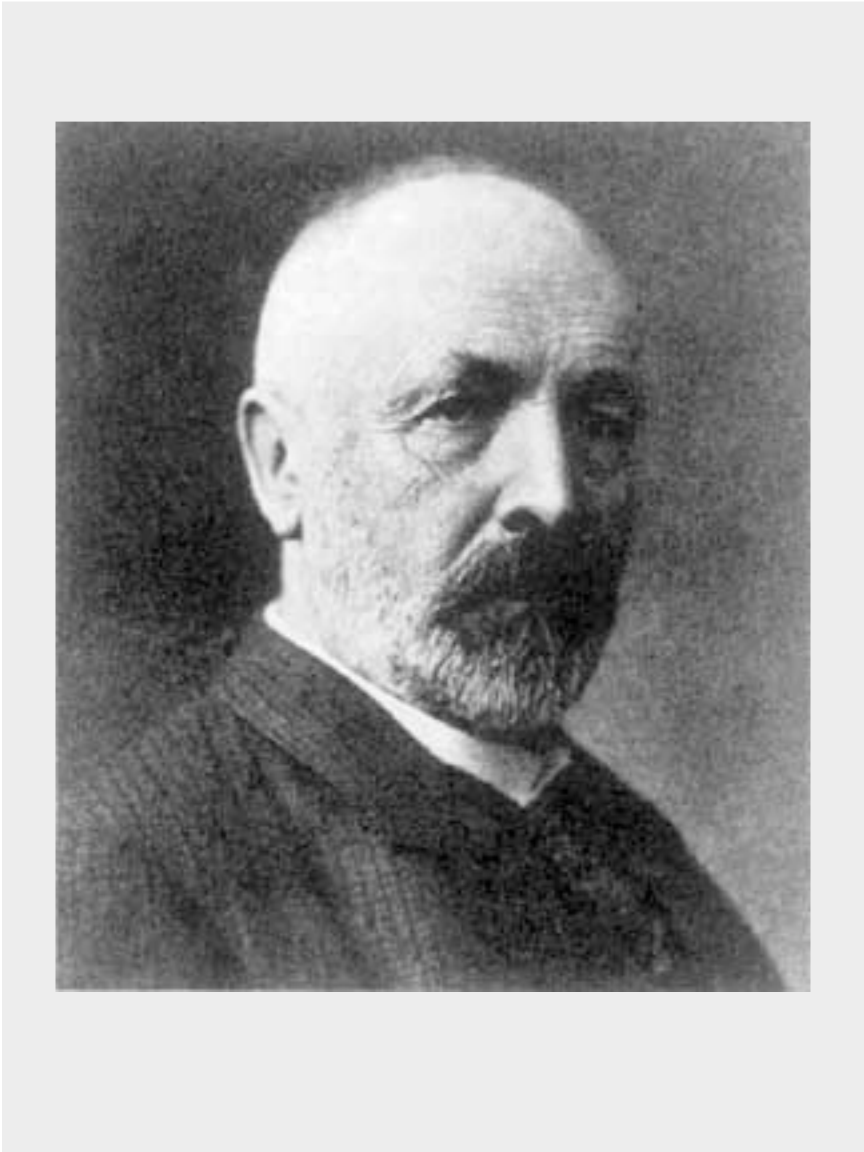

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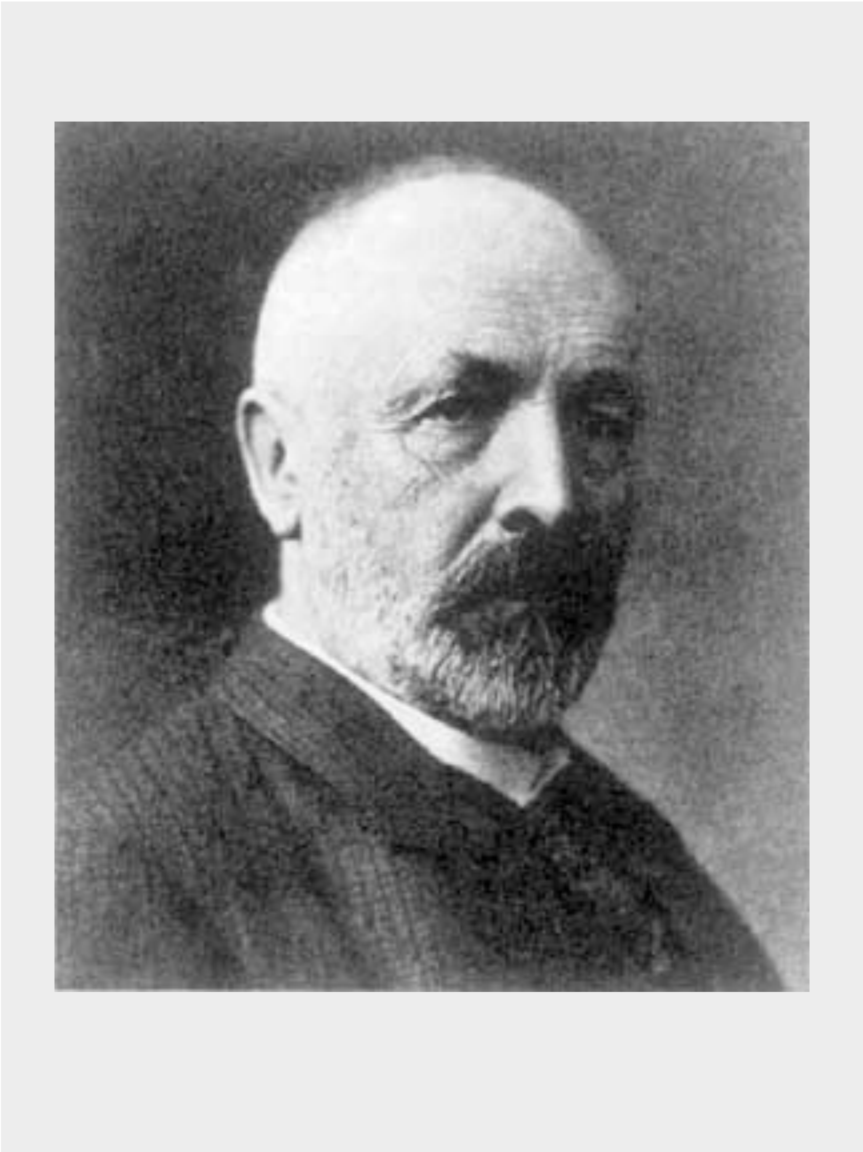

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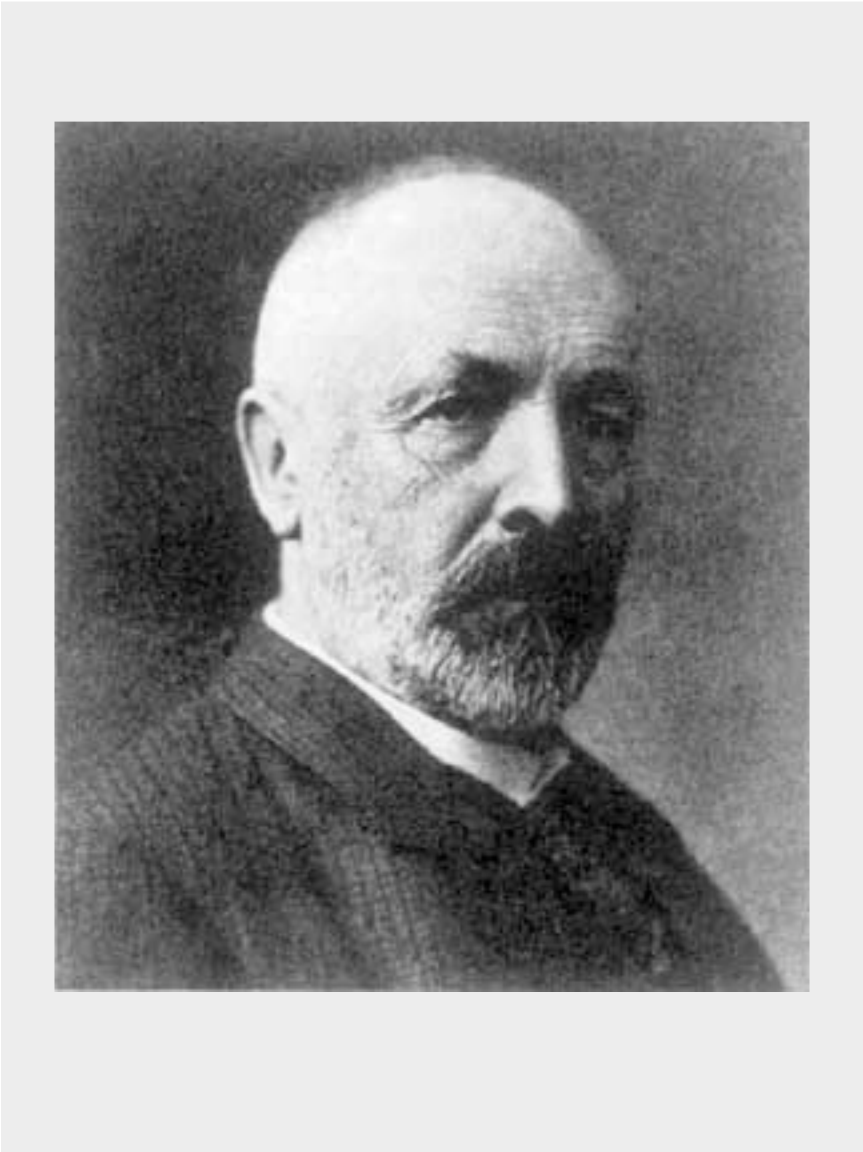

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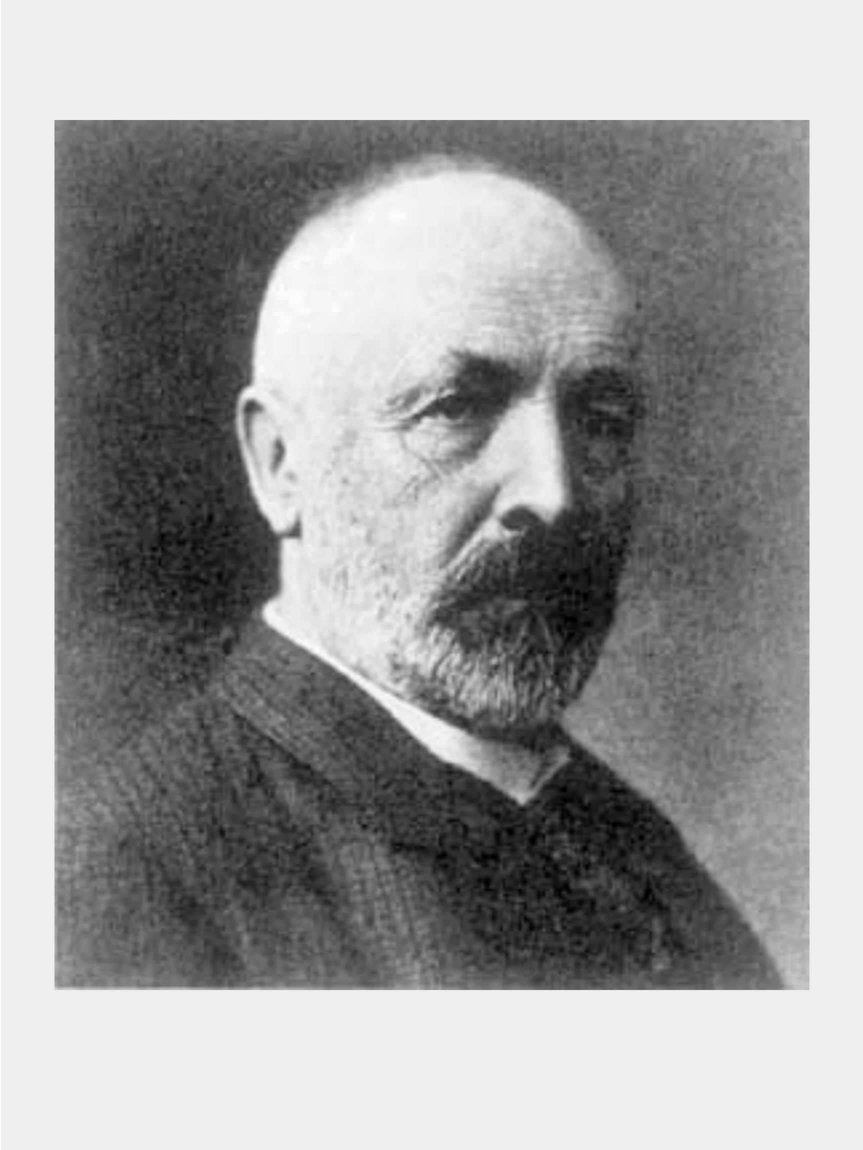

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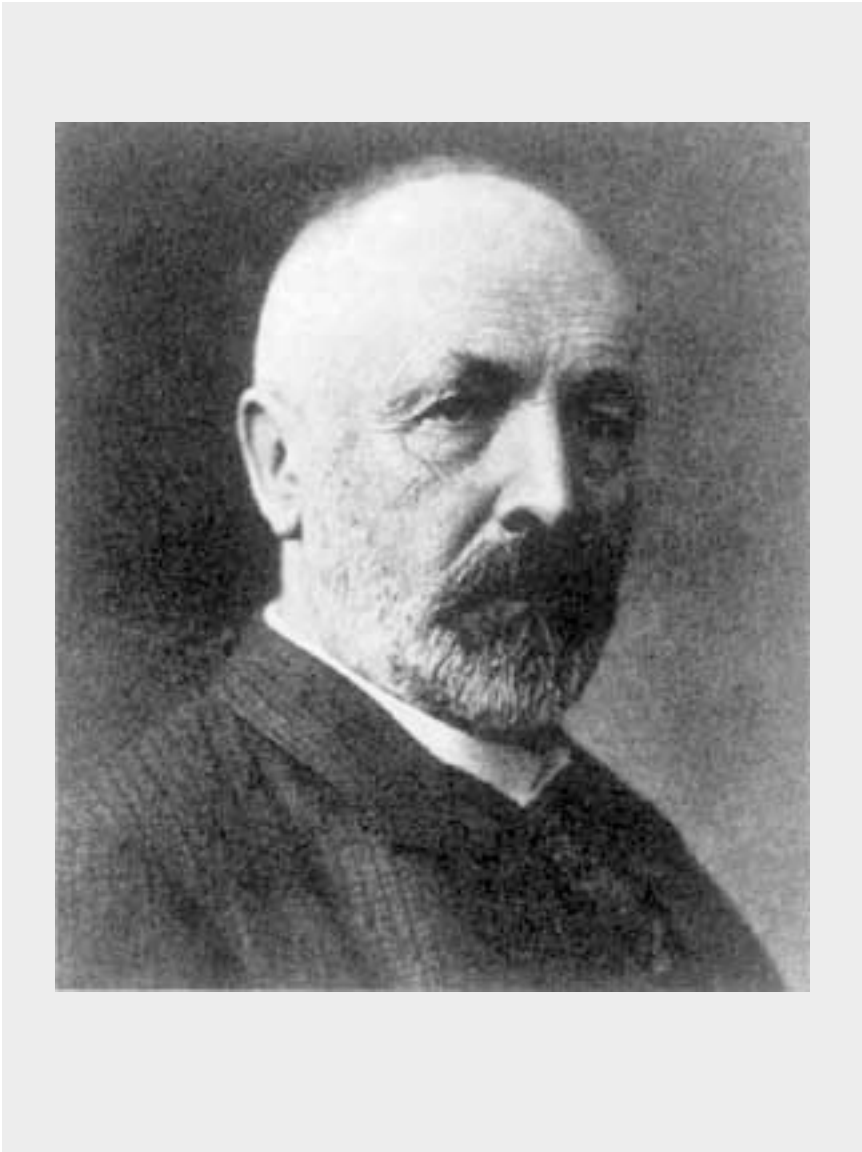

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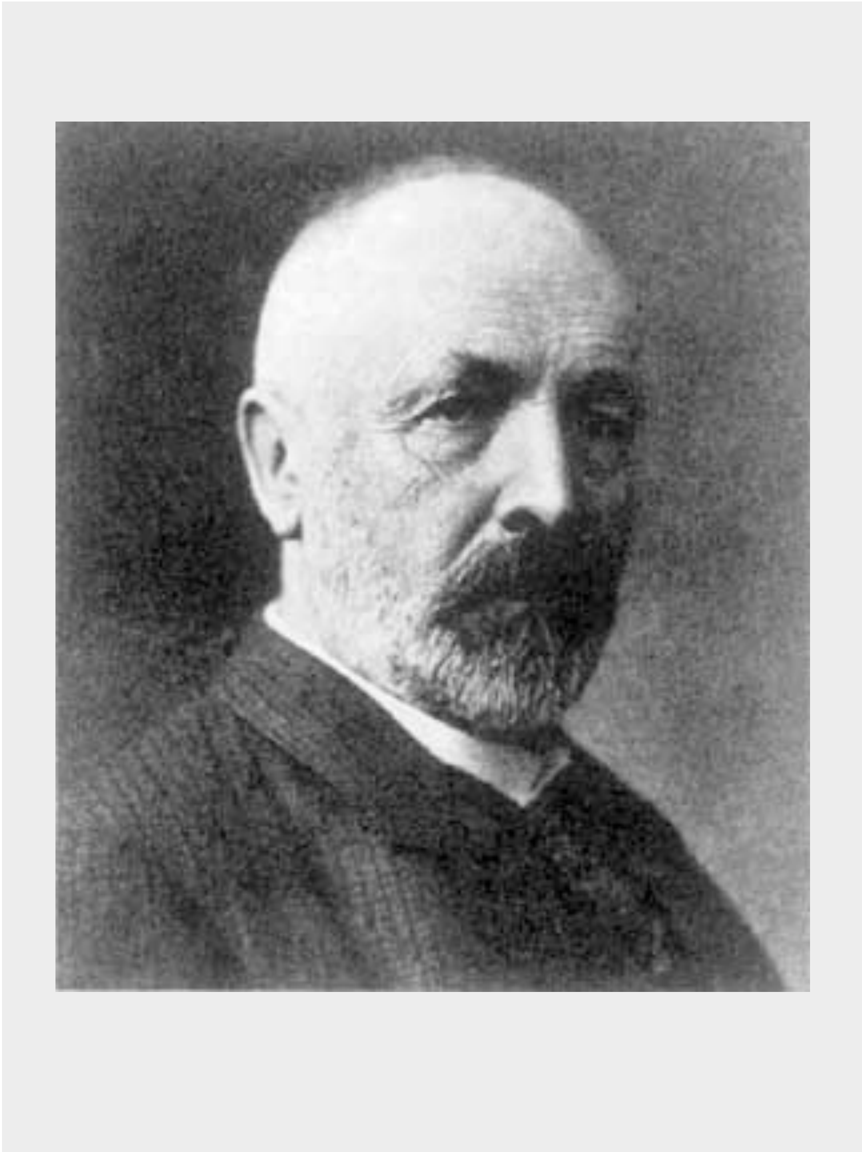
- Russell's paradox (1902)
- There is no set of all sets



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- ``To truth through proof''
- Provable implies true. Easier to study proofs

Truth and Proofs

- The logical connectives (and, or, implication, negation, etc.) carry some intuitive semantics
- For example, $A \wedge B$ (A and B) means that both A and B are true
- But what is the meaning of $A \Rightarrow B$ (A implies B)?


All cats have four legs.
I have four legs.
Therefore, I am a cat.
S. Harris

Dog Logic

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- Unfortunately, there is more than one formalism to define the notion of proof
- Hilbert systems, natural deduction, sequent calculus, categorical logic, etc.

Hilbert

David Hilbert (1862-1943)

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- The axioms are very unnatural!
- That's because they are chosen to yield the deduction theorem
- Unfriendly system for humans.
- Proofs in Hilbert systems are very far from proofs that a human would write

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- Introduction/Elimination

Proofs and Deductions

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- A proof of a proposition, P, does not depend on any assumptions (premises).
- When we construct a proof, we usually introduce extra premises which are later closed (dismissed, discharged).
- Such an ``unfinished'' proof is a deduction.
- We need a mechanism to keep track of closed (discharged) premises (the others are open).

Natural Deduction Rules

- A proof is a tree labeled with propositions
- To prove an implication, $P \Rightarrow Q$, from a list of premises, $\Gamma = (P_1, \ldots, P_n)$, do this:
- Add P to the list Γ and prove Q from Γ and P.
- When this deduction is finished, we obtain a proof of $P \Rightarrow Q$ which does not depend on P, so the premise P needs to be discharged (closed).

Natural Deduction Rules

The axioms and inference rules for *implicational logic* are:

Axioms:

$$\frac{\Gamma, P}{P}$$

The $\Rightarrow$-*elimination rule*:

$$\frac{\dfrac{\Gamma}{P \Rightarrow Q} \qquad \dfrac{\Delta}{P}}{Q}$$

Natural Deduction Rules

The $\Rightarrow$-*introduction rule*:

$$\dfrac{\dfrac{\Gamma, P^x}{Q}}{P \Rightarrow Q}\; x$$

In the introduction rule, the tag x indicates which rule caused the premise, P, to be discharged.

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Every tag is associated with a unique rule but several premises can be labeled with the same tag and all discharged in a single step.

Examples of Proofs

(a)

$$\cfrac{\cfrac{P^x}{P}}{P \Rightarrow P}\; x$$

So, $P \Rightarrow P$ is provable; this is the least we should expect from our proof system!

(b)

$$(Q \Rightarrow R)^y \qquad \frac{(P \Rightarrow Q)^z \qquad P^x}{Q}$$

Examples of Proofs

$$\dfrac{(Q \Rightarrow R)^y \qquad \dfrac{(P \Rightarrow Q)^z \qquad P^x}{Q}}{R}$$

$$\dfrac{\dfrac{(Q \Rightarrow R)^y \qquad \dfrac{(P \Rightarrow Q)^z \qquad P^x}{Q}}{R}}{P \Rightarrow R}\, x$$

Example of Proofs

$$\dfrac{\dfrac{\dfrac{(Q \Rightarrow R)^y \qquad \dfrac{(P \Rightarrow Q)^z \qquad P^x}{Q}}{R}}{P \Rightarrow R}\, x}{(Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\, y$$

Example of Proofs

$$\dfrac{\dfrac{\dfrac{\dfrac{(Q \Rightarrow R)^y \qquad \dfrac{(P \Rightarrow Q)^z \qquad P^x}{Q}}{R}}{P \Rightarrow R}\;x}{(Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\;y}{(P \Rightarrow Q) \Rightarrow ((Q \Rightarrow R) \Rightarrow (P \Rightarrow R))}\;z$$

Examples of proofs

(c) In the next example, the two occurrences of A labeled x are discharged simultaneously.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(A \Rightarrow (B \Rightarrow C))^z \qquad A^x}{B \Rightarrow C} \qquad \dfrac{(A \Rightarrow B)^y \qquad A^x}{B}}{C}}{A \Rightarrow C}\; x}{(A \Rightarrow B) \Rightarrow (A \Rightarrow C)}\; y}{\big(A \Rightarrow (B \Rightarrow C)\big) \Rightarrow \big((A \Rightarrow B) \Rightarrow (A \Rightarrow C)\big)}\; z
$$

More Examples of Proofs

(d) In contrast to Example (c), in the proof tree below the two occurrences of A are discharged separately. To this effect, they are labeled differently.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(A \Rightarrow (B \Rightarrow C))^z \qquad A^x}{B \Rightarrow C} \qquad \dfrac{(A \Rightarrow B)^y \qquad A^t}{B}}{C}}{A \Rightarrow C}\, x}{(A \Rightarrow B) \Rightarrow (A \Rightarrow C)}\, y}{\big(A \Rightarrow (B \Rightarrow C)\big) \Rightarrow \big((A \Rightarrow B) \Rightarrow (A \Rightarrow C)\big)}\, z}{A \Rightarrow \Big(\big(A \Rightarrow (B \Rightarrow C)\big) \Rightarrow \big((A \Rightarrow B) \Rightarrow (A \Rightarrow C)\big)\Big)}\, t$$

Wow, I landed it! (the proof)

Natural Deduction in Sequent-Style

- A different way of keeping track of open premises (undischarged) in a deduction
- The nodes of our trees are now sequents of the form $\Gamma \rightarrow P$, with

$$\Gamma = x_1 : P_1, \ldots, x_m : P_m$$

- The variables are pairwise distinct but the premises may be repeated
- We can view the premise P_i as the type of the variable x_i!

Natural Deduction in Sequent-Style

The *axioms and rules for implication in Gentzen-sequent style*:

$$\Gamma, x\colon P \to P$$

$$\frac{\Gamma, x\colon P \to Q}{\Gamma \to P \Rightarrow Q} \quad (\Rightarrow\text{-}intro)$$

$$\frac{\Gamma \to P \Rightarrow Q \quad \Gamma \to P}{\Gamma \to Q} \quad (\Rightarrow\text{-}elim)$$

Redundant Proofs
Proof Normalization

$$
\dfrac{
\dfrac{\dfrac{\dfrac{((R \Rightarrow R) \Rightarrow Q)^x \qquad (R \Rightarrow R)^y}{Q}}{((R \Rightarrow R) \Rightarrow Q) \Rightarrow Q}\;x}{(R \Rightarrow R) \Rightarrow (((R \Rightarrow R) \Rightarrow Q) \Rightarrow Q)}\;y
\qquad
\dfrac{\dfrac{R^z}{R}}{R \Rightarrow R}\;z
}{((R \Rightarrow R) \Rightarrow Q) \Rightarrow Q}
$$

Redundant Proofs
Proof Normalization

- When an elimination step immediately follows an introduction step, a proof can be normalized (simplified)

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{((R \Rightarrow R) \Rightarrow Q)^x \qquad (R \Rightarrow R)^y}{Q}
    }{((R \Rightarrow R) \Rightarrow Q) \Rightarrow Q}\, x
  }{(R \Rightarrow R) \Rightarrow (((R \Rightarrow R) \Rightarrow Q) \Rightarrow Q)}\, y
  \qquad
  \dfrac{\dfrac{R^z}{R}}{R \Rightarrow R}\, z
}{((R \Rightarrow R) \Rightarrow Q) \Rightarrow Q}
$$

Proof Normalization

- A simpler (normalized) proof:

$$
\dfrac{\dfrac{((R \Rightarrow R) \Rightarrow Q)^x \qquad \dfrac{\dfrac{R^z}{R}}{R \Rightarrow R}\,z}{Q}}{((R \Rightarrow R) \Rightarrow Q) \Rightarrow Q}\,x
$$

Where is that simpler proof?

Normalization and Strong Normalization of Proofs

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- He proved that every proof can be reduced to a normal form (normalization).
- In 1971, he proved that every reduction sequence terminates (strong normalization) and that every proof has a unique normal form.

Propositions as types and proofs as simply-typed lambda terms

$$\Gamma, x : P \rightarrow x : P$$

$$\frac{\Gamma, x : P \rightarrow M : Q}{\Gamma \rightarrow \lambda x : P \cdot M : P \Rightarrow Q} \quad (\Rightarrow\text{-}intro)$$

$$\frac{\Gamma \rightarrow M : P \Rightarrow Q \quad \Gamma \rightarrow N : P}{\Gamma \rightarrow MN : Q} \quad (\Rightarrow\text{-}elim)$$

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$$(\lambda x\colon \sigma \cdot M)N \longrightarrow_\beta M[N/x]$$

- Strong normalization (SN) in the typed lambda-calculus implies SN of proofs.

Adding the connectives and, or, not

- To deal with negation, we introduce falsity (absurdum), the proposition always false:

$$\perp$$

- We view $\neg P,$ the negation of $P,$ as an abbreviation for $P \Rightarrow \perp$

Rules for and

The $\wedge$-*introduction rule*:

$$\frac{\dfrac{\Gamma}{P} \qquad \dfrac{\Delta}{Q}}{P \wedge Q}$$

The $\wedge$-*elimination rule*:

$$\frac{\dfrac{\Gamma}{P \wedge Q}}{P} \qquad \frac{\dfrac{\Gamma}{P \wedge Q}}{Q}$$

Rules for or

The $\vee$-*introduction rule*:

$$\frac{\dfrac{\Gamma}{P}}{P \vee Q} \qquad\qquad \frac{\dfrac{\Gamma}{Q}}{P \vee Q}$$

The $\vee$-*elimination rule*:

$$\frac{\dfrac{\Gamma}{P \vee Q} \qquad \dfrac{\Delta, P^x}{R} \qquad \dfrac{\Lambda, Q^y}{R}}{R}\; x, y$$

Rules for negation

The $\neg$-*introduction rule*:

$$\dfrac{\dfrac{\Gamma, P^x}{\bot}}{\neg P}\; x$$

The $\neg$-*elimination rule*:

$$\dfrac{\dfrac{\Gamma}{\neg P} \qquad \dfrac{\Delta}{P}}{\bot}$$

The ``Controversial ” Rules

The $\perp$-*elimination rule*:

$$\frac{\dfrac{\Gamma}{\perp}}{P}$$

The *proof-by-contradiction rule* (also known as *reductio ad absurdum rule*, for short *RAA*):

$$\frac{\dfrac{\Gamma, \neg P^x}{\perp}}{P}\ x$$

Problems With Negation

- The $\perp$-elimination rule is not so bad.
- It says that once we have reached an absurdity, then everything goes!
- RAA is worse! I allows us to prove double negation elimination and the law of the excluded middle:
- $\neg\neg P \Rightarrow P$ $\neg P \vee P$
- Constructively, these are problematic!

Lack of Constructivity

- The provability of $\neg\neg P \Rightarrow P$ and $\neg P \vee P$ is equivalent to RAA.
- RAA allows proving disjunctions (and existential statements) that may not be constructive; this means that if $A \vee B$ is provable, in general, it may not be possible to give a proof of A or a proof of B
- This lack of constructivity of classical logic led Brouwer to invent intuitionistic logic

That’s too abstract, give me something concrete!

A non-constructive proof

- Claim: There exist two reals numbers, a, b, both irrational, such that a^b is rational.
- Proof: We know that $\sqrt{2}$ is irrational. Either
- (1) $\sqrt{2}^{\sqrt{2}}$ is rational; $a = b = \sqrt{2}$, or
- (2) $\sqrt{2}^{\sqrt{2}}$ is irrational; $a = \sqrt{2}^{\sqrt{2}}, \ b = \sqrt{2}$
- In (2), we use $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2$
- Using the law of the excluded middle, our claim is proved! But, what is $\sqrt{2}^{\sqrt{2}}$?

Non-constructive Proofs

- The previous proof is non-constructive.
- It shows that a and b must exist but it does not produce an explicit solution.
- This proof gives no information as to the irrationality of $\sqrt{2}^{\sqrt{2}}$
- In fact, $\sqrt{2}^{\sqrt{2}}$ is irrational, but this is very hard to prove!
- A ``better" solution: $a = \sqrt{2},\ b = \log_2 9$

Existence proofs are often non-constructive

- Fixed-points Theorems often only assert the existence of a fixed point but provide no method for computing them.
- For example, Brouwer's Fixed Point Theorem.
- That's too bad, this theorem is used in the proof of the Nash Equilibrium Theorem!

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- Also important work in topology

A. Heyting

A. Heyting

- Arend Heyting (1898-1980)

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- Heyting algebras (semantics for intuitionistic logic)

Intuitionistic Logic

- In intuitionistic logic, it is forbidden to use the proof by contradiction rule (RAA)
- As a consequence, $\neg\neg P$ no longer implies P and $\neg P \vee P$ is no longer provable (in general)
- The connectives, and, or, implication and negation are independent
- No de Morgan laws

Intuitionistic Logic

- Fewer propositions are provable (than in classical logic) but proofs are more constructive.
- If a disjunction, $P \vee Q$, is provable, then a proof of P or a proof of Q can be found.
- Similarly, if $\exists t P$ is provable, then there is a term, τ, such that $P[\tau/t]$ is provable.
- However, the complexity of proof search is higher.

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Intuitionistic Logic and Typed lambda-Calculi

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- We now have conjunctive, disjunctive, universal and existential types.
- Falsity can be viewed as an ``error type''
- Strong Normalization still holds, but some subtleties with disjunctive and existential types (permutative reductions)

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- The corresponding lambda-calculus is a polymorphic lambda calculus (first invented by J.Y. Girard, systems F and F-omega, 1971)
- System F was independently discovered by J. Reynolds (1974) for very different reasons.
- Later, even richer typed calculi, the theory of construction (Coquand, Huet)

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- A human prover evolves in a spectrum of formality!

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- In principle, it is possible to write formalized proofs.
- This is desirable if we want to have absolute confidence in a proof.

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- It is important to build tools to check or construct proofs.
- Even if we never write formal proofs, it is important to understand clearly what are the rules of reasoning that we use when we construct informal proofs.

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- Feit and Thompson's paper is 255 pages long.

Proof Verification

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Feit-Thompson theorem has been totally checked in Coq

Thursday 20 September 2012, 18:16. We received following mail from Georges Gonthier (see below).
It concludes the proof in Coq of the Feit-Thompson theorem. This theorem, also named the Odd Order Theorem, is the first main result in the classification of finite groups.
This work was achieved by the team formed by addressees of Georges' mail, team strongly led by Georges Gonthier. It is the end of a 6-year long research effort (almost fulltime work) started in May 2006. After the Four Color theorem, this is the second impressive mathematical theorem totally proved in the Coq proof assistant.
More info can be found in this mail by Laurent Théry.

From Laurent Théry
Date: Thursday 20 September 2012, 20:24
Re: [Coqfinitgroup-commits] r4105 - trunk
Hi,

Just for fun

Feit Thompson statement in Coq:

Theorem Feit_Thompson (gT : finGroupType) (G : {group gT}) : odd #|G| -> solvable G.

How is it proved?

You can see only green lights there:

http://ssr2.msr-inria.inria.fr/~jenkins/current/progress.html

and the final theory graph at:

http://ssr2.msr-inria.inria.fr/~jenkins/current/index.html

How big it is:

Number of lines ~ 170 000
Number of definitions ~15 000
Number of theorems ~ 4 200
Fun ~ enormous!

-- Laurent

A very small piece of the code

```
Proposition coprime_Hall_trans A G H1 H2 :
    A \subset 'N(G) -> coprime #|G| #|A| -> solvable G ->
    pi.-Hall(G) H1 -> A \subset 'N(H1) ->
    pi.-Hall(G) H2 -> A \subset 'N(H2) ->
  exists2 x, x \in 'C_G(A) & H1 :=: H2 :^ x.

 A complement to the above: 'C(A) acts on 'Nby(A)
Lemma norm_conj_cent A G x : x \in 'C(A) ->
  (A \subset 'N(G :^ x)) = (A \subset 'N(G)).

 Strongest version of the centraliser lemma -- not found in textbooks!
 Obviously, the solvability condition could be removed once we have the
 Odd Order Theorem.
Lemma strongest_coprime_quotient_cent A G H :
      let R := H :&: [~: G, A] in
      A \subset 'N(H) -> R \subset G -> coprime #|R| #|A| ->
      solvable R || solvable A ->
  'C_G(A) / H = 'C_(G / H)(A / H).

 A weaker but more practical version, still stronger than the usual form
 (viz. Aschbacher 18.7.4), similar to the one needed in Aschbacher's
 proof of Thompson factorization. Note that the coprime and solvability
 assumptions could be further weakened to H :&: G (and hence become
 trivial if H and G are TI). However, the assumption that A act on G is
 needed in this case.
```

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- For intuitionistic first-order logic: Kripke models.

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- Completeness: Every valid formula is provable.
- Completeness is desirable but not always possible.

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- Complexity of classical prop. validity: co-NP complete (Cook, Karp, 1970)
- Complexity of intuitionistic prop. validity: P-space complete! (Statman, 1979)
- The decision problem (validity problem) for first-order (classical) logic is undecidable (Church, 1936)
- Decision problem for intuitionistic logic also undecidable (double negation translation)

Kurt Godel (1906-1978)
(Right: with A. Einstein)

Alonzo Church
(1903-1995)

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- This leads to logics where the connectives have a double identity: additive or multiplicative.

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- linear logic can be viewed as an attempt to deal with resources and parallelism
- Negation is an involution

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- Temporal logic deals with time (A. Pnueli)
- Process logic (Manna, Pnueli)
- Dynamic logic (Harel, Pratt)
- The world of logic is alive and well!

Searching for that proof!

The proof is hard to reach