On the compression of locational and environmental data in multi-vehicle missions: a control systems approach

José Marcio Luna\textsuperscript{a}, Rafael Fierro\textsuperscript{a}, Chaouki Tanios Abdallah\textsuperscript{a} & Frank Lewis\textsuperscript{b}

\textsuperscript{a} Department of Electrical & Computer Engineering, The University of New Mexico, Albuquerque, NM, USA
\textsuperscript{b} Research Institute UTARI, The University of Texas, Arlington, TX, USA

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On the compression of locational and environmental data in multi-vehicle missions: a control systems approach

José Marcio Lunaa,∗, Rafael Fierroa, Chaouki Tanios Abdallaha and Frank Lewisb

aDepartment of Electrical & Computer Engineering, The University of New Mexico, Albuquerque, NM, USA;
bResearch Institute UTARI, The University of Texas, Arlington, TX, USA

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Mobile agents that take part in multi-vehicle missions usually need to share environmental and locational information with other agents and with control stations through a communication channel. In real scenarios, the agents have to deal with different communication issues, such as interference, loss of connectivity and unexpected reduction of available bandwidth. One way to overcome these issues is by minimising the amount of data in the communication channel, which not only speeds up the sharing of information, but would also avoid the loss of data. We propose a control systems approach that allows the compression of the shared information. Given some problem-dependent mathematical assumptions, our method aims to simplify the calculation of the stationary errors by computing the sign of the errors rather than their exact values and the control law may then be used to stabilise the system. The approach allows for the compression of the outputs involved in the calculation of the errors as such outputs represent the shared information among agents. We carry out the theoretical analysis of our approach and apply it to two case studies, namely a formation control and a coverage control with consensus. We finally validate our theoretical results through simulations in Matlab.

Keywords: data compression; coverage control; formation control; binary consensus

1. Introduction

Multi-vehicle missions with autonomous agents have become a feasible and promising field, thanks to the latest developments in computational power, mobile robots, wireless communications, sensors and power storage. The literature on this topic offers a vast amount of applications such as rendezvous problems (Dimarogonas & Kyriakopoulus, 2007; Lin, Morse, & Anderson, 2003), flocking (Olfati-Saber, 2006; Tanner, Jadbabaie, & Pappas, 2007; Zhou, Wu, Yu, Small, & an Lu, 2012), formation control (Fierro, Song, Das, & Kumar, 2002; Mastellone, Mejia, Stipanović, & Spong, 2011; Spletzer et al., 2001), coverage control (Casbeer et al., 2006; Cortés, Martinez, Karatas, & Bullo, 2004; Gonçalves, Pimenta, & Pereira, 2011; Schwager, Rus, & Slotine, 2009) and boundary tracking (Clark & Fierro, 2007; Matveev, Teimoori, & Savkin, 2012; Susca, Martinez, & Bullo, 2008) among others. All such applications involve the interaction of two or more mobile agents with the capability of taking measurements of their location and the surrounding environment. The agents are able to share their measurements with either a central controller or with their neighbours to fulfil a common goal. Many algorithms have been proposed to address multi-vehicle problems. Some models assume holonomic agents (Cortês et al., 2004; Franceschelli, Egerstedt, Giua, & Mahulea, 2009), while others assume non-holonomic agents (Dirafzoon, Menhaj, & Afšar, 2011; Pimenta et al., 2010).

With the study of decentralised systems, the issue of synchronisation also became an area of strong interest. Linear consensus protocols (Jadbabaie, Lin, & Morse, 2003; Olfati-Saber, 2004), passivity-based synchronisation (Chopra & Spong, 2006) as well as distributed consensus of Lagrangian systems (Ren & Atkins, 2007) offer theoretical tools to coordinate one or several groups of electromechanical systems. Those systems communicate with each other and reach a consensus or agreement about the state of the general system in order to achieve a common objective. Recently, a binary consensus protocol to synchronise Lagrangian systems was proposed in Chen and Lewis (2011). By this approach, the networked agents share their measurements but do not need the complete time-signal measurements. Instead, they just need an estimate of the relative difference between the signals, e.g. the sign of the difference between the states of an agent and its neighbours.

A well-studied issue in computer science and information theory is data compression (Chen, Chandrakasan, & Stojanović, 2012; Marcelloni & Vecchio, 2008; Ziv & Lempel, 1977). The main goal is to store or share information while the necessary amount of data is minimised by getting rid of redundant or non-relevant information. To
the best of the authors’ knowledge, the topic of compression of information on multi-vehicle missions has not been studied from a control systems perspective. Given the communication restrictions that mobile sensor networks may experience in outdoor environments because of the presence of interference, physical obstacles and long distances, the compression of shared information becomes relevant. It might improve the utilisation of the communication channel whenever the bandwidth and channel capacity become compromised under rough environmental circumstances.

1.1 Contributions

Our goal is to provide theoretical tools from a control systems perspective to justify the compression of information shared by agents in a multi-vehicle mission. Our approach suggests the use of the sign of the stationary errors of the system rather than the exact error in a feedback system. By carrying out a theoretical analysis for a formation control and a coverage control with consensus, we prove that under the appropriate conditions, control laws that depend only on the sign of the stationary error can guarantee the stability of the system. One of the simplest ways of estimating the sign is by reducing the precision to quantise the outputs involved in the subtraction, thus compressing the related data.

The paper is organised as follows. Section 2 provides the mathematical background of the classic three-robot shape control. We provide a lemma that shows that the calculation of the control law can be simplified allowing the compression of some of the required outputs. In Section 3, we provide the mathematical background of a decentralised coverage control with consensus. We provide a lemma that proves that by applying a binary consensus protocol the agents are able to reach consensus of the parameters of the environment. Furthermore, we state an additional lemma that assures that the system remains stable even if the environment has time-varying parameters. In Section 4, we present Matlab simulations of both systems where we analyse the effect of the compression of the shared information among agents on the convergence of the system. Finally, in Section 5 we present our conclusions.

2. Case study 1: formation control with compressed communication

In the seminal paper in Das et al. (2002), a stable formation control for a set of non-holonomic vehicles modelled as unicycles is presented. It is assumed that the vehicles are able to measure their own position and their linear and angular speeds. The agents are classified as leaders or followers. A leader is a vehicle that, given its position and linear and angular speeds, determines the trajectory of another vehicle called a follower. A follower can be a leader at the same time, if its position and speed is the reference for another follower. A follower might depend on more than one leader, and a leader might be controlling more than one follower.

In this specific case, we are studying the three-robot shape control known as Separation–Separation–Control or short $S_{13}S_{23}C$ which is illustrated in Figure 1. In this layout, we have a robot labelled $R_1$ which is the leader and is followed by $R_2$ and $R_3$ simultaneously. Robot $R_2$ should keep a desired separation $\rho_{12}^d$ and a constant bearing $\alpha_{12}^d$ with respect to $R_1$. Robot $R_3$ follows $R_1$ and $R_2$ simultaneously with desired separations $\rho_{13}^d$ and $\rho_{23}^d$.

2.1 Model for three-robot shape control

Based on Das et al. (2002), the dynamics of the $S_{13}S_{23}C$ formation are given by

$$\dot{x} = Au + b, \quad \dot{\theta}_2 = w_2, \quad \dot{\theta}_3 = w_3,$$

where the state vector $x$ is defined as

$$x = (\rho_{12}, \alpha_{12}, \rho_{13}, \rho_{23})^T,$$

the input vector is

$$u = (v_2, w_2, v_3, w_3)^T,$$

and the system matrices $A$ and $b$ are given by

$$A = \begin{pmatrix}
\cos \beta_{12} & \sin \beta_{12} & 0 & 0 \\
\sin \beta_{12} & \cos \beta_{12} & 0 & 0 \\
\rho_{12} & \rho_{12} & \cos \beta_{13} & \sin \beta_{13} \\
0 & 0 & \cos \beta_{23} & \sin \beta_{23}
\end{pmatrix},$$

Figure 1. Three unicycle vehicles in a $S_{13}S_{23}C$ formation. The goal is to regulate the shape of the formation by regulating the distances $\rho_{12}$, $\rho_{13}$ and $\rho_{23}$, as well as the desired bearing $\alpha_{12}$. 

β with the angle $\beta_{ij}$ defined as

$$\beta_{ij} = \alpha_{ij} + \theta_i - \theta_j.$$  

The location of the agent $R_i$ with $i = 1, 2, 3$, in the Cartesian plane is given by the triplet $(x_i, y_i, \theta_i)$, where the first two terms correspond to the rectangular coordinates on the plane of the $i$th agent, and the third one to its heading angle, as shown in Figure 1. As illustrated in the same figure, $d$ is the distance from the wheel axis to a reference point on the agent, in this case the front bumper.

As we stated earlier, we aim to compress information shared among the agents involved in multi-vehicle missions. Given that every vehicle in the formation is able to measure its position as well as its angular and linear speeds, we might assume that either the leader is able to transmit its position to the follower through a communication channel, or that the follower is able to measure the position of the leader. In both approaches, either a failure in the communication channel or errors in the sensors of the follower can affect the controller to regulate the entire formation. In the following lemma, which is based on Spletzer et al. (2001) and Fierro et al. (2002), we prove that it is enough for the follower to know the sign of the difference between the desired distances $\rho_{12}^d$ and $\rho_{23}^d$, and the actual distances $\rho_{12}$ and $\rho_{23}$ rather than the exact value to keep the formation.

**Lemma 2.1:** Given the formation control model defined by (1)–(5), and based on Figure 1, assume that the linear and angular speeds of the leader $R_1$ satisfy $v_1 > 0$ and $|w_1| < \omega_{max}$. Furthermore, assume that the linear speeds and relative orientations of $R_1$ and $R_2$ satisfy $|v_1 - v_2| < \epsilon_1$, $|\theta_1 - \theta_2| < \epsilon_2$, where $\epsilon_1, \epsilon_2 \in \mathbb{R}^+$ are small numbers. Moreover, assume that the initial relative orientations fulfill $|\theta_1(0) - \theta_2(0)| < c_1\pi$ where $c_j \in (0, 1) \subset \mathbb{R}$ and $j = 2, 3$.

If the control law

$$u = A^{-1}(r - b)$$

is applied to the system with $A$ and $b$ defined in (4) and (5), and

$$r = \begin{pmatrix}
    k_1 \text{sgn}(\rho_{12}^d - \rho_{12}) \\
    k_2 \text{sgn}(\rho_{12}^d - \alpha_{12}) \\
    k_3 \text{sgn}(\rho_{13}^d - \rho_{13}) \\
    k_4 \text{sgn}(\rho_{23}^d - \rho_{23})
\end{pmatrix},$$

with $k_1, k_2, k_3, k_4 \in \mathbb{R}^+$, then the formation is stable and the outputs $\rho_{12}, \alpha_{12}, \rho_{13}$ and $\rho_{23}$ converge to the desired values $\rho_{12}^d, \alpha_{12}^d, \rho_{13}^d$ and $\rho_{23}^d$, respectively.

**Proof:** By substituting the input vector (6) into (1), we get the closed-loop model

$$\dot{x} = r, \ \dot{\theta}_2 = w_2, \ \dot{\theta}_3 = w_3.$$  

Let us define the vector

$$e = \begin{pmatrix}
    \rho_{12}^d - \rho_{12} \\
    \alpha_{12}^d - \alpha_{12} \\
    \rho_{13}^d - \rho_{13} \\
    \rho_{23}^d - \rho_{23}
\end{pmatrix};$$

therefore, $\dot{e} = -\dot{x} = -r$.

Let us propose the following Lyapunov function candidate $V_1 = e^T \frac{e}{2}$, then

$$\dot{V}_1 = e^T \dot{e} = -e^T r$$

$$= -\left(k_1|\rho_{12}^d - \rho_{12}| + k_2|\alpha_{12}^d - \alpha_{12}| + k_3|\rho_{13}^d - \rho_{13}| + k_4|\rho_{23}^d - \rho_{23}| \right) \leq 0.$$  

Therefore, $e \to 0$ as $t \to 0$. Moreover, from the definition of $u$, we have that

$$w_2 = \frac{\sin \beta_{12}}{d} \left(k_1 \text{sgn}(\rho_{12}^d - \rho_{12}) + v_1 \cos \alpha_{12}\right)$$

$$+ \frac{\rho_{12} \cos \beta_{12}}{d} \left(k_2 \text{sgn}(\alpha_{12}^d - \alpha_{12}) - \frac{v_1 \sin \alpha_{12}}{l_{12}} + w_1\right).$$

Let us define $e_{\theta_2} = \theta_1 - \theta_2$; therefore, $\dot{e}_{\theta_2} = w_1 - w_2$ and after some algebraic and trigonometric manipulation

$$\dot{e}_{\theta_2} = -\frac{v_1 \sin \theta_{\theta_2}}{d} + \eta_1(e_{\theta_2}, w_1, \rho_{12}, \alpha_{12})$$

with

$$\eta_1(e_{\theta_2}, w_1, \rho_{12}, \alpha_{12})$$

$$= w_1 \left(1 - \frac{\rho_{12} \cos(e_{\theta_2} + \alpha_{12})}{d}\right)$$

$$- \frac{k_1 \text{sgn}(\rho_{12}^d - \rho_{12}) \sin(e_{\theta_2} + \alpha_{12})}{d}$$

$$- \frac{\rho_{12} k_2 \text{sgn}(\alpha_{12}^d - \alpha_{12}) \cos(e_{\theta_2} + \alpha_{12})}{d},$$

which can be interpreted as a nominal system affected by a perturbation $\eta_1(e_{\theta_2}, w_1, \rho_{12}, \alpha_{12})$. It is easy to see that the nominal system $\dot{e}_{\theta_2} = -\frac{v_1 \sin \theta_{\theta_2}}{d}$ is asymptotically stable if $|\theta_{\theta_2}| < \pi$.

Since by assumption $|v_1| < \omega_{max}$, $\eta_1(e_{\theta_2}, w_1, \rho_{12}, \alpha_{12}) < \delta_1$. From Khalil (2002), if $|e_{\theta_2}(t_0)| < \epsilon_1 \pi$, with
$c_1 < 1$, then

$$|e_{\theta_1}(t)| \leq \gamma_1, \ t \geq t_1 \text{ for some } t_1 < \infty \text{ and } \gamma_1 \in \mathbb{R}.$$ 

Similarly, if we define, $e_{\theta_1} = \theta_1 - \theta_3$, then $\dot{e}_{\theta_1} = w_1 - w_3$. Following a similar procedure to that followed in Fierro et al. (2002), after some manipulation it gives

$$\dot{e}_{\theta_1} = -\frac{v_1 \sin e_{\theta_1}}{d} + \eta_2(e_{\theta_1}, w_1, \rho_{13}, \rho_{23}, v_1, v_2, \theta_1, \theta_2)$$

which, again, can be interpreted as a nominal system affected by a perturbation $\eta_2(e_{\theta_1}, w_1, \rho_{13}, \rho_{23}, v_1, v_2, \theta_1, \theta_2)$. As before, the nominal system $\dot{e}_{\theta_1} = -\frac{v_1 \sin e_{\theta_1}}{d}$ is asymptotically stable if $|e_{\theta_1}| < \pi$.

Since $|w_1| < w_{\text{max}}, |v_1 - v_2| < \varepsilon_1$ and $|\theta_1 - \theta_2| < \epsilon_2$, after some algebraic and trigonometric operations we conclude that $\eta_2(e_{\theta_1}, w_1, \rho_{13}, \rho_{23}, v_1, v_2, \theta_1, \theta_2) \leq \delta_2$. Therefore, if $|e_{\theta_1}| < c_2 \pi$ with $c_2 < 1$, $|e_{\theta_1}(t)| \leq \gamma_2, \ t \geq t_2$ for some $t_2 < \infty$ and $\gamma_2 \in \mathbb{R}$. □

Remark 1: Notice that the system is stable with just the sign of the differences in vector $\mathbf{r}$ defined in (7). This motivates the simplification of the calculation of the control law (6) by estimating the sign of these differences. One possibility is to reduce the precision used to quantize $\rho_{13}, \rho_{23}, \rho_{d_1}$ and $\rho_{d_2}$. This would compress the information of the position that the follower requires to carry out the calculations for the control law.

Remark 2: Even though the compression of the variables $\rho_{12}$ and $\alpha_{12}$ is suggested by Lemma 2.1, it is unrealistic. Notice that $\rho_{12}$ and $\alpha_{12}$ are required in the control law (6) to calculate the inverse of the matrix $\mathbf{A}$ and to calculate the vector $\mathbf{b}$; therefore, these variables cannot be compressed without having a significant effect on the behaviour of the system.

Remark 3: From the matrix $\mathbf{A}$ and the vector $\mathbf{b}$, notice that the variables that should be known by the followers $\mathbf{R}_2$ and $\mathbf{R}_3$ to calculate the control law in (6) are $\rho_{12}, \rho_{13}, \rho_{23}, \alpha_{12}, \alpha_{13}, \alpha_{23}, \theta_1, \theta_2, w_1$ and $v_1$. By this approach, we are theoretically motivated to compress only $\rho_{13}$ and $\rho_{23}$, which, as we show in the simulation section, might provide a significant reduction on the required amount of shared data.

3. Case study 2: coverage control with compressed communication

In the work presented in Luna, Fierro, Abdallah, and Lewis (2012a) and Luna, Fierro, Abdallah, and Wood (2012b), we study a coverage control for a team of non-holonomic vehicles estimating a parameterised distributed density function by using Voronoi tessellations. Each agent communicates with its neighbour to share the estimated parameters of the distributed density function and reach consensus. We have incorporated binary consensus (Chen & Lewis, 2011) which, similarly to the formation control case in Section 2, allows the agents to reach consensus by requiring only the sign of the difference between the parameters estimated by an agent and by its neighbours, rather than the exact value of the difference. In the following paragraphs, we review some important definitions and theoretical results that support our study.

3.1 Mathematical definitions

Definition 3.1: Based on Du and Gunzburguer (1999), given an open set $Q \subseteq \mathbb{R}^N$, the set $\{V_i\}_{i=1}^k$ is called a Voronoi tessellation of $Q$ if $\bigcap_{i=1}^k V_i = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^k V_i = Q$. Given a set of points $\{p_i\}_{i=1}^k$, belonging to $Q$, the Voronoi region $V_i$ corresponding to the point $p_i$ is defined by

$$V_i = \{ x \in Q \mid \| x - p_i \| < \| x - p_j \| \text{ for } i, j = 1, \ldots, k, j \neq i \},$$

where $\| \cdot \|$ denotes the Euclidean norm on $\mathbb{R}^N$. The points $\{p_i\}_{i=1}^k$ are called generator points.

Definition 3.2: Based on Bullo, Cortés, and Martinez (2009), let us define $Q \subseteq \mathbb{R}^N$ as a convex polytope including its interior. By defining a distributed density function as a mapping $\varphi(q) : Q \mapsto \mathbb{R}_+$ with $q \in Q$, the locational optimisation function is defined as

$$H_{V_i}(P) = \sum_{i=1}^n \int_{V_i} f(\|q - p_i\|)\varphi(q)dq,$$

where $P$ is the set of all the $n$ generator points $\{p_1, \ldots, p_n\} \in Q$ and $V_i$ is the Voronoi cell of the $i$th generator point.

Definition 3.3: The centre of mass $C_{V_i}$ of the $i$th Voronoi partition $V_i$ over a distributed density function $\varphi(q) : Q \mapsto \mathbb{R}_+$ with $q \in Q$ is given by

$$C_{V_i} = \frac{\int_{V_i} q\varphi(q)dq}{M_{V_i}},$$

where $M_{V_i} = \int_{V_i} \varphi(q)dq$ is called the mass of the $i$th Voronoi partition.

Definition 3.4: Based on Cortés et al. (2004), by defining $f(\|q - p_i\|) = \|q - p_i\|^2$ in (10), and by applying a partial derivative with respect to $p_i$, we get

$$\frac{\partial H_{V_i}(P)}{\partial p_i} = \int_{V_i} \frac{\partial}{\partial p_i} f(\|q - p_i\|)\varphi(q)dq$$

$$= 2M_{V_i}(p_i - C_{V_i}).$$
Making (12) equal to zero, we get that \( \mathbf{p}_i = C_{V_i} \). All the Voronoi tessellations in \( Q \) where the generator points are at the same time the centroids of their Voronoi cells minimise (10). These types of tessellations are known as centroidal Voronoi tessellations (Du & Gunzburguer, 1999).

**Definition 3.5:** If the distributed density function depends on the time \( t \in \mathbb{R}_+ \), i.e. \( \varphi = \varphi(t, q) \), then we denote it as a time-varying distributed density function. Otherwise, i.e. \( \varphi = \varphi(q) \), we denote it as a time-invariant distributed density function.

### 3.2 Mathematical assumptions

Let us state some assumptions based on Schwager et al. (2009) and Luna et al. (2012b).

**Assumption 3.6:** There exists a parameter vector \( \mathbf{a} \in \mathbb{R}^m \) and a vector function \( \mathbf{W} : Q \mapsto \mathbb{R}_+^m \) such that

\[
\varphi(q) = \mathbf{W}(q)^T \mathbf{a},
\]

where \( m \in \mathbb{N} \), and \((\cdot)^T\) denotes transpose.

**Remark 4:** The vector function \( \mathbf{W}(q) \) is given to the agents, but the parameter vector \( \mathbf{a} \) is not. To simplify the notation, we use \( \mathbf{W}(q) \equiv \mathbf{W} \) from now on.

**Assumption 3.7:** Let us denote \( a_j \) as the \( j \)th entry of the parameter vector \( \mathbf{a} \), and given a constant value \( \beta \in \mathbb{R}_+ \),

\[ a_j \geq \beta, \quad \forall j = 1, \ldots, m. \]

**Remark 5:** The lower bound \( \beta \) for the parameter vector \( \mathbf{a} \) in Assumption 3.7 avoids \( \mathbf{W}^T \mathbf{a} = \varphi(q) = 0 \) leading to a zero in the denominator of (11).

**Assumption 3.8:** Let us denote the \( j \)th entry of the vector function \( \mathbf{W} \) as \( \mathbf{W}_j \). Assume it satisfies

\[ \mathbf{W}_j > 0 \quad \forall j = 1, \ldots, m. \]

Assumption 3.6 is a rational approach to model some phenomena and chemical concentrations such as concentrations of liquids, light, temperature or magnetic fields. In these cases, the density functions can be approximated by a linear combination of bidimensional bell-shaped functions, whose amplitude is determined by the entries of the parameter vector \( \mathbf{a} \). Some functions that comply with Assumption 3.8 are the well-known bidimensional logistic, Gaussian, Laplacian, Cauchy and Gumbel probability density functions.

### 3.3 Polar unicycle model and control law

For this application, we use the model proposed in Aicardi, Casalino, Bicchi, and Balestrino (1995) which for the \( i \)th agent is given by

\[
\begin{pmatrix}
\dot{\rho}_i \\
\dot{\alpha}_i \\
\dot{\phi}_i
\end{pmatrix} = \begin{pmatrix}
-u_i \cos \alpha_i \\
u_i \sin \alpha_i - w_i \\
\rho_i \frac{u_i \sin \alpha_i}{\rho_i}
\end{pmatrix}.
\]

The position of the \( i \)th agent in its Voronoi cell \( V_i \) is given in polar coordinates, as shown in Figure 2. The angle \( \alpha_i \) is given by \( \alpha_i = \phi_i - \theta_i \), the heading angle of the agent is denoted by \( \theta_i \), and the Euclidean distance between the agent and the estimate of the centroid of \( V_i \) is denoted by \( \rho_i \). The variables \( u_i, w_i \in \mathbb{R} \) are the linear and angular speeds of the \( i \)th agent, respectively.

Finally, in Luna et al. (2012b) we propose the following nonlinear control law:

\[
\begin{pmatrix}
u_i \\
w_i
\end{pmatrix} = \begin{pmatrix}
k_1 \cos \alpha_i \rho_i \\
2k_1 \sin \alpha_i \cos \alpha_i + k_2(\alpha_i + \phi_i)
\end{pmatrix},
\]

where \( k_1, k_2 \in \mathbb{R}_+ \) are control gains.

### 3.4 Adaptive law and binary consensus

We assume that the agents are able to share their estimate of the parameter vector \( \mathbf{a} \), i.e. \( \hat{\mathbf{a}}_i, \forall i = 1, \ldots, n \) with their neighbours. Then, the communication between agents is modelled as an undirected graph, where the sensors might be interpreted as a set of indexed vertices \( V_e = \{v_1, \ldots, v_n\} \) and the links between them as a set of edges \( E = \{e_1, \ldots, e_l\} \), where \( e_l = \{v_j, v_k\} \) with the corresponding symmetric

![Figure 2. Three unicycle vehicles in their respective Voronoi cells. Their respective estimated centroids are indicated by the stars, and the polar coordinates are explicitly indicated for the \( i \)th vehicle.](image)
adjacency matrix $A$. The neighbourhood of the $i$th agent is formally defined as

$$\mathcal{N}_i = \{ j | (v_i, v_j) \in E \}. \quad (16)$$

We propose the following adaptation law to estimate the parameter vector $a_i$:

$$\dot{\hat{a}}_i = \Gamma_2^{-1}(I - I_{\text{proj}})\hat{a}_{p_i}, \quad (17)$$

$$\dot{\hat{a}}_{p_i} = -\xi W W^T \hat{a}_i - \xi \sum_{j \in \mathcal{N}_i} \text{sgn}(\hat{a}_i - \hat{a}_j), \quad (18)$$

where $W$ corresponds to the vector function in (13) and $\Gamma_2^{-1} \in \mathbb{R}^{n \times n}$ is a non-singular positive-definite matrix. $\xi, \gamma \in \mathbb{R}_+$ are the control gains, and $I_{\text{proj}} \in \mathbb{R}^{n \times n}$ is a diagonal matrix that implements a projection law to prevent $\hat{a}_i$ from taking values less than the lower bound $\beta$ (Schwager et al., 2009), and is given by

$$I_{\text{proj}}(j) = \begin{cases} 0 & \text{for } \hat{a}_i(j) > \beta, \\ 0 & \text{for } \hat{a}_i(j) = \beta \text{ and } \hat{a}_{p_i} \geq 0, \\ 1 & \text{otherwise}, \end{cases} \quad (19)$$

where the index $j$ denotes the $j$th diagonal entry of the matrix $I_{\text{proj}}$. $I \in \mathbb{R}^{n \times n}$ is the identity matrix.

The first term in (18) consists of a gradient estimator (Slotine & Li, 1991) and the second term implements a binary consensus protocol (Chen & Lewis, 2011). Notice that all that is needed in this protocol is the sign of the difference between the estimate of the parameter vectors $\hat{a}_i$ and $\hat{a}_j$.

Given $\hat{a}_i$, the estimate of the distributed density function of the $i$th agent is given by $\hat{\phi}_i = W^T \hat{a}_i$ and the estimate of the centre of mass of its Voronoi cell is $\hat{C}_{V_i} = \int_{v_i \in \mathcal{V}} q v d q / \int_{v_i \in \mathcal{V}} q d q$.

Moreover, let us define the parameter error vector as $\tilde{a}_i = \hat{a}_i - a_i$.

### 3.5 Stability analysis

The following lemma from Luna et al. (2012a) incorporates the binary consensus into the coverage control presented in Luna et al. (2012b).

**Lemma 3.9:** Given a system of $n$ non-holonomic unicycle vehicles with dynamics given by (14), feedback control law (15) and adaptive control law (17) and (18), and if Assumptions 3.6 and 3.7 are satisfied, then

$$\lim_{t \to \infty} \| \hat{a}_i - \hat{a}_j \| = 0, \quad \forall i, j \in \mathcal{I}_n,$$

$$\lim_{t \to \infty} W(p_i)^T \hat{a}_i = 0, \quad \forall i \in \mathcal{I}_n,$$

$$\lim_{t \to \infty} \rho_i, |\alpha_i|, |\phi_i| = 0, \quad \forall i \in \mathcal{I}_n,$$

with $\mathcal{I}_n = \{ 1, \ldots, n \}$.

**Proof:** Let us define the variable $m_i$ as

$$m_i = p_i - \hat{C}_{V_i} = -\left( \rho_i \cos(\theta_i + \alpha_i) \right);$$

therefore,

$$\dot{m}_i = \dot{p}_i = \left( \begin{array}{c} u_i \cos \theta_i \\ u_i \sin \theta_i \end{array} \right) \text{ and } \|m_i\| = \rho_i.$$

By defining the state vector for the $i$th agent as

$$z_i = (m_i^T, \tilde{a}_i^T, (\alpha_i, \phi_i))^T,$$

we propose the following Lyapunov function candidate:

$$V(t, z_1, \ldots, z_n) = \sum_i z_i^T \left( \begin{array}{ccc} \Gamma_1 & 0 & 0 \\ 0 & \Gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{array} \right) z_i,$$

where $\Gamma_1 = \text{diag}(\gamma_1, \gamma_1) \in \mathbb{R}^{2 \times 2}$ and $\Gamma_2 \in \mathbb{R}^{n \times n}$ are positive-definite matrices and $\gamma_1, \gamma_3 \in \mathbb{R}_+$ are constants.

Then,

$$\dot{V}(t, z_1, \ldots, z_n) = \sum_i m_i^T \Gamma_1 m_i + \tilde{a}_i^T \Gamma_2 \tilde{a}_i + \gamma_3 (\alpha_i \dot{\alpha}_i + \phi_i \dot{\phi}_i + \alpha_i \dot{\phi}_i + \phi_i \dot{\alpha}_i). \quad (20)$$

Substituting (17) and (18) in (20), we get

$$\dot{V}(t, z_1, \ldots, z_n) = \sum_i m_i^T \Gamma_1 m_i - \xi \tilde{a}_i^T W W^T \hat{a}_i - \tilde{a}_i^T I_{\text{proj}} \hat{a}_{p_i} - \xi \sum_{j \in \mathcal{N}_i} \text{sgn}(\hat{a}_i - \hat{a}_j) \quad (21)$$

The first, second, third and fourth terms in the summation in (21) have already been proven to be negative semidefinite in Luna et al. (2012b). The fifth term in (21) has been proven to be negative semidefinite in Luna et al. (2012a) and the proof, based on Chen and Lewis (2011), goes as follows:

$$-\xi \sum_i \sum_{j \in \mathcal{N}_i} \tilde{a}_i^T \text{sgn}(\hat{a}_i - \hat{a}_j) = -\frac{1}{2} \xi \sum_i \sum_j A_{ij} \tilde{a}_i^T \text{sgn}(\hat{a}_i - \hat{a}_j) + \frac{1}{2} \xi \sum_i \sum_j A_{ij} \tilde{a}_i^T \text{sgn}(\hat{a}_i - \hat{a}_j).$$
\[-\frac{1}{2} \xi \sum_{i \in N_i} A_{ij}(\hat{a}_i - \hat{a}_j)^T \text{sgn}(\hat{a}_i - \hat{a}_j) \]
\[-\frac{1}{2} \xi \sum_{i \in N_i} \|\hat{a}_i - \hat{a}_j\|_1 \]
\[-\frac{1}{2} \xi \sum_{i \in N_i} \|\hat{a}_i - \hat{a}_j\|_\infty \Rightarrow 0, \]
\[-\frac{1}{2} \xi \sum_{i \in N_i} \|\hat{a}_i - \hat{a}_j\|_\infty \leq 0, \]

where \(A_{ij}\) denotes the entry of the adjacency matrix \(A\) at the position \(i, j\), and \(N_i\) is given by (16).

Since \(V(t, z_1, \ldots, z_n)\) is lower bounded and \(\dot{V}(t, z_1, \ldots, z_n) \leq 0\) and is uniformly continuous, then by Barbalat’s lemma, \(\rho_i \to 0, |\alpha_i| \to 0, |\phi_i| \to 0, W^T(p_i)\tilde{a} \to 0\) and \(\|\hat{a}_i - \hat{a}_j\| \to 0\) as \(t \to \infty\).

Now, suppose that the distributed density function has time-varying parameters, then \(\hat{a}_i = \tilde{a}_i - \hat{a}_i\), and by defining the state vector, \(\dot{z}_i = (\rho_i, \alpha_i, \phi_i, \hat{a}_i^T)^T\), the kinematics of the system become

\[\dot{z}_i = (\rho_i, \alpha_i, \phi_i, \hat{a}_i^T)^T + (0 \ 0 \ 0 \ -\hat{a}_i^T)^T.\]

We present the following lemma from Luna et al. (2012a) for time-varying distributed density functions.

**Lemma 3.10:** For a system of \(n\) non-holonomic agents covering a distributed density function \(\varphi = W^T a\) subject to Assumptions 3.6–3.8, suppose that the kinematics of each agent are given by (14) and (22), and that the parameter vector \(a\) is time-varying with its time-derivative satisfying \(\|\dot{a}\| < \delta, \delta \in \mathbb{R}_+\). Moreover, assume that the estimation error of the \(i\)th agent satisfies

\[\hat{a}_i^T(I - I_{proj})WW^T\tilde{a}_i \leq \hat{a}_i^T(I - I_{proj}) \sum_{j \in N_i} \text{sgn}(\hat{a}_i - \hat{a}_j),\]

then we have that if each agent is driven by the nonlinear control law (15) and the adaptive law (17) and (18), and \(|\alpha_i| \leq \frac{\pi}{2}\), then the kinematics of the \(i\)th agent are globally uniformly ultimately bounded (GUUB) with ultimate bound \(b\) given by

\[b = \frac{\lambda_{\text{max}}(B) \lambda_{\text{max}}(\Gamma_1 \Gamma_2 \delta)\xi}{\sqrt{\lambda_{\text{min}}(\Gamma_1) \lambda_{\text{min}}(\Gamma_3)\delta}},\]

where \(\lambda_{\text{max}}(B)\) and \(\lambda_{\text{min}}(B)\) represent the maximum and minimum eigenvalues of the matrix \(B\), respectively, and \(0 < \theta < 1, \theta \in \mathbb{R}_+\). Moreover,

\[\Gamma = \begin{pmatrix} \Gamma_1 & 0 & 0 \\ 0 & \Gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix},\]

where \(\Gamma_1 = \text{diag}(\gamma_1, \gamma_1) \in \mathbb{R}^{2 \times 2}\) and \(\Gamma_2 \in \mathbb{R}^{n \times n}\) is the inverse of the gain matrix of the adaptive law in (17) and (18). \(\gamma_1, \gamma_3 \in \mathbb{R}_+\) are constant. \(\Gamma_3\) is defined as

\[\Gamma_3 = \begin{pmatrix} \frac{n k_1}{2} & 0 & 0 \\ 0 & (\xi + \delta)(I - I_{proj})WW^T & 0 \\ 0 & 0 & \gamma_3 k_2 \end{pmatrix},\]

where \(k_1, k_2 \in \mathbb{R}_+\) are the gains in (15), \(W\) is the vector function in (13), \(I\) is an identity matrix and \(I_{proj}\) is defined by (19).

**Proof:** For a proof, see Luna et al. (2012a).

**Remark 6:** From Lemma 3.9 both sides of the inequality in (23) converge to zero as \(t \to \infty\) and (23) is fulfilled whenever the parameter estimation error of the \(i\)th agent is ‘closer’ to zero compared to the measure of the consensus error. Moreover, since \(\sum_{j \in N_i} \text{sgn}(\hat{a}_i - \hat{a}_j) \geq 0\), we can assert that the parameter estimate of the \(i\)th agent will remain GUUB if, roughly speaking, its neighbours have a ‘better’ estimate of the distributed density function.

**Remark 7:** Assumption 3.8 makes sure that the outer product \(WW^T\) is positive definite. This is a necessary condition to guarantee that the ultimate bound is finite since the minimum eigenvalue of the block diagonal matrix \(\Gamma_3\), namely \(\lambda_{\text{min}}(\Gamma_3)\), is in the denominator of (24).

### 4. Simulation results

In this section, we present a set of simulations to motivate the compression of the information exchanged in the formation control described in Section 2 and the coverage control presented in Section 3. Notice that what makes these analyses relevant is that in both cases the vehicles are assumed to be non-holonomic, and the systems are highly nonlinear.

#### 4.1 Case study 1: formation control with compressed communication

In this simulation in Matlab, we assume that \(R_1\), which is the main leader, is moving at a constant linear speed \(v_1 = 0.5\) and at an angular speed given by \(w_1(t) = 0.1 \sin(0.2t)\) which produces a devious trajectory to test the formation. The initial positions of the agents in the Cartesian plane are \((x_1(0), y_1(0), \theta_1(0))^T = (2, 1, 0)^T\) for \(R_1\), \((x_2(0), y_2(0), \theta_2(0))^T = (2, 2.5, -120)^T\) for \(R_2\) and \((x_3(0), y_3(0), \theta_3(0))^T = (0.64, 1.5, 150)^T\) for \(R_3\). The desired distances are \(\rho_{12} = \rho_{13} = \rho_{23} = 1\), the desired bearing is \(\alpha_{12} = 90^\circ\) and \(d = 0.1\). The control gains in (7) were chosen to be \(k_1 = k_2 = k_3 = k_4 = 0.5\).

We run simulations of the closed-loop system of the formation control (1)–(5) with the control law (6). On each simulation, we vary the precision to quantise \(\rho_{13}\) and \(\rho_{23}\) using 2, 4, 6 and 10 bits.
Figure 3. Error $e_3 = \rho_{13}^t - \rho_{13}$ obtained by implementing the control law (6) with $r$ given by (7). Notice that as we increase the precision, the error decreases, and only with 10 bits the error is $e_3 \approx -0.0019$, in contrast to the 64 bits of double precision.

Figure 4. Error $e_4 = \rho_{23}^t - \rho_{23}$ obtained by implementing the control law (6) with $r$ given by (7). Notice that as we increase the precision, the error decreases, and only with 10 bits the error is $e_4 \approx -0.0017$, which for some practical purposes might be acceptable. The total amount of information required by the control law involves a total of 640 bits (the 10 required variables mentioned in Remark 3 in double precision) and if the errors $e_3$ and $e_4$ obtained with 10 bits are acceptable, then we can reduce the amount of bits to 532, which gives a reduction of 16.9%. This reduction might be significant for some problems.

In Figure 5, we illustrate the response of the error $e_{\theta_{13}}$, which, as indicated by the perturbation function in (9), depends on $\rho_{13}$ and $\rho_{23}$. Notice that as we increase the precision of $\rho_{13}$ and $\rho_{23}$ the bound interval of the error gets narrower, and the system remains stable.

4.2 Case study 2: coverage control with compressed communication

Our implementation in Matlab uses a population of three unicycle vehicles arbitrarily distributed over a sample space $Q$ which corresponds to a unit square. The controller gains in (15) are $k_1 = 3$ and $k_2 = 1$. The parameters of the adaptive law (17) and (18) are $\Gamma_2 = I_{16}$, $\zeta = 1000$ and $\xi = 0.5$, where $I_{16} \in \mathbb{R}^{16 \times 16}$ is an identity matrix. The lower bound defined in Assumption 3.7 is $\beta = 0.1$. All these simulation parameters were calculated according to the assumptions of Lemma 3.9.

The sampling space $Q$ has been divided into a $4 \times 4$ grid where the geometric centre of every square cell determines the mean $\mu_i$ of a bidimensional Gaussian function. The $i$th entry $W_i$ of the vector function $W$ is given by

$$W_i = e^{-\frac{(q - \mu_i)^2}{2 \sigma_i^2}},$$

with $\sigma_i^2 = 0.05$.

We use the team of unicycle vehicles to cover and estimate a distributed density function with a dynamically
expanding circular shape where the highest density is localised at the boundary of the circle. For a detailed description about the parameterisation of the dynamic distributed density function, see section 5.1 of Luna et al. (2012a).

In the following simulations, and based on Lemmas 3.9 and 3.10, we implement a binary consensus protocol with the following precisions for \( \hat{a}_i \), namely (a) 1 bit, (b) 3 bits and (c) 5 bits. Now, let us define

\[
\bar{a}(t) = \frac{1}{n} \left( \sum_{i=1}^{n} \hat{a}_i(t) \right) \quad \forall t > 0,
\]

which is the averaged parameter error vector over all the mobile sensors. In Figure 6, we plot \( W^T \bar{a}(t) \). Furthermore, in Figure 7, we plot the function

\[
\bar{\rho}(t) = \frac{1}{n} \left( \sum_{i=1}^{n} \rho_i(t) \right) \quad \forall t > 0,
\]

which is the averaged position error of all the robots in \( Q \). Notice that, as expected from Lemmas 3.9 and 3.10, \( W^T \bar{a}(t) \) and \( \bar{\rho}(t) \) remain close to the origin as \( t \to \infty \).

The angle \( \alpha_i \) is affected by the noise induced by the approximation by Riemann summations that we carry out to calculate the centroid of the Voronoi partitions defined by (11). Therefore, in Figure 8, we have plotted \( \alpha_i \) for one agent randomly chosen. Even though the asymptotic convergence is not evident, notice that \( \alpha_i(t) \) oscillates around zero as \( t \to \infty \).

The plots in Figures 6–8 validate the mathematical results given in Lemmas 3.9 and 3.10. However, the precision to quantise \( \hat{a}_i \) for the \( i \)th agent does not affect the convergence of \( \rho_i, \alpha_i \) nor \( W(p_i)^T \bar{a}_i \). It only has an effect on the convergence of the consensus error. The quantity \( c_a(t) \) given by

\[
c_a(t) = \sum_{i=1}^{n} \sum_{j \in N_i} \| \hat{a}_j(t) - \hat{a}_i(t) \| \quad \forall t > 0
\]

provides a measurement of the consensus error and is plotted in Figure 9. Notice that the averaged consensus error improves as the precision to quantise the estimate of the parameter vector \( \hat{a}_i \) increases. Notice that with 5 bits the consensus error is in the neighbourhood of the origin as expected from the lemmas. If this error is acceptable, the reduction in the amount of required bits would be of 92.2%.
Figure 9. Plot of $c_e(t)$ for different precisions of $\hat{a}$. Notice that with 5 bits, in contrast to the 64 bits for double precision, the $c_e(t)$ remains close to the origin.

5. Conclusions

We have presented a control systems approach that allows for the compression of environmental and locational information on two different multi-vehicle problems, namely a formation control and a coverage control with consensus. This approach exploits situations where the controller needs access to the difference between the outputs of the system and a reference value, or between two different outputs. Under the appropriate assumptions, the exact difference may be replaced by the sign of the difference while guaranteeing that the system remains stable. The precision of the terms involved can be reduced to provide an estimate of the sign of the difference, thus minimising the amount of information required for the computation.

The simulation results of the formation control reflect that a reduction in the amount of shared information between agents can always be obtained by this approach. Even though the particular model of the formation control allows the compression of just two out of the 10 variables shared by the agents, the simulation results suggest that significant compression rates can be achieved even with the given trade-off on the regulation error of the compressed variables. The simulation results of the coverage control are more encouraging, since the amount of data of the shared information among the agents can be reduced drastically with a better trade-off on the consensus error. The simplicity of the approach makes this methodology implementable for fairly complex nonlinear systems. It is interesting to note that the proposed solution, and the tools developed within, might be useful even for systems with quantised measurements, although the general performance of the system will decrease.

One of the drawbacks of this approach is not only that the stationary errors increase with the compression, but as reflected in the formation control example, if the error is not zero but is bounded to an interval, such interval might be affected negatively by the quantisation error. Another issue is that some or all outputs of the system involved in the sign operation cannot always be compressed, since their exact value might be required in other calculations associated with the control law, as shown in the formation control.

A rigorous analysis that relates the quantisation error to the stationary errors of the system is part of our future research agenda. Other case studies such as boundary tracking, rendezvous problems and flocking are good candidates to be studied through this approach.

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References


