TCOM 370 NOTES 99-2B

FOURIER SERIES (PART II)

1. AMPLITUDE AND PHASE SPECTRUM OF PERIODIC WAVEFORM

We have discussed how for a periodic function x(t) with period T and fundamental frequency $f_0=1/T$, the ***Fourier series*** is a representation of the function in terms of sine and cosine functions as follows:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t) + \sum_{n=1}^{\infty} b_n \sin(2\pi n f_0 t) \qquad (1)$$

Here the a_n and b_n are coefficients defined as integrals in terms of the specific x(t).

[As an example, we considered the periodic rectangular pulse train v(t) of width-τ pulses repeated every T sec., for which the fundamental frequency is $f_0=1/T$. We obtained for it the result that the "dc" or average value $a_0=\frac{A\tau}{T}$ and $a_n = \frac{2A}{n\pi}\sin(\pi n f_0 \tau)$. For this example, $b_n=0$ for all n.]

- Note that a cosine term $\mathbf{a_n\cos(2\pi n f_0 t)}$ and a sine term $\mathbf{b_n\sin(2\pi n f_0 t)}$ (*of the same frequency* nf_0) may be viewed as a ***single*** cosine waveform of frequency nf_0 :

$$a_n\cos(2\pi n f_0 t) + b_n\sin(2\pi n f_0 t) = A_n \cos(2\pi n f_0 t + \phi_n) \qquad (2)$$

where the amplitude $A_n=\sqrt{a_n^2+b_n^2}$ and the phase angle $\phi_n = -\tan^{-1}(\frac{b_n}{a_n})$.

This follows easily from the identity

$$\cos(2\pi n f_0 t + \phi_n) = \cos(\phi_n)\cos(2\pi n f_0 t) - \sin(\phi_n)\sin(2\pi n f_0 t),$$

because $\phi_n = -\tan^{-1}(\frac{b_n}{a_n})$ implies that $\sin(\phi_n) = \frac{-b_n}{\sqrt{a_n^2 + b_n^2}}$ and $\cos(\phi_n) = \frac{a_n}{\sqrt{a_n^2 + b_n^2}}$ (consider a right triangle with sides $-b_n$, a_n, and $\sqrt{a_n^2 + b_n^2}$)

Thus we may alternatively write Eq.(1) as

$$x(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(2\pi n f_0 t + \phi_n) \qquad (3)$$

- In any Fourier series for a real periodic function, each pair of a_n and b_n coefficients leads to a single cosine with **frequency nf_0.** The **phase ϕ_n** of each cosine may be different, just as the **non-negative amplitudes A_n** are generally different. The phase relationships are important because they correspond to having different amounts of **"time shifts" or "delays"** for each of the sinusoidal waveforms relative to a zero-phase waveform.

Illustrating the importance of ***phase***, in the figure below are shown two waveforms,

$$x_1(t) = \cos(2\pi 2 f_0 t) - 0.5\cos(2\pi 3 f_0 t + \pi/4)$$

$$= \cos(2\pi 2 f_0 t) + 0.5\cos(2\pi 3 f_0 t + 5\pi/4)$$

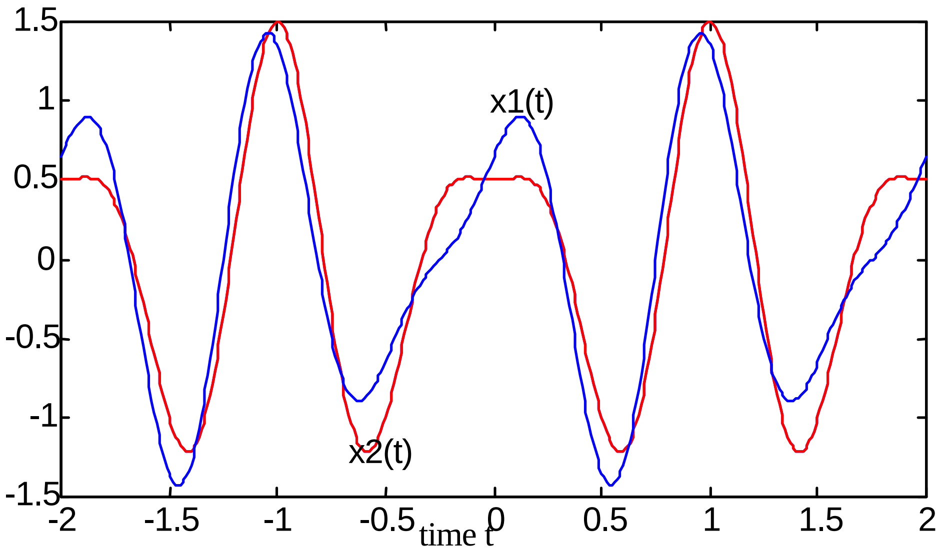
and

$$x_2(t) = \cos(2\pi 2 f_0 t) - 0.5\cos(2\pi 3 f_0 t)$$

$$= \cos(2\pi 2 f_0 t) + 0.5\cos(2\pi 3 f_0 t + \pi)$$

with f_0= 0.5 Hz.

Thus each waveform is a combination of 1 Hz and 1.5 Hz cosine terms, with the same amplitudes (1 and 0.5) for each, but with the phase of the second cosine differing by $\pi/4$ in the two waveforms. The waveforms are significantly different from each other.

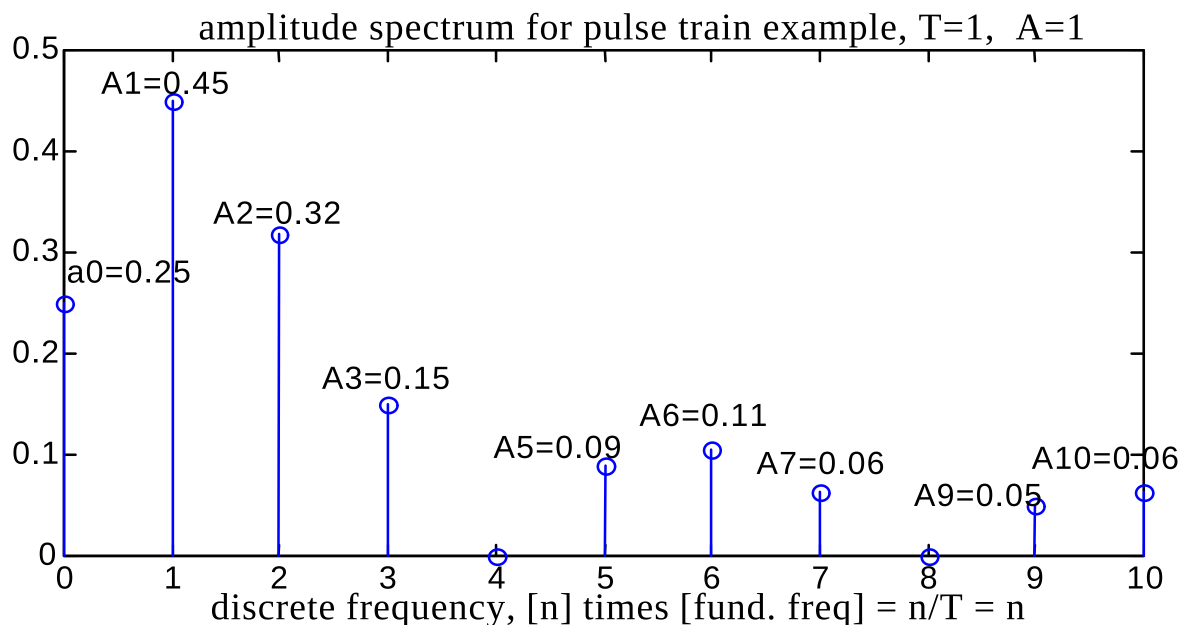


For any periodic waveform, the ***Fourier spectrum*** is the set of A_n coefficients together with their respective phases ϕ_n, where the index n=0,1,2,... of course corresponds to frequency nf_0. Note that the *dc term a_0* is the average value of the periodic function, and may be considered to be the zero-frequency (n=0) term; we define $A_0 = |a_0|$ (there is no b_0 term).

The ***amplitude spectrum*** refers only to the amplitudes A_n. It may be *plotted* as a function of n. Similarly, the ***phase spectrum*** is the phase ϕ_n as a function of n.

2. AMPLITUDE SPECTRUM OF RECTANGULAR PULSE TRAIN

Consider again the example of the *rectangular pulse train*, and let T=1, τ=0.25 and pulse amplitudes A=1. Then we have $A_n = \sqrt{a_n^2 + b_n^2} = \sqrt{a_n^2} = |a_n|$ $= \frac{2A}{n\pi} |\sin(\pi n f_0 \tau)| = \frac{2}{n\pi} |\sin(\pi n \frac{1}{4})|$ for n ≥ 1. Note that the *amplitude* A_n is always ≥ 0. Evaluating the A_n for this example, we get the following plot of A_n vs. n:



Such a plot helps us decide the highest frequency that we need for a good approximation to the original periodic function, in this case for the pulse train. From the plot above we may decide that beyond frequency 4 Hz, none of the Fourier coefficients have significant magnitude and may be neglected.

For this example, the term $|\sin(\pi n f_0 \tau)|$ is *always* between +1 and –1, whereas 2A/π is a constant. The $\frac{1}{n}$ part therefore makes A_n decrease in value with n.

In general, for index n beyond some integer N the A_n amplitudes remain small in magnitude and may be neglected to get a good finite-term representation of the periodic function. For such an approximation the highest frequency used is Nf_0. We say that the ***bandwidth*** of the periodic function is Nf_0.

Of course, the larger f_0 is the larger this highest frequency Nf_0 is that we need in the approximate representation of the periodic function. This also means that if we are transmitting the waveform over a communication link, the link has to be able to deliver to the receiver all frequencies between 0 and Nf_0 without significant change. We say that the channel ***bandwidth*** needs to be Nf_0.

We have already noted the importance of the phase values of each cosine or sine frequency in preserving the shape of the waveform. Thus in order to preserve the shape of a periodic waveform that is transmitted over some communication link, the channel has to be able to transmit ***each frequency within the signal bandwidth without significant attenuation (amplitude change) and with no significant phase shift (delay).***

3. SINGLE RECTANGULAR PULSE OF DURATION τ

In communication systems we are generally interested in transmitting sequences of some particular pulse shape, say with amplitudes that are different from pulse to pulse, rather than a periodic repetition of the pulse. We therefore have to consider what frequencies are present in representing a *single* pulse of duration τ.

- This situation is approached if we take our *periodic* pulse train and let the repetition period T go to ∞.

We already have the Fourier series representation of the pulse train with period T, so let's see what happens when we let T approach ∞.

For the rectangular pulse train, we have

$$a_n = \frac{2A}{n\pi} \sin(\pi n f_0 \tau)$$

$$= 2Af_0\tau \, \frac{\sin(\pi n f_0 \tau)}{\pi n f_0 \tau} \qquad \text{for } n \geq 1$$

Consider a fixed width τ for the pulse and let T approach ∞. As T becomes larger and f_0 becomes smaller, the product nf_0 becomes a multiple of a smaller and smaller quantity f_0. With n taking on all integer values starting from 1, the quantity nf_0 acts as a ***continuous*** variable between 0 and ∞ , because it actually takes on a whole range of very finely spaced values.

The amplitude spectrum for the pulse train is $|a_n| = 2A\tau f_0 \left|\frac{\sin(\pi n f_0 \tau)}{\pi n f_0 \tau}\right|$ as a function of the frequency nf_0; but nf_0 acts as a continuous variable f, so that the amplitude spectrum can be interpreted as the function $2A\tau f_0\left|\frac{\sin(\pi f \tau)}{\pi f \tau}\right|$ of f. Note that the part $2A\tau f_0$ is a constant, even though it becomes very small as f_0 decreases. (Each frequency has a very small amplitude, but then there are a very large number of individual frequencies present.) The *shape* of the amplitude spectrum is determined by the function $\left|\frac{\sin(\pi f \tau)}{\pi f \tau}\right|$.

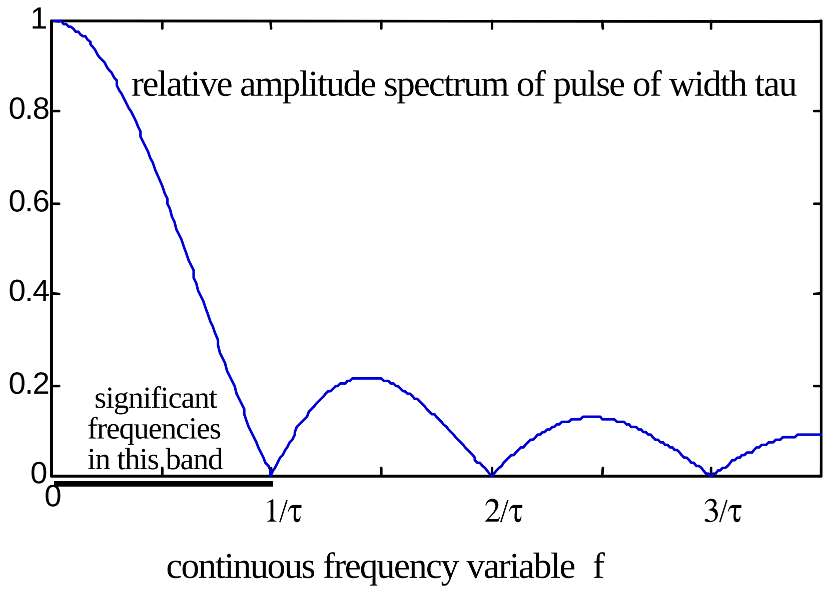
(We may also argue that the amplitude a_n is that of a sinusoid at frequency nf_0 and that since the frequency spacing is f_0, dividing the amplitude spectrum by f_0 gives us the amplitude *density* (per unit of frequency width). In this way we can remove the vanishingly small f_0 from the amplitude spectrum and get the amplitude spectral density.)

In any case, we find that

- the ***single rectangular pulse of width t*** contains ***all*** frequencies between 0 and ∞.

- the ***relative amplitudes*** (ignoring the overall amplitude factor) of these frequencies is given by the function $\left|\frac{\sin(\pi f \tau)}{\pi f \tau}\right|$.

The plot below depicts this function $\left|\frac{\sin(\pi f \tau)}{\pi f \tau}\right|$ as a function of f.



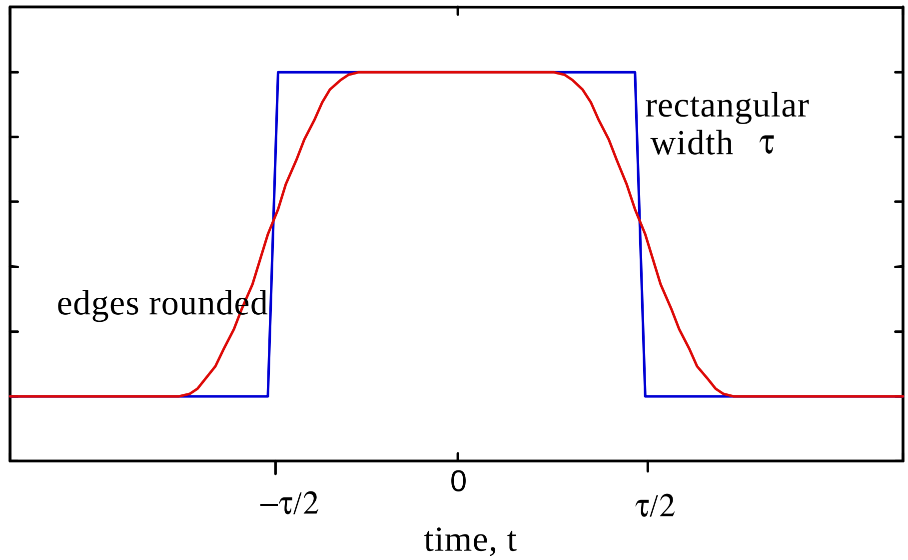
From this we may conclude that the highest frequency we need in the representation of a single rectangular pulse of width τ is approximately $\frac{1}{\boldsymbol{t}}$, because roughly speaking the significant frequencies are those below this limit (the others have relatively low amplitudes).

Now by packing duration-τ rectangular pulses right next to each other, and making their amplitudes take on values of +1 or -1 (or $+A$ and $-A$) in accordance with some data bit sequence, we can transmit a waveform which is a sequence of rectangular pulses with apparently random amplitudes. Each such pulse requires the channel to have a bandwidth of $W=\frac{1}{\tau}$ Hz. Note that the actual *pulse amplitude A* does not affect the relative *Fourier amplitude spectrum*, and a succession of pulses with random amplitudes will pass through a channel if the channel has sufficient bandwidth to pass any one of the pulses through.

- This leads to the idea that if we have a **channel model with bandwidth W**, then we may send **rectangular pulses** of minimum duration $\tau=\frac{1}{W}$ packed close to each other, i.e. **at a rate of** 1 pulse every $\frac{1}{W}$ sec. or **W pulses per second**, and reconstruct a good approximation of pulse amplitudes and the pulse sequence at the receiver. Thus the pulse amplitudes may be used to carry data bit values and provide a **data rate of W bits/sec**.

4. SINGLE ROUNDED-OFF PULSE

If we round-off the edges of the rectangular pulse of width τ, we get a pulse like that shown in the figure below.



The less abrupt rise and fall in this pulse leads to it having its Fourier amplitude spectrum more concentrated within 0 and $\frac{1}{\tau}$ on the frequency axis, with smaller components outside this range, compared to the rectangular pulse. Such pulses are desirable in keeping the transmitted power more tightly within the "bandwidth" of $\frac{1}{\tau}$. While there is some time-domain overlap in transmitting such pulses at rate $\frac{1}{\tau}$ pulses per sec., if the overlap is not too large then decisions about the individual pulse amplitudes in a train of pulses can still be made at the receiver.

- The **theoretical maximum "Nyquist" rate** at which we are able to transmit pulses over a channel with bandwidth W is **2W pulses per sec**., if we require that the pulse amplitudes be exactly recoverable by sampling at the receiver. This can be achieved with a very special type of non-rectangular "spread-out" pulse that is very hard to use in practice. This theoretical Nyquist pulse (not shown) is a special pulse with all its amplitude spectrum **strictly** within the limit of $\frac{1}{\tau}$, and if used at the rate of $\frac{1}{\tau}$ pulses/sec. this pulse overlaps with other pulses in such a way that individual pulse amplitudes can still be recovered **perfectly** (in theory!)

[5. Significance of Negative Frequencies

Each *real sinusoid* in the general Fourier Series representation may be written as a sum of *complex exponentials,* i.e.

$$A_n \cos(2\pi n f_0 t + \phi_n) = \frac{A_n}{2} \left[e^{j(2\pi n f_0 t + \phi_n)} + e^{-j(2\pi n f_0 t + \phi_n)}\right]$$

Thus each real cosine frequency nf_0 may be viewed in the domain of complex exponentials as being composed of a positive frequency nf_0 and a negative frequency $-nf_0$, each with an amplitude which is one-half of A_n (and corresponding phase angles ϕ_n and $-\phi_n$). We may therefore think of the Fourier spectrum for real periodic signals (or individual real pulses) as being symmetrically placed around the origin, with one-half the amplitudes A_n at both negative and positive values of each frequency.**]**