Stable, Scalable, Fair Congestion Control and AQM Schemes that Achieve High Utilization in the Internet

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Abstract

Virtual queue-based active queue management schemes have been proposed to provide low-loss, low-delay service in the Internet. In an earlier work, we had proposed a particular scheme called the Adaptive Virtual Queue (AVQ) algorithm where the capacity of the virtual queue is adapted to the traffic conditions to achieve a desired level of utilization in the network. Here, we study the choice of the parameters of the congestion-controllers at the sources and the AVQ scheme at the links that is required to ensure stability. In particular, we consider a system in which users with diverse round-trip delays and fairness requirements access a general topology network. For this system, we show that, by choosing the speed of adaptation at the sources and the links appropriately, one can guarantee the stability of the network.

I. Introduction

An explosive growth in multi-media and peer to peer applications in the Internet has resulted in packet losses and delays due to congestion in the network. As a result, there has been a surge of interest in designing best-effort service networks that can deliver low-loss, low-delay service by encouraging the users to adapt to network congestion using minimal information from the network [4], [8], [13], [9]. Recently, *Explicit Congestion Notification* (ECN) marks have been proposed to allow the network to notify users of incipient congestion. On detecting an onset of congestion, the routers in the network sets

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a bit in the packet header of the source which is then echoed back by the destination to the source. On receiving the mark, the source reduces its rate.

To provide ECN marks, the routers have to detect the onset of congestion intelligently. Algorithms which help the router detect an onset of congestion are called *Active Queue Management* (AQM) schemes. AQM schemes can either drop or mark a packet on detecting incipient congestion. In this paper, we assume that marking is employed.

Several Active Queue Management schemes have been proposed in recent literature to provide early congestion notification to users. The main motivation behind this area of research is to provide a low-loss, low-delay service over the current Internet. Some of the recently proposed AQM schemes include the *Random Early Detect* (RED) algorithm [3], the *Gibbens and Kelly Virtual Queue* (GKVQ) algorithm [4], the *Random Early Marking* (REM) algorithm [1], the *Adaptive Virtual Queue* algorithm [9], [11], [10] and the *Proportional Integral* (PI) algorithm [5] to name a few. While the RED, REM and PI algorithms detect congestion based on the queue lengths at the link, the GKVQ algorithm and the AVQ algorithm detect congestion based on the arrival rate into the link. In this paper we will consider the stability properties of the AVQ algorithm. We note that a dual algorithm based on [13] has been shown to be stable and scalable in [15]. However, unlike our results, the controller in [15] cannot achieve arbitrary fair resource allocation. The result in [15] requires that the utility function of a user be of a particular form. Specifically the utility of User r depends on its RTT. Thus, the resource allocation is limited by the specific utility function and is not fair in general. Since the publication of the original version of this paper in [12], a stable, scalable algorithm that achieves fairness and high utilization has been presented in [2]. However, our analysis is for a general topology network, whereas the result in [2] is only for a single link and single source.

A key issue in the design of AVQ is choosing the speed of adaptation of the virtual-queue capacity at the links. The speed of adaptation is determined by a *step size* parameter, which is denoted by α. Motivated by the recent work of Vinnicombe in [16], we explored the issue of stability in a single-link network with *users of diverse round-trip delays* in [11]. In this paper we extend these results to a general network topology. We give a simple rule to design the speed of adaptation at the link that ensures the stability of the system when users have diverse round-trip delays. In Section II, we will give the system model and give a brief overview of previous results and in Section III we will discuss the stability of the system under heterogeneous time delays.

II. SYSTEM MODEL

We adopt the system model as given in [8]. Consider a network with a set $\mathcal{L}$ of links and let C_l and γ_l be the capacity and the desired utilization respectively of each $l \in \mathcal{L}$. By **desired utilization**, we refer to the ratio of the maximum arrival rate (that can be supported by the link to guarantee a desired small loss probability) to the link capacity. Let a route r be a non-empty set of links and $\mathcal{R}$ be the set of all routes in the network. We will associate a flow with each route and hence we will use the terms flow and route interchangeably throughout this paper. Let flow r generate traffic at rate x_r. The rate x_r is assumed to have a utility $\Delta_r \log(x_r)$ to user r. Following the notation of [7], for each flow r, let $d_1(r,j)$ be the delay from the source of route r to the link j, and $d_2(r,j)$ be the feedback delay from the link j back to the source. Let T_r be the total round-trip delay for route r. Note that for all $j \in \mathcal{L}$ such that route r traverses link j we have $T_r = d_1(r,j) + d_2(r,j)$.

Let each link l in the network generate feedback in the form of Explicit Congestion Notification (ECN) marks. Assume that the fraction of packets marked is a function of the total arrival rate (λ_l) and a parameter called the virtual capacity ($\tilde{C}_l$) of the link and that the total marks are distributed among the users in proportion to their flow rates. Let $p_l(\lambda_l, \tilde{C}_l)$ be the fraction of total flow that is marked by link l. The marking function at each link $l \in \mathcal{L}$, $p_l(q,s), q \geq 0, s \geq 0$ is assumed to be strictly increasing in q and strictly decreasing in s, and continuously differentiable in both its arguments.

Now, let each User r employ the weighted proportionally fair congestion-control algorithm [8]:

$$\dot{x}_r = \kappa_r \left[\Delta_r - x_r(t - T_r) \sum_{l:l \in r} p_l \left(\sum_{j:l \in j} x_j(t - d_1(j,l) - d_2(r,l)), \tilde{C}_l(t - d_2(r,l)) \right) \right] \qquad \forall r \in \mathcal{R}, n \tag{1}$$

where κ_r determines the speed of the congestion-controller, and Δ_r is the weight or the willingness to pay of User r. The virtual capacity at each link is simultaneously updated using the adaptive algorithm (called the *Adaptive Virtual Queue* (AVQ) Algorithm [9]):

$$\dot{\tilde{C}}_l = \alpha_l(\gamma_l C_l - \sum_{j:l \in j} x_j(t - d_1(j,l))), \qquad \forall l \in \mathcal{L}, \tag{2}$$

where α_l is the step-size that determines the speed of adaptation. Note that the AVQ algorithm reduces the virtual capacity whenever the total flow into the link exceeds the desired utilization (γC) at the link. This results in more marks and forces the users to reduce their rates. When the total flow into the link is smaller than the desired utilization at the link, the virtual capacity is increased, thereby reducing the number of marks sent back and hence allowing the users to increase their rates. In essence, the AVQ algorithm tries to match the achieved utilization to the desired utilization at the link in steady state. We note that the congestion controllers at the sources do not use TCP-Reno, which is the most widely used version of TCP today. However, TCP-Reno is not stable for all link speeds and RTTs. Our work should be viewed as a new algorithm that is stable for all link speeds and RTTs.

In the absence of feedback delays, i.e., $d_1(r,l) = d_2(r,l) = 0$ for all routes r and links l, it was shown in [11] that, for sufficiently small values of α, the system of differential equations given by (1)-(2) converges asymptotically to the unique solution of the convex optimization problem given by:

$$\max_{\{x_r\}} \sum_r \Delta_r \log(x_r) \tag{3}$$

subject to $\sum_{j:l\in j} x_j \leq \gamma_l C_l$, $\forall l$ and $x_j \geq 0$, $\forall j$.

In [10], we considered a single-link network with TCP-like congestion-controllers accessing the link. It was shown in [9], that a TCP user can be approximated by a $\frac{-1}{T^2 x}$ utility function, where T is the round-trip delay of the user and x is the flow rate of the user. We assumed that all users have the same round-trip delay and that the system comprising of only the congestion-controllers are locally stable. Under this scenario, we showed that if we choose the speed of adaptation at the link α to be inversely proportional to the round-trip delay, then the system is locally stable. In other words, the link adaptation has to slower than the round-trip delay. In [12], we considered a single-link network with proportional congestion-controllers and the AVQ scheme at the link. We then addressed the problem of jointly choosing the speed of adaptation of the congestion-controllers (κ) and the speed of adaptation at the links (α). In this paper, we extend this result to a general network setting in the presence of heterogeneous feedback delays. We show that the speed of adaptation at the links has to be slower than the maximum round-trip delay of the flows for the system to be locally stable.

III. Stability Result

Our stability result is a local stability result. Thus, we linearize (1)-(2) about the equilibrium point. Let $y_r(t) := x_r(t) - \hat{x}_r$, where $\hat{x}_r$ denotes the equilibrium value of $x_r(t)$, and $z(t) := \tilde{C}_l(t) - \hat{C}_l$, where $\hat{C}_l$ is the equilibrium value of $\tilde{C}_l(t)$. Denote,

$$p_l(\sum_{j:l\in j} \hat{x}_j, \hat{C}_l) \text{ by } \hat{p}_l, \qquad \frac{\partial p_l}{\partial x}(\sum_{j:l\in j} \hat{x}_j, \hat{C}_l) \text{ by } \hat{p}_l^x,$$

$$|\frac{\partial p_l}{\partial \tilde{C}_l}(\sum_{j:l\in j} \hat{x}_j, \hat{C}_l)| \text{ by } \hat{p}_l^{\tilde{c}}, \quad \text{and} \quad \sum_{l:l\in r} \hat{p}_l \text{ by } q_r.$$

Let $y_k(s)$ and $z_l(s)$ denote the Laplace transforms of $y_k(t)$ and $z_l(t)$ respectively. Define $Y(s) := [y_1(s), y_2(s), \cdots, y_{|\mathcal{R}|}(s)]^T$ and $Z(s) := [z_1(s), z_2(s), \cdots, z_{|\mathcal{L}|}(s)]^T$. Let $Y(0)$ and $Z(0)$ denote the initial conditions. It is straightforward to see that the Laplace transform of the linearized version of (1) and (2) can be written as:

$$\begin{pmatrix} sI_{|\mathcal{R}|\times|\mathcal{R}|}+A(s) & B(s) \\ C(s) & sI_{|\mathcal{L}|\times|\mathcal{L}|} \end{pmatrix} \begin{pmatrix} Y(s) \\ Z(s) \end{pmatrix} = \begin{pmatrix} Y(0) \\ Z(0) \end{pmatrix},$$

where

$$A(s)=\text{diag}\{\frac{\kappa_i\Delta_i}{\hat{x}_i}\}+\text{diag}\{\kappa_i\hat{x}_i\}R^T(s)\text{diag}\{\hat{p}_l^x\}R(s), \qquad C(s)=\text{diag}\{\alpha_l\}R(s),$$

$$B(s)=\text{diag}\{\kappa_i\hat{x}_i\}R^T(s)\text{diag}\{\hat{p}_l^{\tilde{c}}\}, \quad \text{and} \quad R_{lr}(s)=\begin{cases} e^{-sd_1(r,l)} & \text{if r traverses link l} \\ 0 & \text{otherwise} \end{cases}.$$

We will also assume that $R(0)$ is full row-rank. For the linear system to be stable, we need all the roots of the equation given by

$$\det\begin{pmatrix} sI_{|\mathcal{R}|\times|\mathcal{R}|}+A(s) & B(s) \\ C(s) & sI_{|\mathcal{L}|\times|\mathcal{L}|} \end{pmatrix}=0 \tag{4}$$

to lie in the left half-plane. In the appendix, we show that $s=0$ is not a solution to (4) and that the roots of (4) are the same as the roots of $\det(sI+A(s)+\frac{1}{s}B(s)C(s))=0$. Let $G(s)=\frac{1}{s}(A(s)+\frac{1}{s}B(s)C(s))$. We can write

$$G(s)=\text{diag}\{\frac{e^{-sT_i}}{sT_i}\}\text{diag}\{\kappa_iT_i\}X\left[WX^{-2}+\tilde{M}(s)+\frac{1}{s}\bar{M}(s)\right],$$

where

$$\tilde{M}(s)=R^*(s)\text{diag}\{\hat{p}_l^x\}R(s),\ W=\text{diag}\{\Delta_i\},\ \bar{M}(s)=R^*(s)\text{diag}\{\alpha_l\hat{p}_l^{\tilde{c}}\}R(s) \ \text{ and } \ X=\text{diag}\{\hat{x}_i\}.$$

Define,

$$K(s):=\text{diag}\{\frac{e^{-sT_i}}{sT_i}\}\text{diag}\{\kappa_iT_i\}X\left[WX^{-2}+\tilde{M}(s)\right].$$

Therefore, $K(s)$ describes the system when $\alpha=0$, i.e, $K(s)$ describes the system of congestion-controllers without the AVQ scheme at the routers. For this system, it was shown in [16] that one can choose $\{\kappa_r\}$ in a decentralized manner that will ensure the local stability of the congestion-control algorithms. The result in [16] was earlier conjectured in [7]. We also refer the reader to [14] for a weaker version of the result in [16].

We now state the main result of the paper now and outline its proof. Define, $T_{\max}:=\max_{r\in\mathcal{R}}T_r$, and $x_{\max}=\max_{r\in\mathcal{R}}\hat{x}_r$.

Theorem 3.1: Given a $0<\epsilon<1$, if

$$\kappa_r(\hat{q}_r+\sum_{j\in r}\hat{p}_j^x\sum_{q:j\in q}\hat{x}_q)=\epsilon\frac{\pi}{2T_r} \qquad \forall r\in\mathcal{R}, \tag{5}$$

and

$$\alpha_l \le \frac{\hat{p}_l^x}{\hat{p}_l^{\tilde{c}} T_{\max}} \min\{\frac{1}{\sqrt{2}}, \frac{(1-\epsilon)}{\epsilon} \frac{\pi}{8(\hat{x}_{\max} T_{\max})} \min_k \left[\frac{\hat{q}_k}{\hat{q}_k + \sum_{l:l\in k} \hat{p}_l^x \sum_{j:l\in j} \hat{x}_j} \right] \} \qquad \forall l \in \mathcal{L}, \tag{6}$$

then the system comprising of the congestion-controllers at the sources and the AVQ algorithm at the link is locally asymptotically stable.

Proof: To show the stability of the system, we first show that the eigenvalues of $G(j\omega)$ do not encircle -1 for all values of ω. Towards this, we show:

(i) There exists a ω^* such that no eigenvalue of $G(j\omega)$ is real for all $\omega < \omega^*$. (Lemma 3.1)

(ii) We know from [16] that one can choose $\{\kappa_r\}$ such that the eigenvalues of $K(j\omega)$ do not enclose -1. We can show that, given $\epsilon > 0$, each user can choose $\{\kappa_r\}$ according to (5) such that a $(1-\epsilon)$ neighborhood of the eigenvalues of $K(j\omega)$ do not enclose -1 for all ω. (Lemma 3.2)

(iii) Finally, using Lemma 3.3 we show that, if α is chosen according to (6), the eigenvalues of $G(j\omega)$ can be bounded within a $(1-\epsilon)$ neighborhood around the eigenvalues of $K(j\omega)$ for all $\omega > \omega^*$.

Using (i), (ii) and (iii), we can now easily show that the eigenvalues of $G(j\omega)$ do not enclose -1. Appealing to the Generalized Nyquist criteria, the system is locally asymptotically stable. ■

Lemma 3.1: If

$$\alpha_l < \frac{\hat{p}_l^x}{\sqrt{2}\hat{p}_l^{\tilde{c}} T_{\max}} \qquad \forall l \in \mathcal{L},$$

then, for all $\omega < \omega^* := \frac{\pi}{4T_{\max}}$, the eigenvalues of $G(j\omega)$ cannot be real.

Proof: Define

$$H(j\omega) := \text{diag}\{\frac{e^{-j\omega T_i}}{j\omega T_i}\}\text{diag}\{\kappa_i T_i\}\sqrt{X}\left[WX^{-2} + \tilde{M}(j\omega) + \frac{1}{j\omega}\bar{M}(j\omega)\right]\sqrt{X}.$$

Note that $\sigma(H(j\omega)) = \sigma(G(j\omega))$. Let $\lambda(\omega)$ be an eigenvalue of $H(j\omega)$ and μ be the corresponding normalized eigenvector. Therefore,

$$\begin{aligned}
\lambda\mu &= (\text{diag}\{\frac{e^{-j\omega T_i}}{j\omega T_i}\kappa_i T_i\})\sqrt{X}([WX^{-2} + \tilde{M}(j\omega)] + \frac{1}{j\omega}\bar{M}(j\omega))\sqrt{X}\mu \\
\lambda\text{diag}\{e^{j\omega T_i}\frac{j\omega}{\kappa_i}\}\mu &= \sqrt{X}([WX^{-2} + \tilde{M}(j\omega)] + \frac{1}{j\omega}\bar{M}(j\omega))\sqrt{X}\mu \\
\lambda &= \frac{\mu^*\sqrt{X}([WX^{-2} + \tilde{M}(j\omega)] + \frac{1}{j\omega}\bar{M}(j\omega))\sqrt{X}\mu}{\mu^*\text{diag}\{e^{j\omega T_i}\frac{j\omega}{\kappa_i}\}\mu}
\end{aligned}$$

$$\lambda = -\frac{(\mu^*\sqrt{X}(j\omega[WX^{-2} + \tilde{M}(j\omega)] + \bar{M}(j\omega))\sqrt{X}\mu)\left[\sum_{k=1}^N \frac{|\mu_k^2|}{\kappa_k}\left(\cos(\omega T_k) - j\sin(\omega T_k)\right)\right]}{\omega^2\left(\sum_{k=1}^N \frac{|\mu_k^2|\cos(\omega T_k)}{\kappa_k}\right)^2 + \omega^2\left(\sum_{k=1}^N \frac{|\mu_k^2|\sin(\omega T_k)}{\kappa_k}\right)^2}. \tag{7}$$

Defining

$$\Lambda := \left(\sum_{k=1}^{N} \frac{|\mu_k^2| \cos(\omega T_k)}{\kappa_k}\right)^2 + \left(\sum_{k=1}^{N} \frac{|\mu_k^2| \sin(\omega T_k)}{\kappa_k}\right)^2 \text{ and } \Sigma := WX^{-2} + \tilde{M}(j\omega),$$

we can write the imaginary part of (7) as

$$\text{Im}(\lambda) = -\frac{\mu^* \sqrt{X} \Sigma \sqrt{X} \mu}{\omega \Lambda} \sum_{k=1}^{N} \frac{|\mu_k^2|}{\kappa_k} \left[\cos(\omega T_k) - T_k \frac{\mu^* \sqrt{X} \bar{M}(j\omega) \sqrt{X} \mu}{\mu^* \sqrt{X} \Sigma \sqrt{X} \mu} \frac{\sin(\omega T_k)}{\omega T_k}\right]. \tag{8}$$

Let $\alpha_l = \hat{\alpha} \frac{\hat{p}_l^x}{\hat{p}_l^{\tilde{c}}}$. Therefore,

$$\frac{\mu^* \sqrt{X} \bar{M}(j\omega) \sqrt{X} \mu}{\mu^* \sqrt{X} \Sigma \sqrt{X} \mu} = \hat{\alpha} \frac{\mu^* \sqrt{X} \tilde{M} \sqrt{X} \mu}{\mu^* \sqrt{X} [WX^{-2} + \tilde{M}(j\omega)] \sqrt{X} \mu} < \hat{\alpha}.$$

Hence $\hat{\alpha} \leq \frac{1}{\sqrt{2} T_{\max}}$ is sufficient to ensure that (8) can never be zero if $\omega < \frac{\pi}{4 T_{\max}}$. ■

Lemma 3.2: Let $\eta(\omega)$ be any eigenvalue of $K(j\omega)$. For a given $0 < \epsilon < 1$, if

$$\kappa_r (q_r + \sum_{j \in r} \hat{p}_j^x \sum_{q: j \in q} \hat{x}_q) \leq \epsilon \frac{\pi}{2 T_r} \qquad \forall r \in \mathcal{R},$$

then, for all ω, a $(1 - \epsilon)$ neighborhood of $\eta(\omega)$ will not enclose -1.

Proof: Using a result from [16], we know that

$$\sigma(K(j\omega) \subset \rho(Q) \left(\text{ Convex Hull } \{\frac{e^{-j\omega T_i}}{j\omega T_i}\} \right),$$

where

$$Q = \text{diag}\{\sqrt{\kappa_i T_i \hat{x}_i}\}(WX^{-2} + \tilde{M}(j\omega))\text{diag}\{\sqrt{\kappa_i T_i \hat{x}_i}\},$$

where $\rho(Q)$ stands for the spectral radius of Q. From the conditions of the lemma, $\rho(Q) < \epsilon \frac{\pi}{2}$. Since the real part of $\frac{e^{-j\omega T_i}}{j\omega T_i}$ is greater than or equal to $-\frac{2}{\pi}$ for all ω, the result follows. ■

Lemma 3.3: Let $\lambda(\omega)$ be an eigenvalue of $G(j\omega)$. If

$$\alpha_l \leq \frac{(1 - \epsilon)}{\epsilon} \frac{\pi \hat{p}_l^x}{8 \hat{p}_l^{\tilde{c}} T_{\max}} \frac{1}{(\hat{x}_{\max} T_{\max})} \min_k \left[\frac{\hat{q}_k}{\hat{q}_k + \sum_{l: l \in k} \hat{p}_l^x \sum_{j: l \in j} \hat{x}_j} \right] \qquad \forall l \in \mathcal{L},$$

and

$$\kappa_r (\hat{q}_r + \sum_{j \in r} \hat{p}_j^x \sum_{q: j \in q} \hat{x}_q) = \epsilon \frac{\pi}{2 T_r} \qquad \forall r \in \mathcal{R},$$

then, for all $\omega > \omega^*$, $Re(\lambda(\omega)) > -1$.

Proof: For ease of exposition we define the following variables:

$$Q = \text{diag}\{\sqrt{\kappa_i T_i \hat{x}_i}\}(WX^{-2} + \tilde{M}(j\omega))\text{diag}\{\sqrt{\kappa_i T_i \hat{x}_i}\},$$

$$\tilde{Q} = \text{diag}\{\sqrt{\kappa_i T_i \hat{x}_i}\}\bar{M}(j\omega)\text{diag}\{\sqrt{\kappa_i T_i \hat{x}_i}\}, \quad \text{and } L = \text{diag}\{\frac{e^{-j\omega T_i}}{j\omega T_i}\}.$$

Let $\alpha_l = \hat{\alpha}\frac{\hat{p}_l^x}{\hat{p}_l^{\bar{c}}}$. Note that

$$\sigma(G(j\omega)) = \sigma((Q + \frac{\hat{\alpha}}{j\omega}\tilde{Q})L).$$

Let $\lambda(\omega)$ be an eigenvalue of $G(j\omega)$ and let μ be the corresponding eigenvector. Also note Q is a Hermitian positive definite matrix and $\tilde{Q}$ is a Hermitian matrix. Therefore

$$\begin{aligned}
& (Q + \frac{\hat{\alpha}}{j\omega}\tilde{Q})L\mu &=& \ \lambda\mu \\
(\Leftrightarrow) \qquad & Q(I + \frac{\hat{\alpha}}{j\omega}Q^{-1}\tilde{Q})L\mu &=& \ \lambda\mu \\
(\Leftrightarrow) \qquad & (I + \frac{\hat{\alpha}}{j\omega}Q^{-1}\tilde{Q})L\mu &=& \ \lambda Q^{-1}\mu \\
(\Rightarrow) \qquad & \mu^*(I + \frac{\hat{\alpha}}{j\omega}Q^{-1}\tilde{Q})L\mu &=& \ \lambda\mu^* Q^{-1}\mu \\
(\Leftrightarrow) \qquad & \frac{\mu^* L\mu}{\mu^* Q^{-1}\mu} + \frac{\hat{\alpha}}{j\omega}\frac{\mu^* Q^{-1}\tilde{Q}L\mu}{\mu^* Q^{-1}\mu} &=& \ \lambda.
\end{aligned}$$

Note that from Lemma 3.2, we know that the real part of the first term $\frac{\mu^* L\mu}{\mu^* Q^{-1}\mu}$ does not encircle $-\epsilon$. Therefore, we need to show that

$$\left|\frac{\hat{\alpha}}{j\omega}\frac{\mu^* Q^{-1}\tilde{Q}L\mu}{\mu^* Q^{-1}\mu}\right| \leq (1-\epsilon).$$

Next, note that

$$\begin{aligned}
\left|\frac{\hat{\alpha}}{j\omega}\frac{\mu^* Q^{-1}\tilde{Q}L\mu}{\mu^* Q^{-1}\mu}\right| &\leq \frac{\hat{\alpha}}{\omega^*}\frac{\| Q^{-1} \|_2 \| \tilde{Q} \|_2 \| L \|_2}{\lambda_{\min}(Q^{-1})} \\
&\leq \frac{\hat{\alpha}\rho(Q)^2}{\lambda_{\min}(Q)(\omega^*)^2}\max_k(\frac{1}{T_i}),
\end{aligned}$$

where $\| \, . \, \|_2$ is the matrix induced spectral norm given by $\| A \|_2 = \sqrt{\lambda_{\max}(A^* A)}$. The first inequality follows directly from the Cauchy-Schwartz inequality and the last inequality is due to the fact that $\rho(\tilde{Q}) < \rho(Q)$. We know that

$$\begin{aligned}
\lambda_{\min}(Q) &= \lambda_{\min}(\mathrm{diag}\{\sqrt{\kappa_k T_k \hat{x}_k}\}(WX^{-2} + \tilde{M}(j\omega))\mathrm{diag}\{\sqrt{\kappa_k T_k \hat{x}_k}\}) \\
&\geq \lambda_{\min}(\mathrm{diag}\{\sqrt{\kappa_k T_k \hat{x}_k}\}WX^{-2}\mathrm{diag}\{\sqrt{\kappa_k T_k \hat{x}_k}\}) \\
&= \min_k(\frac{\Delta_k \kappa_k T_k}{\hat{x}_k}).
\end{aligned}$$

Since $\rho(Q) = \epsilon\frac{\pi}{2}$ and $\omega^* = \frac{\pi}{4T_{\max}}$, we have

$$\left|\frac{\hat{\alpha}}{j\omega}\frac{\mu^* Q^{-1}\tilde{Q}L\mu}{\mu^* Q^{-1}\mu}\right| \leq \hat{\alpha}\frac{4\epsilon^2}{\min_k(\frac{\Delta_k \kappa_k T_k}{\hat{x}_k})T_{\min}T_{\max}^2}.$$

If x_{max} is measured in packets-per-second, $T_{\min} \geq \frac{1}{x_{\max}}$ due to processing delays. Therefore,

$$\alpha_l \leq \frac{(1-\epsilon)}{\epsilon} \frac{\pi \hat{p}_l^x}{8\hat{p}_l^{\tilde{c}} T_{\max}} \frac{1}{(\hat{x}_{\max} T_{\max})} \min_k \left[\frac{\hat{q}_k}{\hat{q}_k + \sum_{l: l \in k} \hat{p}_l^x \sum_{j: l \in j} \hat{x}_j} \right] \qquad \forall l \in \mathcal{L},$$

ensures that

$$\left| \frac{\hat{\alpha}}{j\omega} \frac{\mu^* Q^{-1} \tilde{Q} L \mu}{\mu^* Q^{-1} \mu} \right| \leq (1-\epsilon).$$

Therefore, the eigenvalues do not enclose -1. ■

The following corollary specializes the result of Theorem 3.1 to a marking function which can be viewed as the overflow probability in an $M/M/1$ queue.

Corollary 3.1: Suppose that the marking function at each node is given by

$$p_l(\lambda, \tilde{C}) = \left(\frac{\lambda}{\tilde{C}}\right)^B,$$

for some $B > 0$. Given a $0 < \epsilon < 1$, if

$$\kappa_r \hat{q}_r (1 + B) = \epsilon \frac{\pi}{2T_r} \qquad \forall r \in \mathcal{R}, \tag{9}$$

and

$$\alpha_l \leq \frac{1}{p^{1/B} T_{\max}} \min\{\frac{1}{\sqrt{2}}, \frac{(1-\epsilon)}{\epsilon} \frac{\pi}{8\hat{x}_{\max} T_{\max}(1+B)}\} \qquad \forall l \in \mathcal{L}, \tag{10}$$

then the system comprising of the congestion-controllers at the sources and the AVQ algorithm at the link is locally asymptotically stable. ■

IV. Simulations

In this section we simulate the system given by (1) and (2). The aim of our simulations to verify that the system given by (1) and (2) converges to the equilibrium point even when the parameters are chosen using the linearized system (Theorem 3.1). Towards this, we study a single link of capacity $10Mbps$, desired utilization $\gamma = 0.90$, and a packet size of $8,000$ bits, which is equivalent to a link of capacity 2500 packets per second. This link is used by users in two classes, one with $\Delta_i = 1$ and round-trip time (RTT) equal to 100 msecs (Class 1) and the other with $\Delta_i = 2$ and RTT=200 msecs (Class 2). The marking function used is $(\lambda/\tilde{C})^B$, where λ is the total arrival rate into the link, and $B = 5$. We assume that there are 100 users in each class. The congestion-control algorithm at the sources as well as the link adaptation algorithm are simulated using a discrete-time implementation in MATLAB. We use Theorem 3.1 to choose the parameters of the congestion-control and link adaptation algorithms. We choose $\epsilon = 0.4$, $\kappa_i = 0.01$ for users in class 1, $\kappa_i = 0.005$ for users in class 2 and $\alpha = 0.011$. The initial

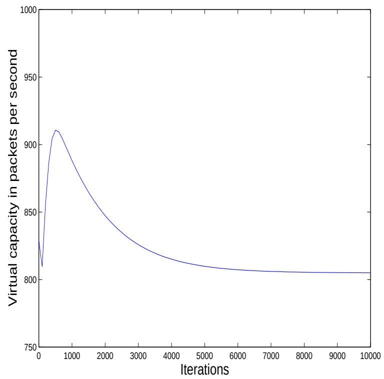

Fig. 1. Evolution of the virtual capacity

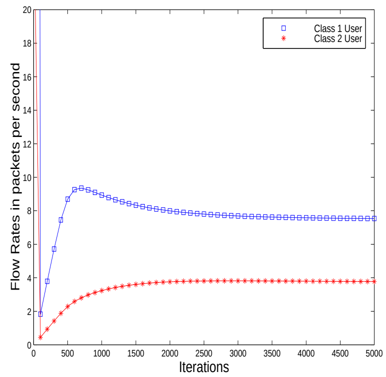

Fig. 2. Evolution of a typical user's flow rate

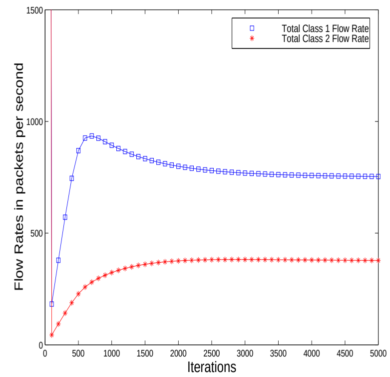

Fig. 3. Evolution of the total flow rate of each class

rates of each user are randomly chosen in the interval $(0, 300)$ packets per second. Figure 1 shows the evolution of the virtual capacity with time. Figure 2 shows the evolution of the flow rates of a typical user (in this figure, we consider User 3 in Class 1 and User 7 in Class 2). We can see that the flow rates of each user converge to the equilibrium point. In Figure 3, we show the evolution of the total flow rates of each class. We can see that even in a non-linear setting the system converges to the equilibrium point when the parameters are chosen to satisfy the local stability condition.

V. Conclusions

In this paper we show that one can jointly choose the speed of adaptation κ of the congestion controllers and the speed of adaptation α in the AVQ scheme at the links to ensure stability of the entire system comprising of the congestion-controllers at the sources and the AVQ algorithm at the link. The choice of α depends on the maximum round-trip delay of the flows. Hence it is sufficient to estimate just an upper bound on the round-trip delays of the flows in the network.

VI. Appendix

Lemma 6.1: Assuming that the matrix H is invertible,

$$\det \begin{pmatrix} E & F \\ G & H \end{pmatrix} = 0$$

if and only if

$$\det(E - FH^{-1}G) = 0,$$

where E, F, G and H are matrices of compatible dimensions.

Proof: Let

$$\det\begin{pmatrix} E & F \\ G & H \end{pmatrix} = 0.$$

Therefore, there exists a $[\mu_1 \quad \mu_2]^T \neq 0$ such that

$$\begin{pmatrix} E & F \\ G & H \end{pmatrix}\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = 0.$$

Therefore, $G\mu_1 + H\mu_2 = 0$ which implies that $\mu_2 = -H^{-1}G\mu_1$. Therefore, $(E - FH^{-1}G)\mu_1 = 0$. Note that if $\mu_1 = 0$, this implies $\mu_2 = -H^{-1}G\mu_1 = 0$. But we know that both μ_1 and μ_2 cannot be equal to the zero vector. Hence, $\mu_1 \neq 0$ and thus, $\det(E - FH^{-1}G) = 0$.

To prove the converse, assume $\det(E - FH^{-1}G) = 0$. Therefore, there exists a $\mu_1 \neq 0$ such that $(E - FH^{-1}G)\mu_1 = 0$. We can now find a $\mu_2 = -H^{-1}G\mu_1$ such that

$$\begin{pmatrix} E & F \\ G & H \end{pmatrix}\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = 0.$$

Therefore,

$$\det\begin{pmatrix} E & F \\ G & H \end{pmatrix} = 0.$$

Hence proved. ■

Lemma 6.2: $s = 0$ is not a solution to (4).

Proof: Suppose that $s = 0$ is a solution. Then, there exists $[\mu_1 \quad \mu_2]^T \neq 0$ such that

$$\begin{aligned} \text{diag}\{\frac{\kappa_i\Delta_i}{\hat{x}_i}\}\mu_1 + \text{diag}\{\kappa_i\hat{x}_i\}R^T(0)\text{diag}\{\hat{p}_l^x\}R(0)\mu_1 + \text{diag}\{\kappa_i\hat{x}_i\}R^T(0)\text{diag}\{\hat{p}_l^{\tilde{c}}\}\mu_2 &= 0 \\ \text{diag}\{\alpha_l\}R(0)\mu_1 &= 0. \end{aligned}$$

Assuming that $R(0)$ has full row rank, neither $\mu_1 = 0$ nor $\mu_2 = 0$ since

$$\text{diag}\{\frac{\kappa_i\Delta_i}{\hat{x}_i}\} + \text{diag}\{\kappa_i\hat{x}_i\}R^T(0)\text{diag}\{\hat{p}_l^x\}R(0)$$

is invertible. Further, since $R(0)\mu_1 = 0$,

$$\begin{aligned} \text{diag}\{\frac{\Delta_i}{\hat{x}_i}\}\mu_1 + \text{diag}\{\hat{x}_i\}R^T(0)\text{diag}\{\hat{p}_l^x\}R(0)\mu_1 + \text{diag}\{\hat{x}_i\}R^T(0)\text{diag}\{\hat{p}_l^{\tilde{c}}\}\mu_2 &= 0 \\ \text{diag}\{\frac{\Delta_i}{\hat{x}_i}\}\mu_1 + \text{diag}\{\hat{x}_i\}R^T(0)\text{diag}\{\hat{p}_l^{\tilde{c}}\}\mu_2 &= 0. \\ \text{diag}\{\frac{\Delta_i}{(\hat{x}_i)^2}\}\mu_1 + R^T(0)\text{diag}\{\hat{p}_l^{\tilde{c}}\}\mu_2 &= 0. \end{aligned}$$

Multiplying by μ_1^*, we get

$$\begin{aligned}\mu_1^* \text{diag}\{\frac{\Delta_i}{\hat{x}_i^2}\}\mu_1 + \mu_1^* R^T(0)\text{diag}\{\hat{p}_l^{\tilde{c}}\}\mu_2 &= 0 \\ \mu_1^* \text{diag}\{\frac{\Delta_i}{\hat{x}_i^2}\}\mu_1 &= 0.\end{aligned}$$

But diag$\{\frac{\Delta_i}{\hat{x}_i^2}\}$ is a positive-definite matrix. Hence, we have a contradiction and the result is proved. ∎

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