Rotating Reference Frames and Relative Motion

Sections 6.3-6.4

Any Vector fixed to *S*

If a vector is fixed in *S*

$$\mathbf{r} = r_1\,\mathbf{b}_1 + r_2\,\mathbf{b}_2$$

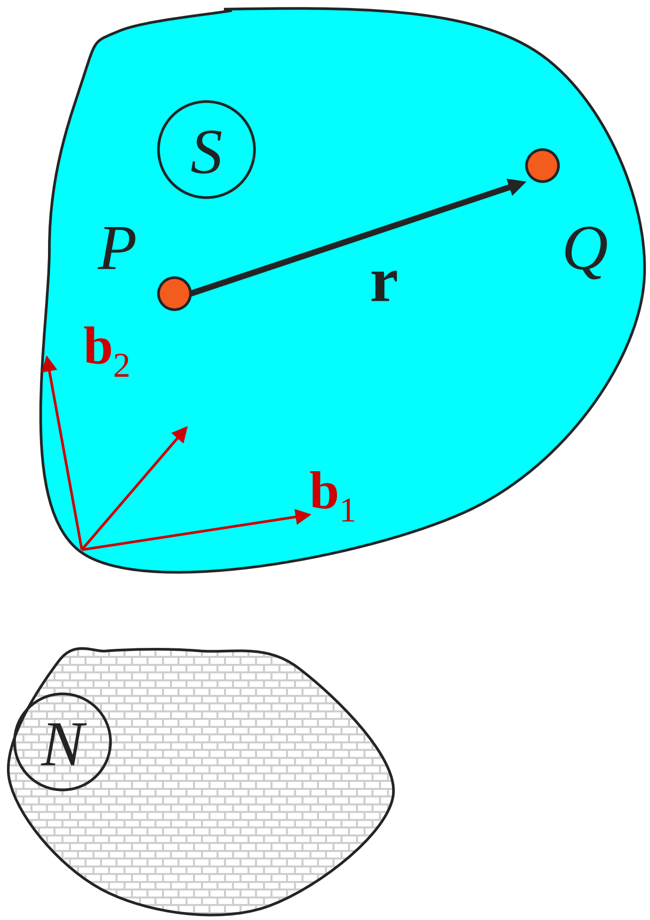
r_1 , r_2 are constant. Why?

$$\frac{d\mathbf{r}}{dt} = r_1\frac{d\mathbf{b}_1}{dt} + r_2\frac{d\mathbf{b}_2}{dt}$$

Rate of change of a unit vector

$$\frac{d\mathbf{b}_i}{dt} = \omega \times \mathbf{b}_i$$

$$\frac{d\mathbf{r}}{dt} = \omega \times \mathbf{r}$$



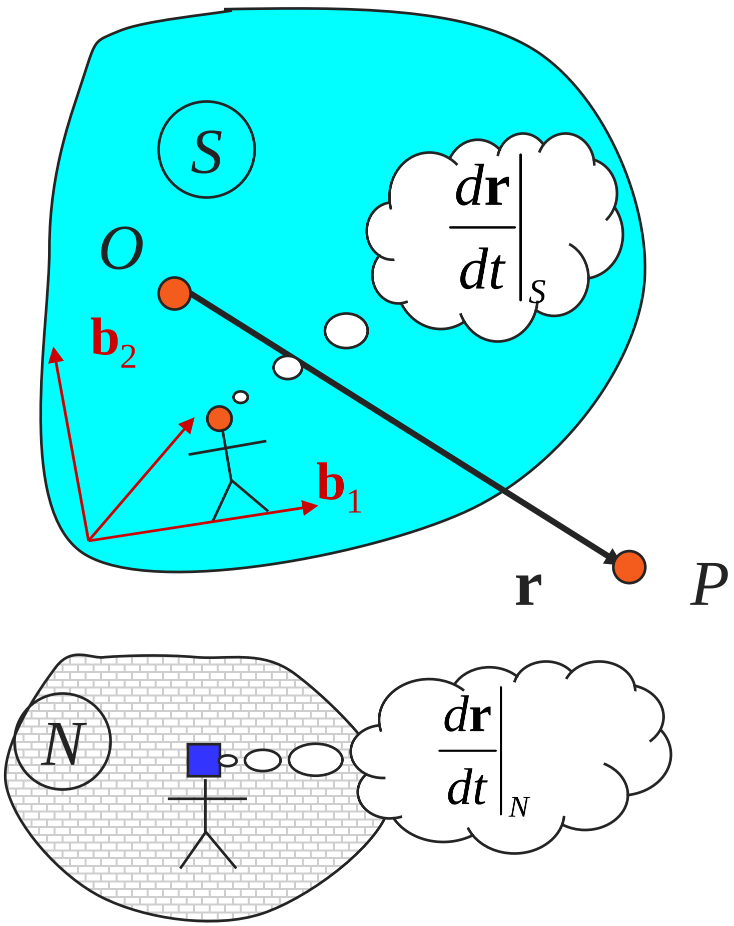
Differentiation of a Vector not fixed to *S*

$$\left.\frac{d\mathbf{r}}{dt}\right|_N = \frac{dr_1}{dt}\mathbf{b}_1 + \frac{dr_2}{dt}\mathbf{b}_2 + r_1\left.\frac{d\mathbf{b}_1}{dt}\right|_N + r_2\left.\frac{d\mathbf{b}_2}{dt}\right|_N$$

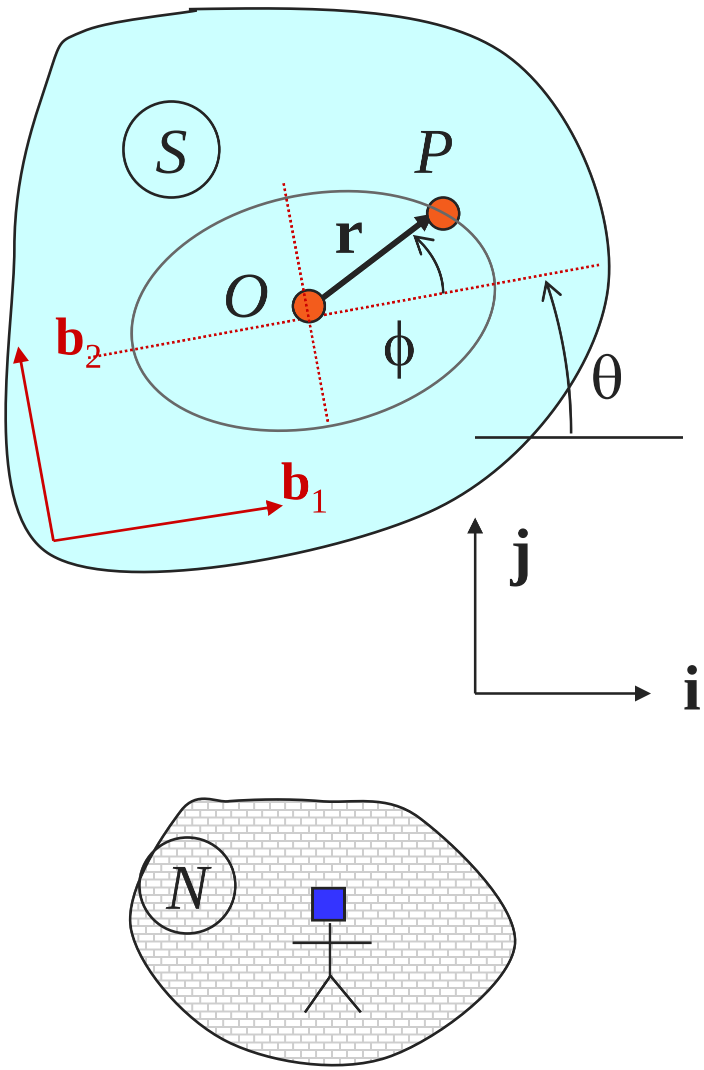
$$\left.\frac{d\mathbf{r}}{dt}\right|_S = \frac{dr_1}{dt}\mathbf{b}_1 + \frac{dr_2}{dt}\mathbf{b}_2 + r_1\left.\frac{d\mathbf{b}_1}{dt}\right|_S + r_2\left.\frac{d\mathbf{b}_2}{dt}\right|_S$$

$$\left.\frac{d\mathbf{r}}{dt}\right|_N = \left.\frac{d\mathbf{r}}{dt}\right|_S + \omega\times\mathbf{r}$$

r can be *any vector*



Example

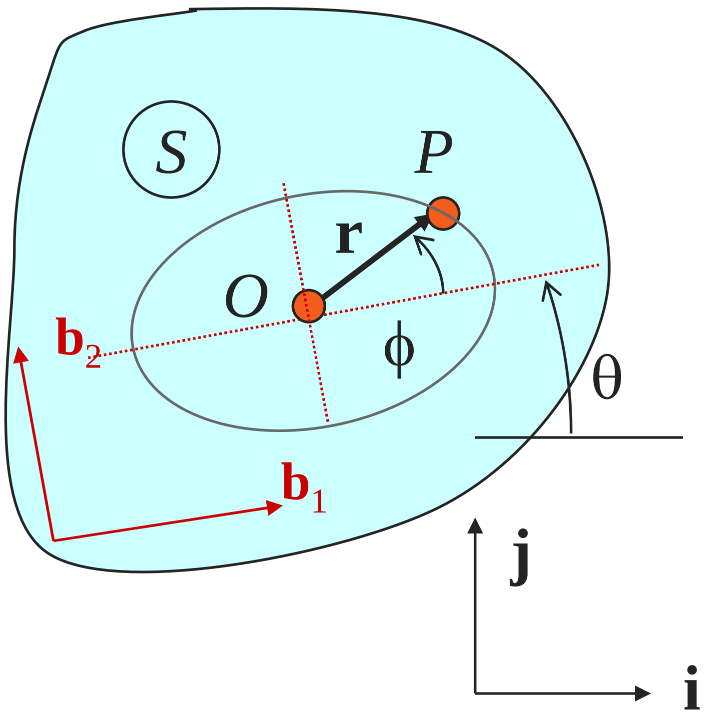


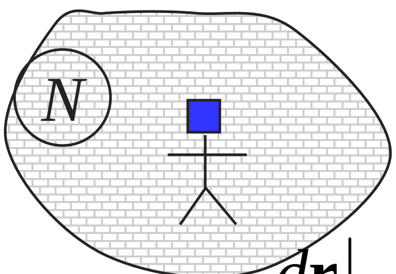
The point P travels in an elliptical slot attached to body S with major and minor axes $2a$ and $2b$ respectively so that $\phi = \gamma t$ where γ is a constant. The angular velocity of B is given by $\omega\mathbf{k}$.

$$\mathbf{r} = a\cos\gamma t\mathbf{b}_1 + a\sin\gamma t\mathbf{b}_2$$

Find the velocity of P as seen by an observer attached to an inertial frame

Example





$$\mathbf{r} = a\cos\gamma t\,\mathbf{b}_1 + b\sin\gamma t\,\mathbf{b}_2$$

$$\left.\frac{d\mathbf{r}}{dt}\right|_S = -a\gamma\sin\gamma t\,\mathbf{b}_1 + b\gamma\cos\gamma t\,\mathbf{b}_2$$

$$\left.\frac{d\mathbf{r}}{dt}\right|_N = \left[-a\gamma\sin\gamma t\,\mathbf{b}_1 + b\gamma\cos\gamma t\,\mathbf{b}_2\right]$$

$$+\,\omega\mathbf{k}\times\left(a\cos\gamma t\,\mathbf{b}_1 + b\sin\gamma t\,\mathbf{b}_2\right)$$

$$\left.\frac{d\mathbf{r}}{dt}\right|_N = -\left(a\gamma\sin\gamma t + \omega b\sin\gamma t\right)\mathbf{b}_1 + \left(b\gamma\cos\gamma t + \omega a\cos\gamma t\right)\mathbf{b}_2$$

Vectors are independent of choice of unit vectors used to write the vectors

Absolute velocity

- velocity measured in some designated inertial frame

$$\left.\frac{d\mathbf{r}}{dt}\right|_N$$

Relative velocity

- velocity measured in some other (moving) frame

$$\left.\frac{d\mathbf{r}}{dt}\right|_S$$

Components of absolute velocity

- along unit vectors fixed to the designated inertial frame

$$\left.\frac{d\mathbf{r}}{dt}\right|_N = (\cdots)\mathbf{i} + (\cdots)\mathbf{j}$$

- along unit vectors fixed to some other (moving) frame

$$\left.\frac{d\mathbf{r}}{dt}\right|_N = (\cdots)\mathbf{b}_1 + (\cdots)\mathbf{b}_2$$

Components of relative velocity

- along unit vectors fixed to the designated inertial frame

$$\left.\frac{d\mathbf{r}}{dt}\right|_S = (\cdots)\mathbf{i} + (\cdots)\mathbf{j}$$

- along unit vectors fixed to some other (moving) frame

$$\left.\frac{d\mathbf{r}}{dt}\right|_S = (\cdots)\mathbf{b}_1 + (\cdots)\mathbf{b}_2$$

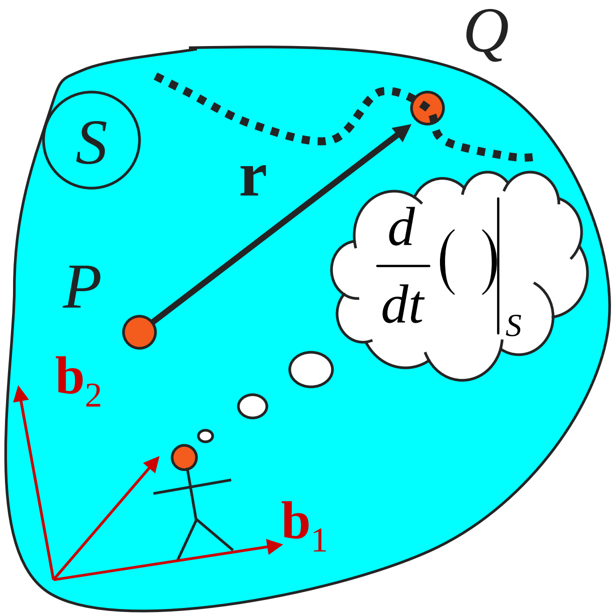
Notation

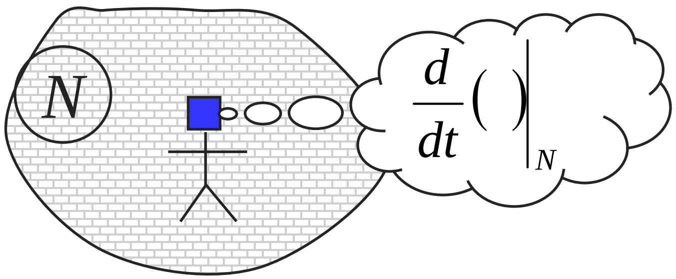
$\mathbf{v}_{rel}$ $\quad \left.\dfrac{d\mathbf{r}_{Q/P}}{dt}\right|_S$ derivative of $\mathbf{r}_{Q/P}$ measured by an observer in S

$\mathbf{v}_{Q/P}$ $\quad \left.\dfrac{d\mathbf{r}_{Q/P}}{dt}\right|_N$ derivative of $\mathbf{r}_{Q/P}$ measured by an observer fixed to an inertial frame N

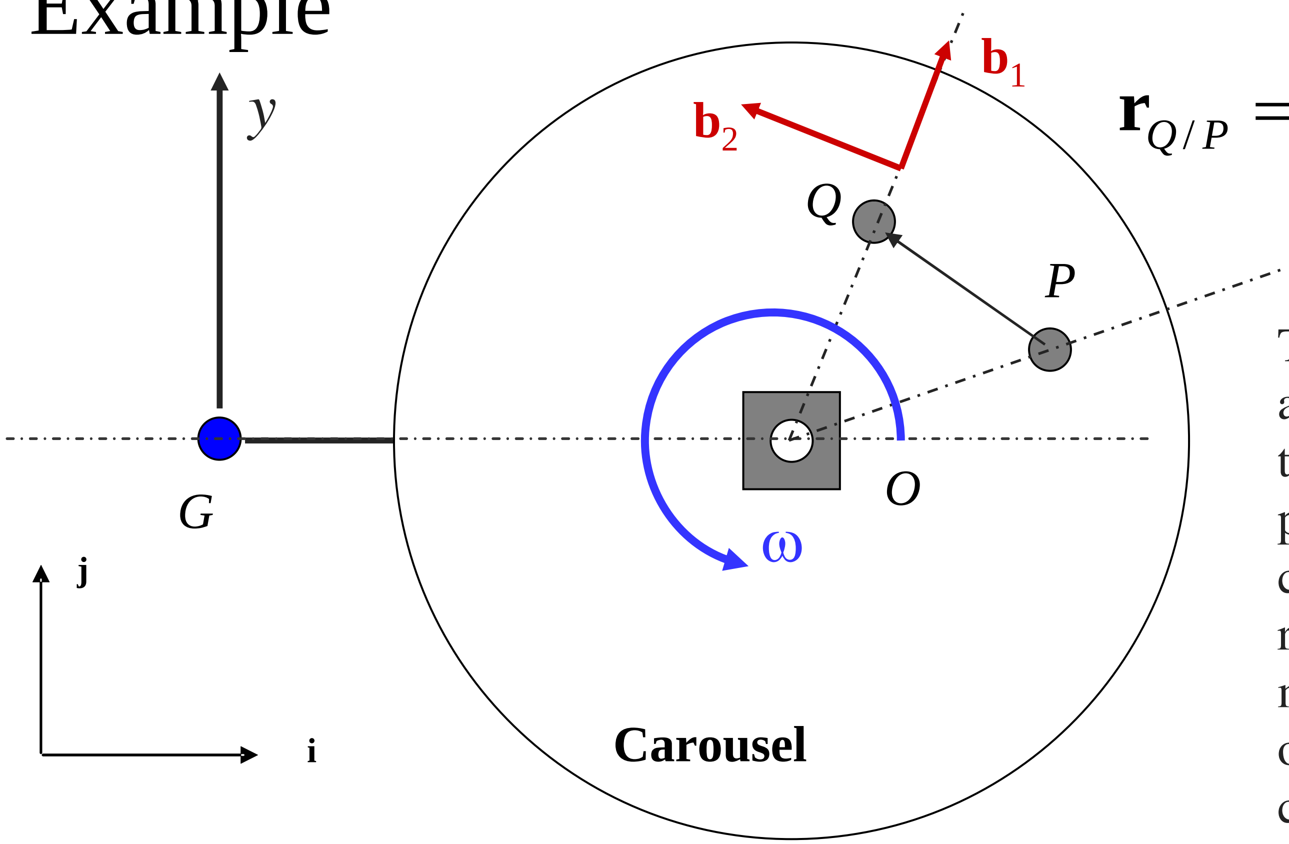
$\mathbf{a}_{rel}$ $\quad \left.\dfrac{d\mathbf{v}_{Q/P}}{dt}\right|_S$ derivative of $\mathbf{v}_{Q/P}$ measured by an observer in S

$\mathbf{a}_{Q/P}$ $\quad \left.\dfrac{d\mathbf{v}_{Q/P}}{dt}\right|_N$ derivative of $\mathbf{v}_{Q/P}$ measured by an observer fixed to an inertial frame N





Example



$$\mathbf{r}_{Q/P} = \alpha \mathbf{b}_1 + \beta \mathbf{b}_2$$

The carousel rotates about an axis perpendicular to the plane of this paper passing through *O* at a constant rate of ω rads/sec. Your friend is moving in radial direction outward relative to the carousel at point *Q* at the rate of *s* m/sec, while accelerating at *a* m/sec^2 (also in a radial direction). You are standing at *P* fixed to the carousel.

$\mathbf{v}_{rel} =$

$\mathbf{v}_{Q/P} =$

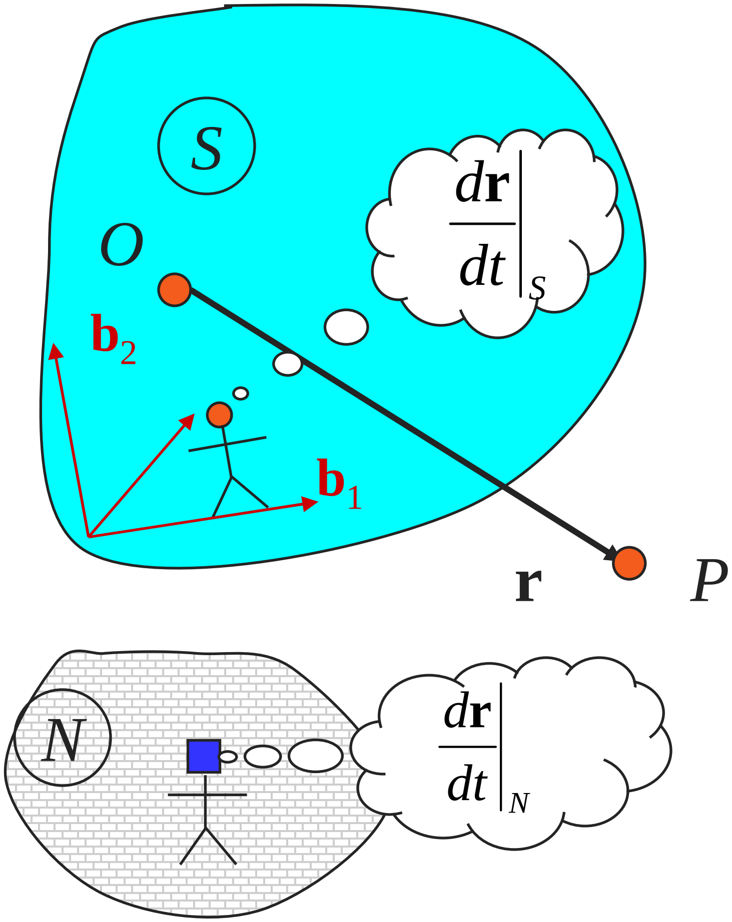
$\mathbf{a}_{rel} =$

Recall…

Differentiation of a Vector not fixed to *S*

Take **r** to be *any vector*

$$\left.\frac{d\mathbf{r}}{dt}\right|_N = \left.\frac{d\mathbf{r}}{dt}\right|_S + \omega \times \mathbf{r}$$



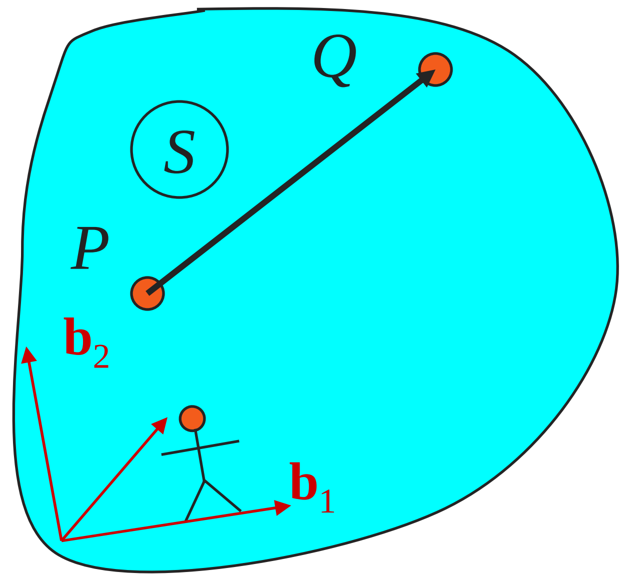
Velocity and Acceleration equations

Let **r** be the position vector $\overrightarrow{PQ}$

$$\left.\frac{d\mathbf{r}_{Q/P}}{dt}\right|_N = \left.\frac{d\mathbf{r}_{Q/P}}{dt}\right|_S + \omega \times \mathbf{r}_{Q/P}$$

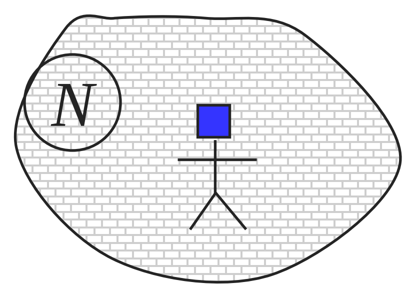
P and *Q*, and therefore $\mathbf{r}_{Q/P}$ are fixed to *S*

$$\mathbf{v}_{Q/P} = \left.\frac{d\mathbf{r}_{Q/P}}{dt}\right|_N = \omega \times \mathbf{r}_{Q/P}$$



What if we replace **r** by the velocity $\mathbf{v}_{Q/P}$?

$$\mathbf{a}_{Q/P} = \left.\frac{d\mathbf{v}_{Q/P}}{dt}\right|_N = \left.\frac{d\mathbf{v}_{Q/P}}{dt}\right|_S + \omega \times \mathbf{v}_{Q/P}$$



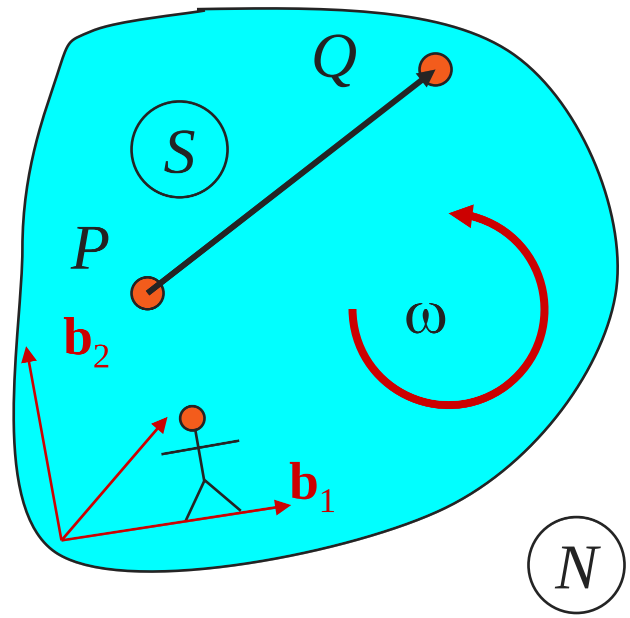
Velocity and Acceleration equations (continued)

So far…

- $\mathbf{r}_{Q/P}$ is the position vector $\overrightarrow{PQ}$

$$\mathbf{v}_{Q/P} = \left.\frac{d\mathbf{r}_{Q/P}}{dt}\right|_N = \omega \times \mathbf{r}_{Q/P}$$

$$\mathbf{a}_{Q/P} = \left.\frac{d\mathbf{v}_{Q/P}}{dt}\right|_N = \left.\frac{d\mathbf{v}_{Q/P}}{dt}\right|_S + \omega \times \mathbf{v}_{Q/P}$$



$$\left.\frac{d\mathbf{v}_{Q/P}}{dt}\right|_S = \boxed{\left.\frac{d\omega}{dt}\right|_S} \times \mathbf{r}_{Q/P} + \omega \times \underbrace{\left.\frac{d\mathbf{r}_{Q/P}}{dt}\right|_S}_{0}$$

Define α , the angular acceleration of *S*

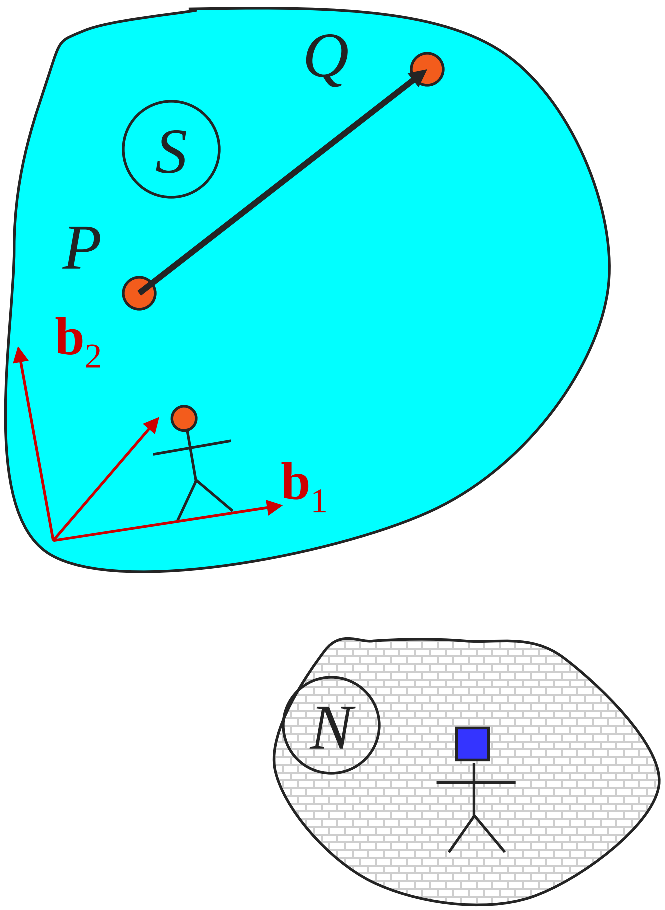
$$\alpha = \left.\frac{d\omega}{dt}\right|_N = \left.\frac{d\omega}{dt}\right|_S$$

Summary: Velocity and Acceleration equations for *P* and *Q* fixed to a moving body *S*

$$\mathbf{v}_Q - \mathbf{v}_P = \omega \times \mathbf{r}_{Q/P}$$

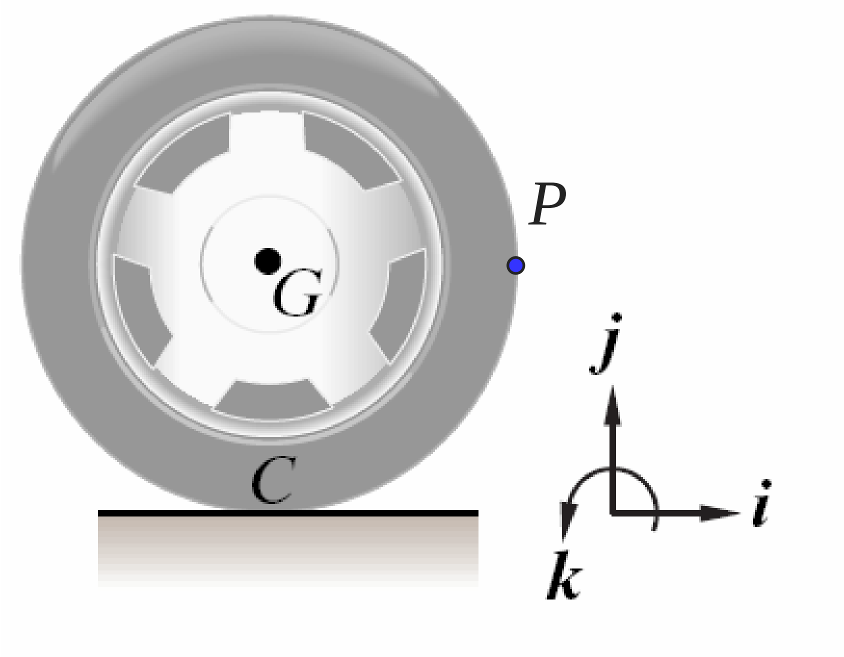
$$\mathbf{a}_Q - \mathbf{a}_P = \alpha \times \mathbf{r}_{Q/P} + \omega \times \left(\omega \times \mathbf{r}_{Q/P}\right)$$

$$\alpha = \left.\frac{d\omega}{dt}\right|_N = \left.\frac{d\omega}{dt}\right|_S$$



α is the angular acceleration of *S*

Example



The wheel rolls to the right with clockwise angular velocity ω and a clockwise angular acceleration α. Find the velocity and acceleration of points *C*, *G*, and *P*.