Two Approaches

Newton-Euler Equations of Motion

For each rigid body
- Newton’s second law (translation)
  \[
  \sum \mathbf{F}_i = m\mathbf{a}
  \]
- Euler’s equations of motion for rotations
  \[
  \sum \mathbf{r}_i \times \mathbf{F}_i + \sum \mathbf{M}_j = I_G \alpha
  \]

Analytical dynamics (Lagrange’s equations of motion)
- Principle of virtual work (Galileo, Bernouli)
  - Ignore reaction forces and focus on the forces and moments that do work
- D’Alembert’s principle
  - Incorporate inertial forces
- One equation of motion for each degree of freedom
Lagrange’s Equation of Motion

Derivation

- Single degree of freedom system
- Lumped masses (as in particle dynamics)
- Planar system

Basic principle

- Net power (due to forces and moments acting on the system = rate of increase in kinetic energy of the system)
Lagrange’s Equation of Motion (continued)

- Degrees of freedom: \( n=1 \)
- Generalized coordinate: \( q = \theta \)
- Applied forces and moments: \( F, \tau \)
- Net power
  \[ P = \tau \dot{\theta} - F_x \dot{x} \]
  \[ = \tau \dot{\theta} - F_x \left( -r \sin(\theta + \phi) \right) \dot{\theta} \]
  \[ = Q \dot{\theta} \]
- Generalized force,
  \[ Q = \tau + F_x \left( \frac{r \sin(\theta + \phi)}{l \cos \phi} \right) \]
- Kinetic energy
  \[ T = \frac{1}{2} m x^2 \]
  \[ = \frac{1}{2} m \left( \frac{m r^2 \sin^2(\theta + \phi)}{\cos^2 \phi} \right) \dot{\theta}^2 \]
  \[ = \frac{1}{2} J(q) q^2 \]
Lagrange’s Equation of Motion (continued)

- Net power \( P = Q\dot{\theta} \)
- Generalized force,
  \[ Q = \tau + F_x \left( \frac{mr^2 \sin^2(\theta + \phi)}{l \cos \phi} \right) \]
- Kinetic energy
  \[ T = \frac{1}{2} J(q)\dot{q}^2 \]
- Generalized inertia
  \[ J(\theta) = \frac{1}{2} \left( \frac{mr^2 \sin^2(\theta + \phi)}{\cos^2 \phi} \right) \]
- Power = rate of change of energy
  \[ Q\dot{q} = \left( J(q)\ddot{q} + \frac{1}{2} \frac{dJ(q)}{dq} \dot{q}^2 \right) \dot{q} \]
Lagrange’s Equation of Motion (continued)

- Net power
  \[ P = Q \dot{\theta} \]

- Kinetic energy
  \[ T = \frac{1}{2} J(q) \dot{q}^2 \]

- Power = rate of change of energy
  \[ Q \dot{q} = \left( J(q) \dot{q} + \frac{1}{2} dJ(q) \right) \dot{q}^2 \]

- Fundamental form of Lagrange’s equation of motion
  \[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}} \right) = \frac{dJ(q)}{dq} \dot{q}^2 + J(q) \ddot{q} \]
  \[ \frac{\partial T}{\partial q} = \frac{1}{2} dJ(q) \dot{q}^2 \]
Standard Form of Lagrange’s Equation

- Fundamental form

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}} \right) - \frac{\partial T}{\partial q} = Q'
\]

- Divide applied (external) forces into conservative forces and other (non conservative) forces,

\[
Q' = Q_{cons} + Q
\]

- A conservative force can be expressed as a derivative of a scalar potential, called the potential energy, \( V(q) \)

\[
Q_{cons} = -\frac{dV(q)}{dq}
\]

- Define the Lagrangian,

\[
L = T - V
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = Q
\]
Configuration 4
Trebuchet with Fixed Counterweight and Sling
Lagrange’s Equation for $n$ DOF systems

- $n$ equations of motion
  \[
  \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = Q_i, \quad i = 1, \ldots, n
  \]

- Conservative forces, $Q_{\text{cons},i}$, are included in the Lagrangian
  \[
  Q_{\text{cons},i} = -\frac{\partial V(q)}{\partial q_i}
  \]

- $Q_i$ does not include conservative forces

\[
\begin{align*}
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} &= 0, \quad i = 1, 2
\end{align*}
\]