Two Approaches

Newton-Euler Equations of Motion

For each rigid body

- Newton's second law (translation)

$$\sum \mathbf{F}_i = m\mathbf{a}$$

- Euler's equations of motion for rotations

$$\sum \mathbf{r}_i \times \mathbf{F}_i + \sum \mathbf{M}_j = I_G \alpha$$

Analytical dynamics (Lagrange's equations of motion)

- Principle of virtual work (Galileo, Bernouli)
  - Ignore reaction forces and focus on the forces and moments that do work
- D'Alembert's principle
  - Incorporate inertial forces
- One equation of motion for each degree of freedom

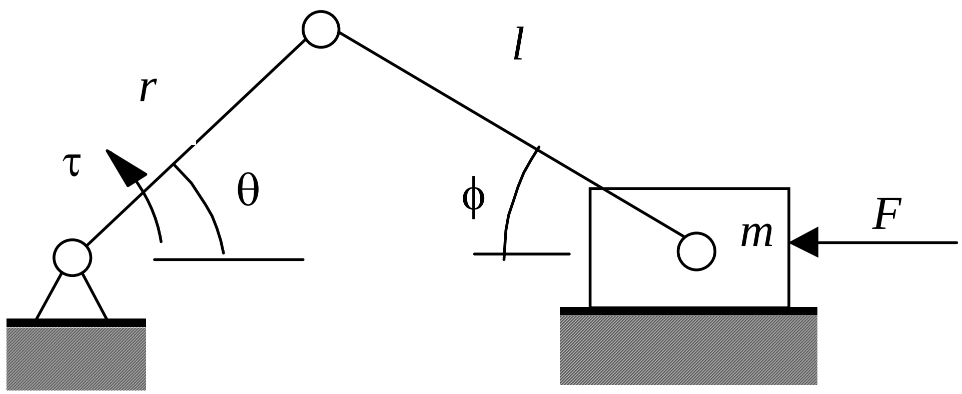
Lagrange's Equation of Motion

Derivation

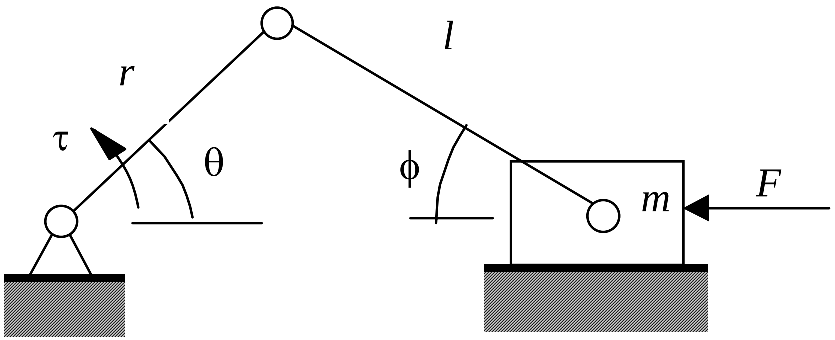
- Single degree of freedom system
- Lumped masses (as in particle dynamics)
- Planar system

Basic principle

- Net power (due to forces and moments acting on the system = rate of increase in kinetic energy of the system



Lagrange's Equation of Motion (continued)



- Degrees of freedom: n=1
- Generalized coordinate: $q = \theta$
- Applied forces and moments: F, τ
- Net power

$$P = \tau\dot{\theta} - F_x \dot{x}$$

$$= \tau\dot{\theta} - F_x \left(\frac{-r\sin(\theta+\phi)}{l\cos\phi} \right)\dot{\theta}$$

$$= Q\dot{\theta}$$

- Generalized force, $Q = \tau + F_x \left(\frac{r\sin(\theta+\phi)}{l\cos\phi} \right)$
- Kinetic energy

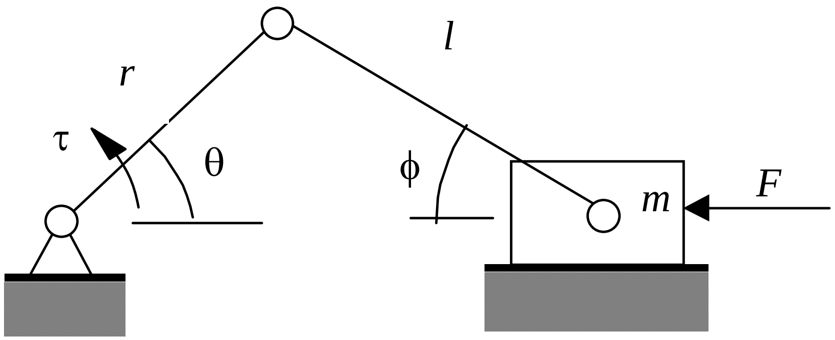
$$T = \tfrac{1}{2} m\dot{x}^2$$

$$= \tfrac{1}{2} m \left(\frac{mr^2 \sin^2(\theta+\phi)}{\cos^2\phi} \right)\dot{\theta}^2$$

$$= \tfrac{1}{2} J(q)\dot{q}^2$$

Generalized inertia
$J(q)$

Lagrange's Equation of Motion (continued)



- Net power $P = Q\dot{\theta}$
- Generalized force,

$$Q = \tau + F_x\left(\frac{r\sin(\theta+\phi)}{l\cos\phi}\right)$$

- Kinetic energy

$$T = \tfrac{1}{2}J(q)\dot{q}^2$$

- Generalized inertia

$$J(\theta) = \tfrac{1}{2}\left(\frac{mr^2\sin^2(\theta+\phi)}{\cos^2\phi}\right)$$

- Power = rate of change of energy

$$Q\dot{q} = \left(J(q)\ddot{q} + \tfrac{1}{2}\frac{dJ(q)}{dq}\dot{q}^2\right)\dot{q}$$

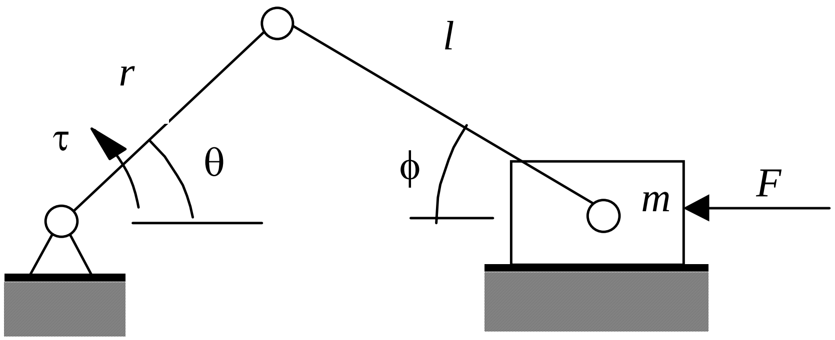
Lagrange’s Equation of Motion (continued)

- Net power

$$P = Q\dot{\theta}$$

- Kinetic energy

$$T = \tfrac{1}{2} J(q)\dot{q}^2$$



- Power = rate of change of energy

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) = \frac{dJ(q)}{dq}\dot{q}^2 + J(q)\ddot{q}$$

$$Q\dot{q} = \left(J(q)\ddot{q} + \tfrac{1}{2}\frac{dJ(q)}{dq}\dot{q}^2 \right)\dot{q}$$

$$\frac{\partial T}{\partial q} = \tfrac{1}{2}\frac{dJ(q)}{dq}\dot{q}^2$$

- Fundamental form of Lagrange’s equation of motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = Q$$

Standard Form of Lagrange's Equation

- Fundamental form $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = Q'$$
- Divide applied (external) forces into conservative forces and other (non conservative) forces,

$$Q' = Q_{cons} + Q$$

- A conservative force can be expressed as a derivative of a scalar potential, called the potential energy, $V(q)$

$$Q_{cons} = -\frac{dV(q)}{dq}$$

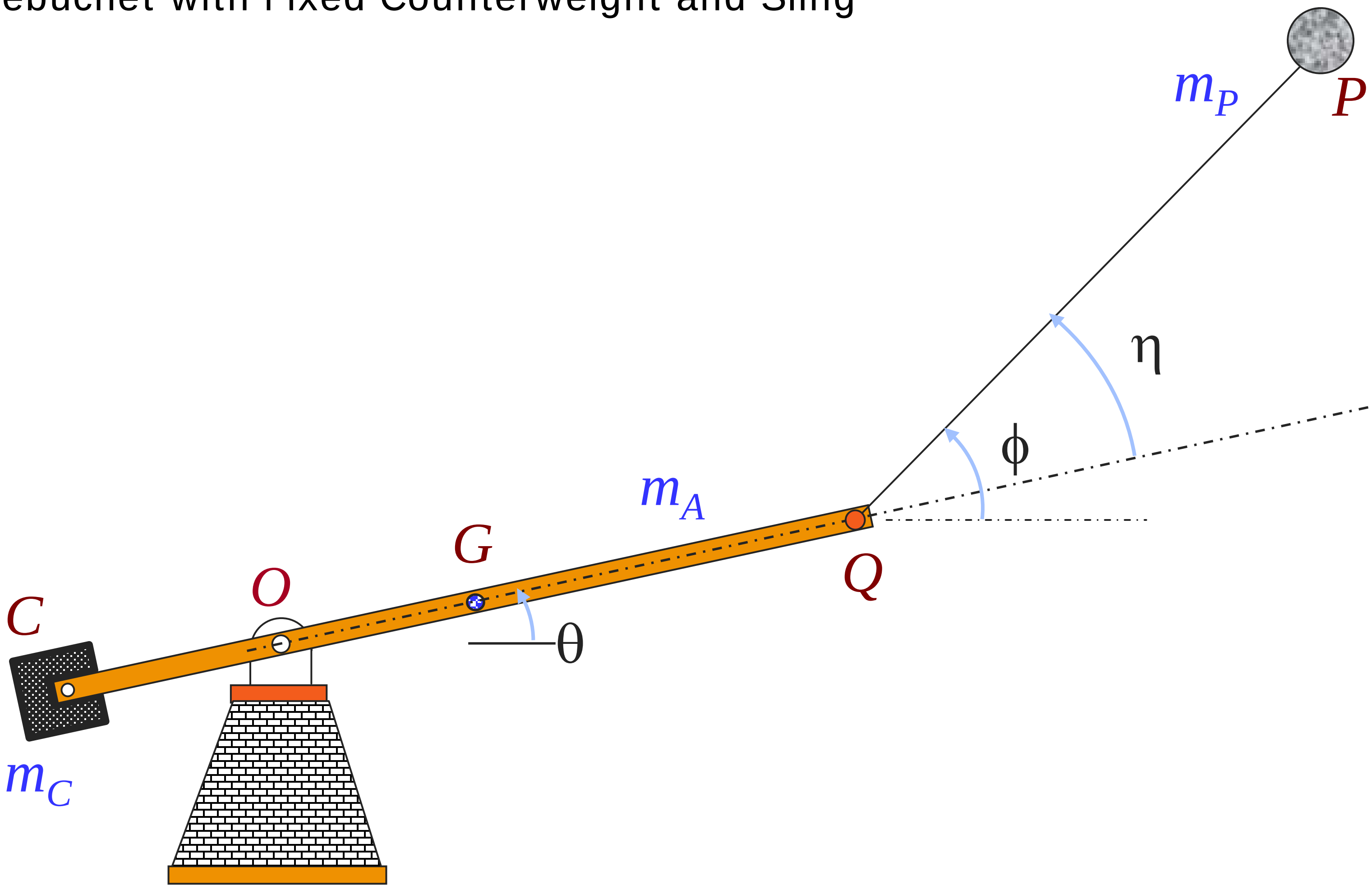
- Define the Lagrangian,

$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q$$

Configuration 4

Trebuchet with Fixed Counterweight and Sling



Lagrange's Equation for *n* DOF systems

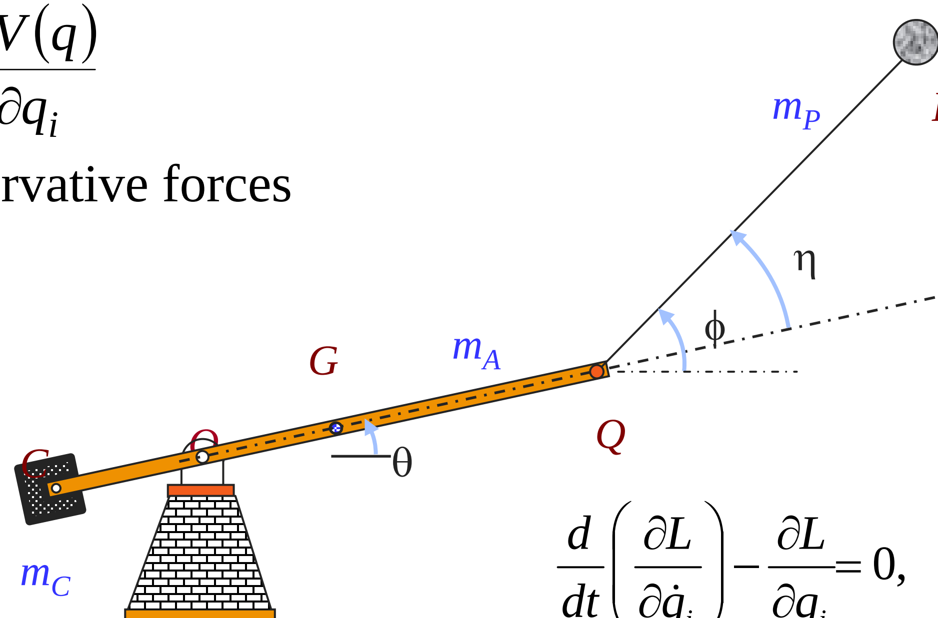
- *n* equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i, \quad i = 1, \ldots, n$$

- Conservative forces, $Q_{cons,\,i}$, are included in the Lagrangian

$$Q_{cons,i} = -\frac{\partial V(q)}{\partial q_i}$$

- Q_i does not include conservative forces



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1,2$$