Introduction To Multibody Dynamics

Analytical dynamics

Broadly speaking, there are two approaches to rigid body dynamics. The first approach is based on drawing free body diagrams for each rigid body and writing Newton's second law

$$\begin{aligned} \sum \mathbf{F}_i &= m\mathbf{a} \\ \sum_i \mathbf{r}_i \times \mathbf{F}_i + \sum_j \mathbf{M}_j &= \mathbf{I}\boldsymbol{\alpha} \end{aligned} \tag{1}$$

where **a** is the acceleration of the center of the mass of the rigid body, $\boldsymbol{\alpha}$ is the angular acceleration of the rigid body, *m* is the mass and **I** is the mass moment of inertia about the center of mass of the rigid body. This approach introduces many extraneous unknowns including the reaction forces and involves the solution of 3*n* equations of motion for a system with *n* rigid bodies.

The second approach is based on principles of analytical dynamics. It starts with the principle of virtual work originating from the works of Galileo and Bernouli. The principle of virtual work lends itself to static analysis of multibody systems. It allows us to ignore reaction forces and simply focus on the specific unknown forces (moments) that we are interested in. Often this number is much smaller than 3*n*, and this is the motivation for using the principle of virtual work for the analysis of robot manipulators. The work of D'Alembert and Lagrange addresses dynamic analysis in the same spirit. Rather than write 3*n* equations of force balance for a multibody system with mobility (number of degrees of freedom) *M*, we want to write *M* equations of motion that describe how the generalized coordinates of the system evolve as a function of time. The final result that we will be interested in allows us to do precisely this. These equations of motion are called Lagrange's equations of motion.

Dynamics of a single degree-of-freedom system

We will first consider a single degree-of-freedom system and derive the equations of motion for the system. Let us adopt the following notation:

q	generalized coordinate (joint angle in the example)
Q	generalized force (moment about the pin joint O in the example)
J	generalized inertia (moment of inertia about O in the example)
T	kinetic energy of the system

We will assume that we can compute the generalized force from all the applied (working) forces. For the sake of convenience, we will assume there is no friction in the system.

The schematic of the single degree-of-freedom is shown in Figure 1. The system can be pictured as a single link whose displacement is described by the generalized coordinate q, with a generalized force Q acting on it. However, its inertia (or shape) changes as the system configuration changes. This concept becomes fairly straightforward when we consider an example.

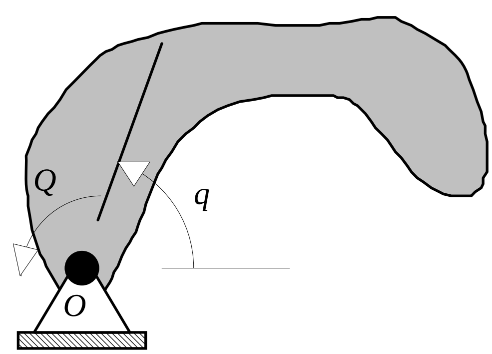

Figure 1 A schematic of a single degree-of-freedom system

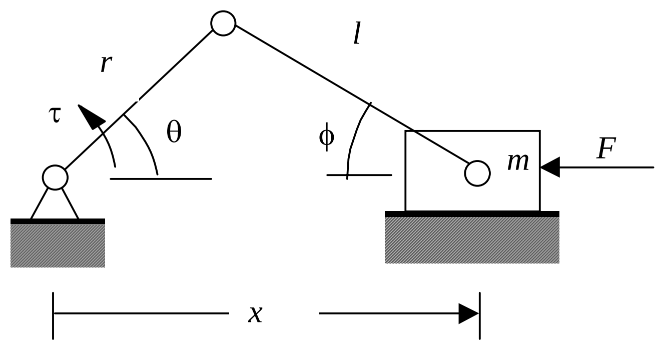

Figure 2 The slider crank mechanism

Consider a slider crank mechanism subject to a torque τ on the crank and a force F on the piston. For the sake of simplicity, let us assume that the inertia of the crank and the connecting rod are negligible compared to the mass of the piston (m), and there is no gravity or friction in the problem. The kinetic energy of the system is given by:

$$\begin{aligned} T &= \tfrac{1}{2} m\dot{x}^2 \\ &= \tfrac{1}{2} m \left(\frac{mr^2 \sin^2(\theta+\phi)}{\cos^2\phi} \right) \dot{\theta}^2 \\ &= \tfrac{1}{2} J(q)\dot{q}^2 \end{aligned} \qquad (2)$$

where $q = \theta$, and $J(q)$ is the generalized inertia of the system given by:

$$J(\theta) = m\left(\frac{r^2 \sin^2(\theta+\phi)}{\cos^2\phi} \right) \qquad (3)$$

Note that ϕ is a function of the generalized coordinate $q = \theta$ in this problem. $J(q)$ depends on the generalized coordinate and therefore changes with the configuration of the system. This quantity is really the inertia felt at the crank if one were to try to rotate the crank of a slider crank mechanism with a massless crank and connecting rod.

Next we will use the fact that the work done on the system by the generalized force (effectively the applied or working external forces) must equal the change of kinetic energy. Alternatively, the power associated with the generalized force must equal the rate of change of kinetic energy.

$$Q\,\dot{q} = \frac{d}{dt}\left(\tfrac{1}{2}\,J(q)\,\dot{q}^2\right) \tag{4}$$

Starting from this point we will derive the single (as opposed to 3 times the number of rigid bodies in the system) equation of motion for the system:

$$\begin{aligned} Q\,\dot{q} &= \frac{d}{dt}\left(\tfrac{1}{2}\,J(q)\,\dot{q}^2\right) \\ &= \left(J(q)\,\ddot{q} + \tfrac{1}{2}\frac{dJ(q)}{dt}\,\dot{q}\right)\dot{q} \\ &= \left(J(q)\,\ddot{q} + \tfrac{1}{2}\frac{dJ(q)}{dq}\,\dot{q}^2\right)\dot{q} \end{aligned}$$

In other words, we have an equation relating the generalized force to the second derivative of the generalized coordinate:

$$J(q)\,\ddot{q} + \tfrac{1}{2}\frac{dJ(q)}{dq}\,\dot{q}^2 = Q \tag{5}$$

Lagrange's equation of motion

We want to write (5) in terms of the kinetic energy, *T*, as opposed to writing it in terms of *J*(*q*) and its derivative. If we note the following identities:

$$\begin{aligned} \frac{\partial T}{\partial q} &= \tfrac{1}{2}\frac{dJ(q)}{dq}\,\dot{q}^2 \\ \frac{\partial T}{\partial \dot{q}} &= J(q)\,\dot{q} \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) &= \frac{dJ(q)}{dq}\,\dot{q}^2 + J(q)\,\ddot{q} \end{aligned}$$

we can see that the left hand side of (5) can be written in terms of $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right)$ and $\frac{\partial T}{\partial q}$:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = Q \qquad (6)$$

This is the fundamental form of **Lagrange's equation of motion** for a single degree of freedom system.

This equation of motion can be extended for multiple degrees of freedom in a straight forward way. For such a system, T is a function of n generalized coordinates and their time derivatives, and we can write one equation of motion for each degree of freedom. Without proof, the standard form of Lagrange's equations of motion is:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j, \quad j=1,...,n.$$

If we further assume that the forces acting on the system are conservative, we can find a potential function, $V(q_1, q_2,..., q_n, t)$ such that all generalized active forces can be expressed as partial derivatives of the potential function:

$$Q_j = -\frac{\partial V}{\partial q_j}, \qquad j=1,...,n.$$

Armed with this expression, we can show that Eqn. (6) reduces to:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \quad j=1,...,n.$$

(7)

where $L=T-V$ is called the *Lagrangian* for the system. This is the *standard form of Lagrange's equations of motion.*

If we label the generalized force accounting for all non conservative forces as Q_j, we can include the effect of conservative forces in the Lagrangian and write:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j, \quad j=1,...,n.$$