Some useful results on rotations and other displacements

- Transformations and Displacements
- Similarity transformations
- Rotations or spherical displacements
  - Euler’s theorem for rigid body rotations
  - The axis-angle representation for rotations
- General displacements
  - Chasles’ theorem for rigid body displacements
  - The screw axis representation for displacements
Coordinate transformation from \{B\} to \{A\}

- position vector of \(P\) in \{B\} is transformed to position vector of \(P\) in \{A\}
- description of \{B\} as seen from an observer in \{A\}

\[ A \mathbf{r}^P = A \mathbf{R}_B B \mathbf{r}^P + A \mathbf{r}^{O'} \]
Displacements

The same equation can have two interpretations:

- It transforms the position vector of any point in \{B\} to the position vector in \{A\}

- It transforms the position vector of any point in the first position/orientation (described by \{A\}) to its new position vector in the second position orientation (described by \{B\}).

\[
^A r^{P'} = ^A R_B^B r^P + ^A r^{O'}
\]

Coordinate transformation from \{B\} to \{A\}

\[
^A r^{P'} = ^A R_B^A r^P + ^A r^{O'}
\]

Displacement of a body-fixed frame from \{A\} to \{B\}
Composition of Displacements

Displacement from \{A\} to \{B\}

\[ A_A B = \begin{bmatrix} A_R B & A_r O' \\ 0_{1x3} & 1 \end{bmatrix} \]

Displacement from \{B\} to \{C\}

\[ B_A C = \begin{bmatrix} B_R C & B_r O' \\ 0_{1x3} & 1 \end{bmatrix} \]

Displacement from \{A\} to \{C\}

\[ A_A C = \begin{bmatrix} A_R C & A_r O'' \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A_R B & A_r O' \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} B_R C & B_r O'' \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A_R B \times B R_C & A_R B \times B r'' + A_r O' \\ 0 & 1 \end{bmatrix} \]

Note \( X_A Y \) describes the displacement of the body-fixed frame from \{X\} to \{Y\} in reference frame \{X\}
Kinematics

What happens when you want to describe the displacement of the body-fixed frame from \{A\} to \{B\} in reference frame \{F\}?

Displacement is described in \{A\} by the homogeneous transform, \( A^A_B \).

Want to describe the same displacement in \{F\}. The position and orientation of \{A\} relative to \{F\} is given by the homogeneous transform, \( F^A_A \).

The same displacement which moves a body-fixed frame from \{A\} to \{B\}, will move another body-fixed frame from \{F\} to \{G\}:

\[
F^A_G = F^A_A A^A_B B^B_A G
\]

\[
F^A_G = F^A_A A^A_B (F^A_A)^{-1}
\]
Euler’s and Chasles’ Theorems

**Rotations**

Any displacement of a rigid body such that a point on the rigid body, say O, remains fixed, is equivalent to a rotation about a fixed axis through the point O.

**General Displacements**

The most general rigid body displacement can be produced by a translation along a line followed (or preceded) by a rotation about that line.
Proof of Euler’s Theorem for Spherical Displacements

Displacement from \{F\} to \{M\}

\[ P = Rp \]

Solve the eigenvalue problem:

\[ Rp = \lambda p \]

\[ |R - \lambda I| = 0 \]

\[ -\lambda^3 + \lambda^2 a_{11} + R_{22} + R_{33} \]

\[ -\lambda[ a_{22} R_{33} - R_{32} R_{23} + a_{11} R_{33} - R_{13} R_{31} + a_{11} R_{22} - R_{12} R_{21}] + |R| = 0 \]

Three eigenvalues and eigenvectors are:

\[ \lambda_1 = e^{i\phi}, \ p_1 = x \]

\[ \lambda_2 = e^{-i\phi}, \ p_2 = x \]

\[ \lambda_3 = 1, \ p_3 = u \]

where \[ \cos \phi = \frac{1}{2} a_{11} + R_{22} + R_{33} - 1 \]
The Axis/Angle for a Rotation Matrix

Define

\[ F R_{F'} = Q = \begin{bmatrix} v_2 & u \end{bmatrix} \]

and look at the displacement in the new reference frame, \{F'\}.

Can show:

\[ R = Q \Lambda Q^T \]

where,

\[ \Lambda = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ v_1 = \frac{1}{2} a + \lambda \]

\[ v_2 = \frac{i}{2} a - \lambda \]

\[ R v_1 = \frac{1}{2} a + \lambda \]

\[ = v_1 \cos \phi + v_2 \sin \phi \]

\[ R v_2 = \frac{i}{2} a - \lambda \]

\[ = -v_1 \sin \phi + v_2 \cos \phi \]
Kinematics

Between Rotation Matrix angle Axis/Angle

Rotation about \( \mathbf{u} \) through \( \phi \)

\[
\mathbf{R}_p = p \cos \phi + \mathbf{u} \mathbf{u}^T p \mathbf{a} - \cos \phi + \mathbf{u} \sin \phi \times \mathbf{p}
\]

Rodrigues-Euler-Lexell formula

\[
\mathbf{R} = \mathbf{I} \cos \phi + \mathbf{u} \mathbf{u}^T \mathbf{a} - \cos \phi + \mathbf{U} \sin \phi
\]

where,

\[
\mathbf{U} = \begin{bmatrix}
0 & -u_3 & u_2 \\
-u_2 & 0 & -u_1 \\
u_1 & u_2 & 0
\end{bmatrix}
\]

Extracting the axis and the angle from the rotation matrix

1. Find the eigenvector corresponding to \( \lambda = 1 \).
2. From Rodrigues’ formula:

\[
\cos \phi = \frac{1}{2} \mathbf{a}_{11} + R_{22} + R_{33} - 1
\]

\[
\mathbf{U} = \frac{1}{2 \sin \phi} \mathbf{a} - \mathbf{R}^T \mathbf{j}
\]

Notes:

1. Map from \( \mathbf{R} \) to \((\mathbf{u}, \phi)\) is one to many.
   - restrict \( \phi \) to the interval \([0, \pi]\)
2. Singular
   - \( \mathbf{R} = \mathbf{I} \)
   - \( \text{trace}(\mathbf{R}) = -1 \)
Chasles’ Theorem for Planar Displacements

Displacements in the $x$-$y$ plane

$$A = \begin{bmatrix} R & d \\ 1 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \quad d = \begin{bmatrix} d_x \\ d_y \end{bmatrix}$$

If $R$ is not the identity matrix, there is one fixed point on the rigid body for any displacement called the *pole* (or the instantaneous center) of the displacement.

$$c = a - R^{-1}d$$

This point corresponds to the eigenvector of the matrix $A$ for a unit eigenvalue.

$$\begin{bmatrix} \cos \theta & -\sin \theta & d_x \\ \sin \theta & \cos \theta & d_y \\ 0 & 1 \end{bmatrix}$$

Pure translations?

Canonical representation of a planar displacement (except pure translations)
Chasles Theorem for General Displacements

Want a special reference frame in which we have a simple description of the displacement.

**Translation part:**

If the top 2×2 submatrix of \((Q^T R Q - I)\) is nonsingular, we can always find \(c'\) such that

\[
\Lambda = \begin{bmatrix}
    c^T & -Q^T c & d \\
    0 & 1 & 1
\end{bmatrix}
\]

so that

\[
Q^T R Q - Q^T c + Q^T d = \Theta^T R Q - I Q^T c + d'
\]

This implies \(\Lambda\) has the form

\[
\Lambda = \begin{bmatrix}
    \cos \phi & -\sin \phi & 0 & 0 \\
    \sin \phi & \cos \phi & 0 & 0 \\
    0 & 0 & 1 & k \\
    0 & 0 & 0 & 1
\end{bmatrix}
\]

**Rotational part:**

Choose

\[
Q = \begin{bmatrix}
    v_2 \\
v_1
\end{bmatrix}
\]

so that

\[
Q^T R Q = \begin{bmatrix}
    \cos \phi & -\sin \phi & 0 \\
    \sin \phi & \cos \phi & 0 \\
    0 & 0 & 1
\end{bmatrix}
\]
Chasles’ Theorem: Canonical Representation of Displacements

In \{G_1\},
\[ G_1 A_{G_2} = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \]

In new frame \{E_1\}, given by:
\[ G_1 A_{E_1} = \begin{bmatrix} R & c \\ 0 & 1 \end{bmatrix} \]

the transformation consists of a translation along and a rotation about the \(z\)-axis:
\[ E_1 A_{E_2} = \Lambda = \begin{bmatrix} \cos \phi & -\sin \phi & 0 & 0 \\ \sin \phi & \cos \phi & 0 & 0 \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{bmatrix} \]
Screw Axis from Homogeneous Transformation Matrix

1. Find an appropriate proper orthogonal matrix \( Q \) such that its third column is \( u \) (the axis of rotation).

2. Find the projection of the vector \( d \) on a plane perpendicular to \( u \),
   \[ d_p = d - k \ u, \quad k = (d \cdot u) \]

3. Choose \( Q \)
   \[ Q = \begin{pmatrix} w & u \\ \hline v & \phi \end{pmatrix} \]
   where
   \[ v = \frac{d_p}{d_p}, \quad w = u \times v \]

4. Choose \( c \)
   \[ c' = \frac{d_p}{2} \begin{pmatrix} 1 & \sin \phi & 0 \\ \cos \phi & \cos \phi & 0 \\ 0 & 0 & 0 \end{pmatrix} \]
   \[ c = Qc' = \frac{d_p}{2} \begin{pmatrix} 1 & \sin \phi & 0 \\ \cos \phi & \cos \phi & 0 \\ 0 & 0 & 0 \end{pmatrix} + a \frac{\sin \phi}{\cos \phi} f v j \]
Homogeneous Transform from Axis, Pitch, and Angle

Given the screw axis, we know $\mathbf{u}$, $\phi$, and $h$

Let $\mathbf{Q}$ be any orthogonal matrix of the form:

$$
\mathbf{Q} = \begin{pmatrix}
\cos \phi & -\sin \phi & 0 & 0 \\
\sin \phi & \cos \phi & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

and $\mathbf{c}$ be a position vector of any point on the screw axis.