Holonomic and Nonholonomic Constraints

Holonomic Constraints

Constraints on the position (configuration) of a system of particles are called *holonomic* constraints.

- Constraints in which time explicitly enters into the constraint equationare called *rheonomic*.
- Constraints in which time is not explicitly present are called *scleronomic*.

Note: Inequalities do not constrain the position in the same way as equality constraints do. Rosenberg classifies inequalities as nonholonomic constraints.

- Particle is constrained to lie on a plane:

$$A\, x_1 + B\, x_2 + C\, x_3 + D = 0$$

- A particle suspended from a string in three dimensional space.

$$(x_1 - a)^2 + (x_2 - b)^2 + (x_3 - c)^2 - r^2 = 0$$

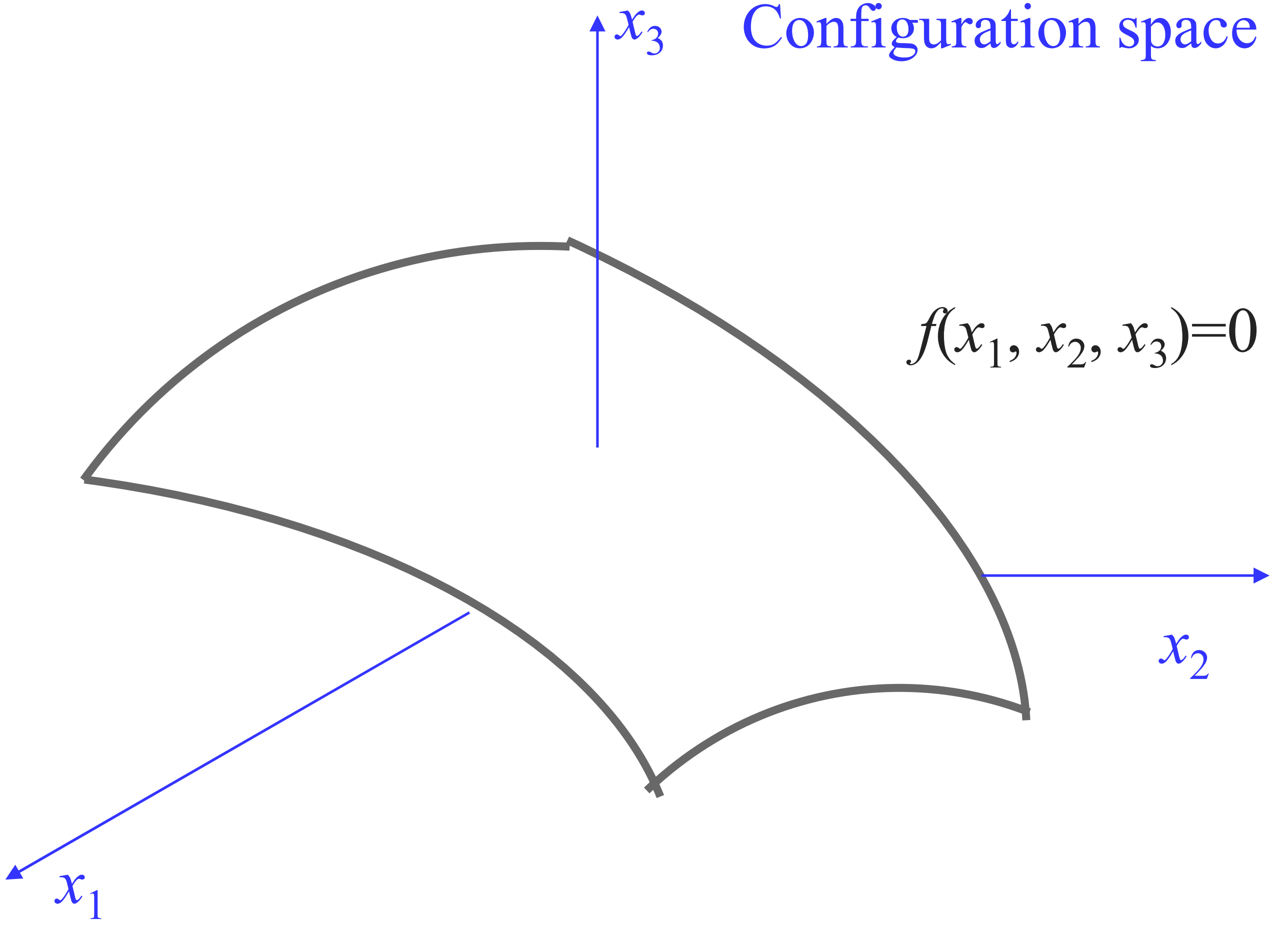
- A particle on spinning platter (carousel)

$$x_1 = a \cos(\omega t + \phi);\ x_2 = a \sin(\omega t + \phi)$$

- A particle constrained to move on a circle in three-dimensional space whose radius changes with time t.

$$x_1\ dx_1 + x_2\, dx_2 + x_3\, dx_3 - c^2\, dt = 0$$

Holonomic Constraint

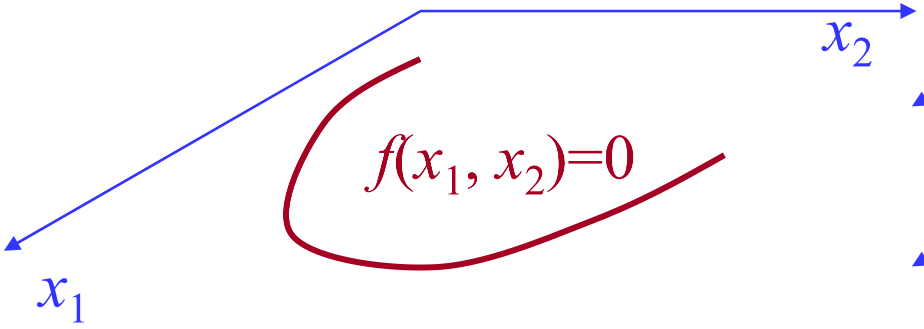


Scleronomic and Rheonomic

Scleronomic

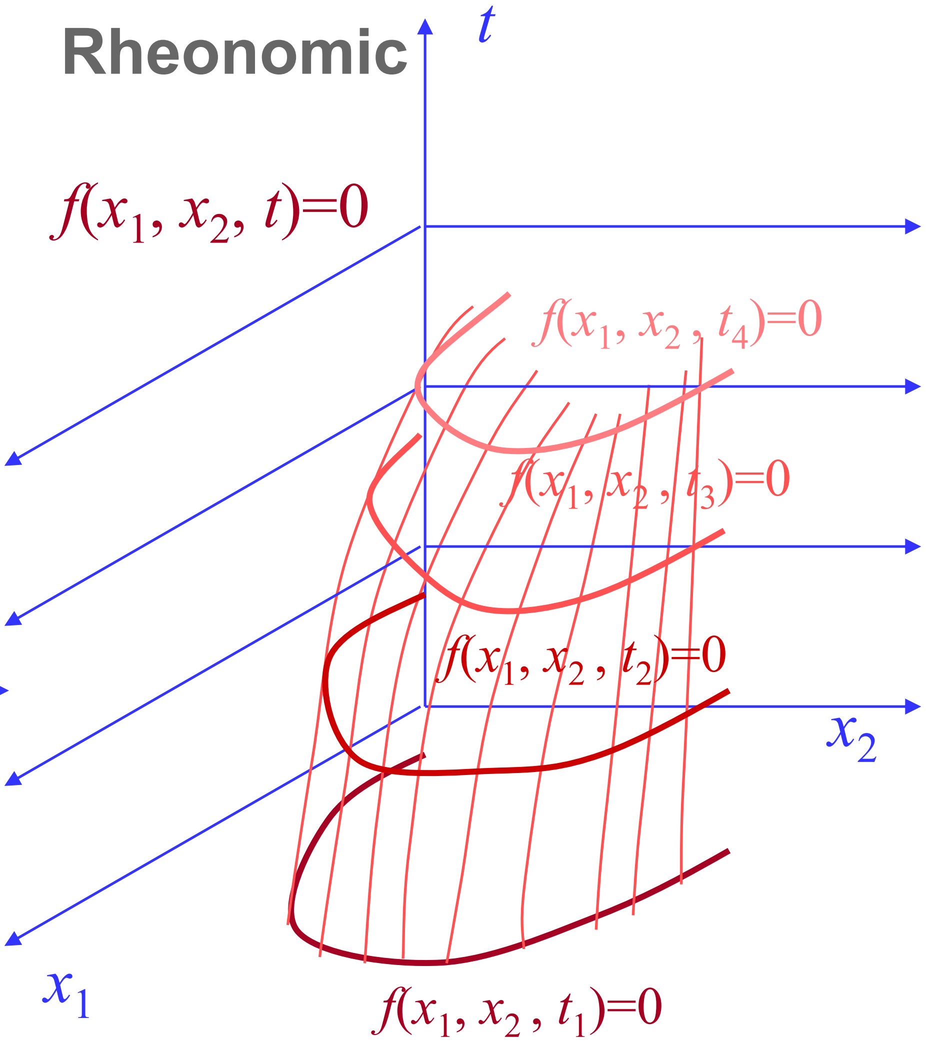
$f(x_1, x_2)=0$

Configuration space



Rheonomic

$f(x_1, x_2, t)=0$



Nonholonomic Constraints

Definition 1

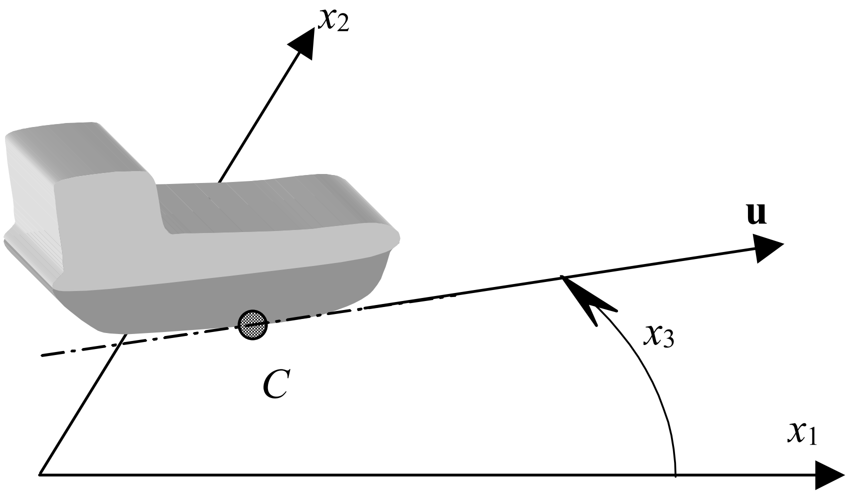
- All constraints that are not holonomic

- A particle constrained to move on a circle in three-dimensional space whose radius changes with time *t*.

$$x_1 \; dx_1 + x_2 \, dx_2 + \; x_3 \, dx_3 \; - c^2 \, dt = 0$$

- The *knife-edge constraint*

$$\dot{x}_1 \sin x_3 - \dot{x}_2 \cos x_3 = 0$$



When is a constraint on the motion nonholonomic?

Velocity constraint

$$a_1\dot{x}_1 + a_2\dot{x}_2 + \ldots + a_{n-1}\dot{x}_{n-1} + a_n = 0$$

3 dimensional case

$$P\,dx + Q\,dy + R\,dz = 0 \qquad \mathbf{v} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

Or constraint on instantaneous motion

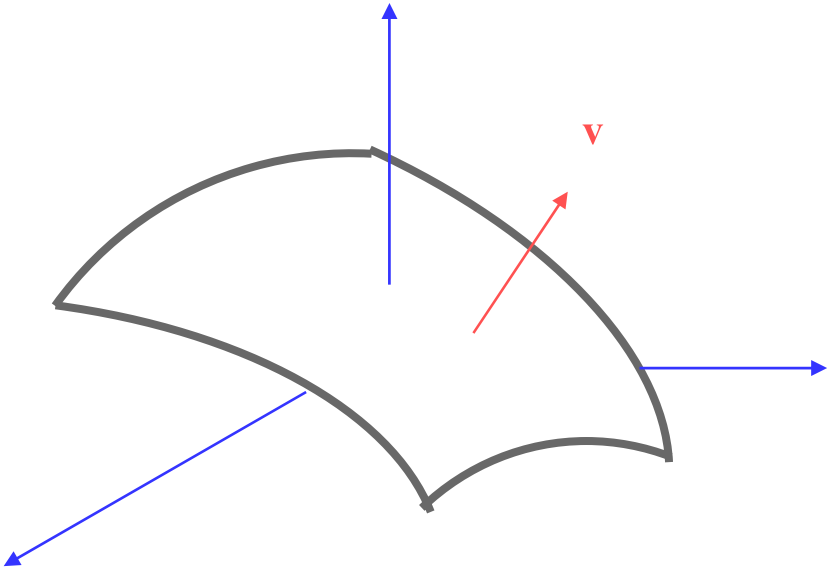
$$a_1 dx_1 + a_2 dx_2 + \ldots + a_{n-1} dx_{n-1} + a_n dt = 0$$

Pfaffian Form

Question

Can the above equation can be reduced to the form:

$$f(x_1, x_2, \ldots, x_{n-1}, t) = 0$$



When is a scleronomic constraint on motion in a three-dimensional configuration space nonholonomic?

Velocity constraint

$$P\dot{x} + Q\dot{y} + R\dot{z} = 0$$

Or constraint in the Pfaffian form

$$P\,dx + Q\,dy + R\,dz = 0 \qquad (1)$$

Question

Can the above equation can be reduced to the form:

$$f(x, y, z) = 0$$

Or,

Can we at least say when the differential form (1) an exact differential?

$$df = P\,dx + Q\,dy + R\,dz$$

- A **sufficient** condition for (1) to be integrable is that the differential form is an exact differential.
- If it is an exact differential, there must exist a function *f*, such that

$$P = \frac{\partial f}{\partial x}, \qquad Q = \frac{\partial f}{\partial y}, \qquad R = \frac{\partial f}{\partial z}$$

- The necessary and sufficient conditions for this to be true is that the first partial derivatives of *P*, *Q*, and *R* with respect to *x*, *y*, and *z* exist, and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \qquad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}.$$

Recall Stokes Theorem!

When is a scleronomic constraint on motion in a three-dimensional configuration space nonholonomic?

Constraint in the Pfaffian form

$$P\,dx + Q\,dy + R dz = 0 \qquad (1)$$

Question

Can the above equation can be reduced to the form:

$$f(x, y, z)=0$$

For the constraint to be *integrable*, it is necessary and sufficient that there exist an integrating factor $\alpha(x, y, z)$, such that,

$$\alpha P\,dx + \alpha Q\,dy + \alpha R dz = 0 \ (2)$$

be an exact differential.

- If (2) is an exact differential, there must exist a function g, such that

$$\alpha P = \frac{\partial g}{\partial x}, \qquad \alpha Q = \frac{\partial g}{\partial y}, \qquad \alpha R = \frac{\partial g}{\partial z}$$

- The necessary and sufficient conditions for this to be true is that the first partial derivatives of P, Q, and R with respect to x, y, and z exist, and

$$\frac{\partial(\alpha \mathrm{P})}{\partial y} = \frac{\partial(\alpha Q)}{\partial x},$$

$$\frac{\partial(\alpha P)}{\partial z} = \frac{\partial(\alpha R)}{\partial x},$$

$$\frac{\partial(\alpha R)}{\partial y} = \frac{\partial(\alpha Q)}{\partial z}.$$

When is a scleronomic constraint on motion in a three-dimensional configuration space nonholonomic?

$$\frac{\partial(\alpha \mathrm{P})}{\partial y} = \frac{\partial(\alpha Q)}{\partial x},$$

$$\frac{\partial(\alpha P)}{\partial z} = \frac{\partial(\alpha R)}{\partial x},$$

$$\frac{\partial(\alpha R)}{\partial y} = \frac{\partial(\alpha Q)}{\partial z}.$$

$$\left(\frac{\partial \alpha}{\partial y}\right)P - \left(\frac{\partial \alpha}{\partial x}\right)Q = \alpha\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right),$$

$$\left(\frac{\partial \alpha}{\partial z}\right)P - \left(\frac{\partial \alpha}{\partial x}\right)R = \alpha\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right),$$

$$\left(\frac{\partial \alpha}{\partial y}\right)R - \left(\frac{\partial \alpha}{\partial z}\right)Q = \alpha\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right).$$

$$\mathbf{v} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

$$\nabla\alpha \times \mathbf{v} = -\alpha\nabla \times \mathbf{v}$$

Necessary and sufficient condition for (2) to be holonomic, provided **v** is a well-behaved vector field and

$$\mathbf{v} \cdot \nabla \times \mathbf{v} = 0$$

Examples

1. $\sin x_3\, dx_1 - \cos x_3\, dx_2 = 0$

$$\mathbf{v} = \begin{bmatrix} -\sin x_3 \\ \cos x_3 \\ 0 \end{bmatrix}$$

2. $2x_2x_3\, dx_1 + x_1x_3\, dx_2 + x_1x_2\, dx_3 = 0$

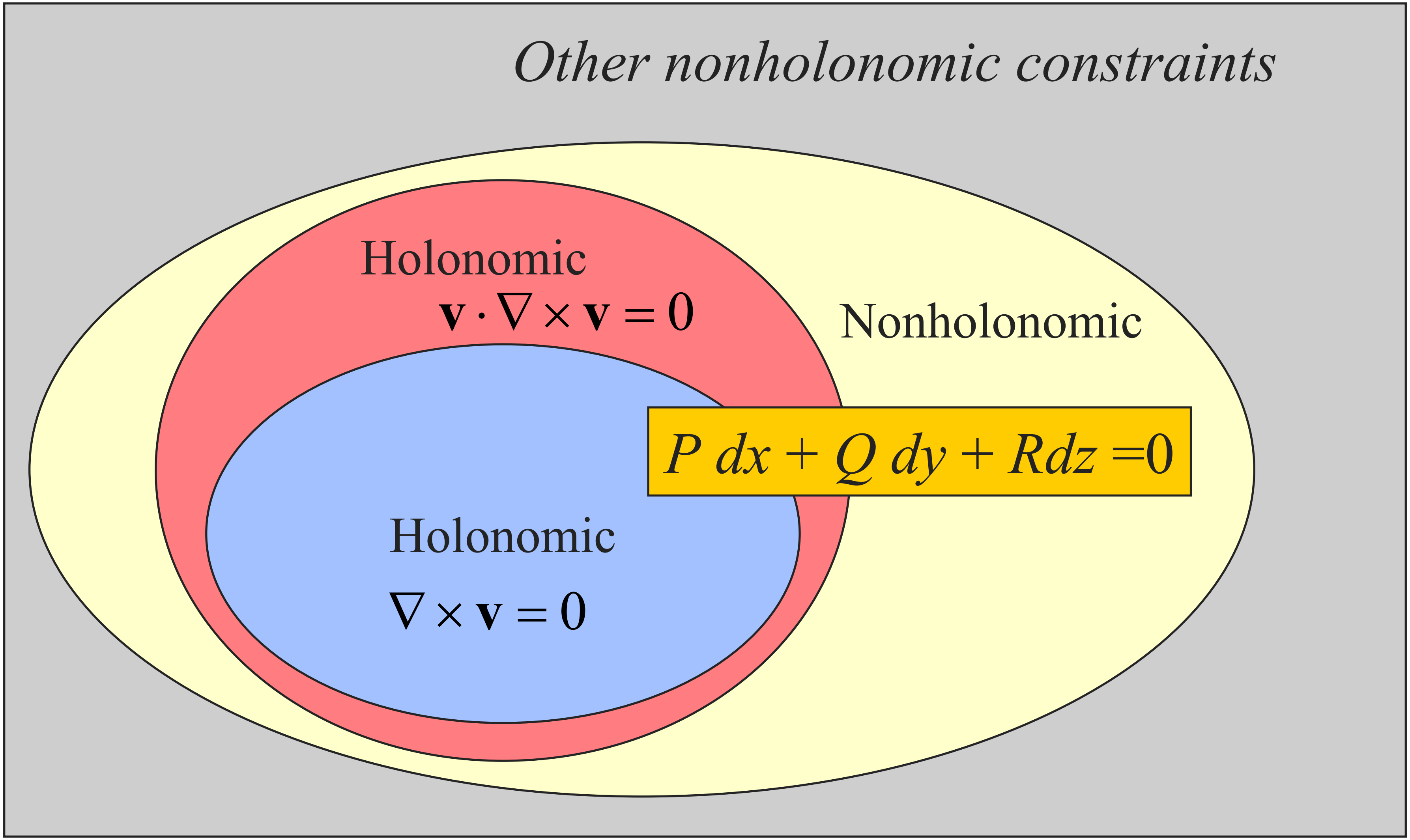
$$x_1\,(2x_2x_3\, dx_1 + x_1x_3\, dx_2 + x_1x_2\, dx_3) = 0$$

$$\mathbf{v} = \begin{bmatrix} 2x_2x_3 \\ x_1x_3 \\ x_1x_2 \end{bmatrix}$$

$$d((x_1)^2\, x_2\, x_3) = 0$$

3. $\dot{x}_1\dot{x}_2 - \dot{x}_3 = 0$

Nonholonomic constraints in 3-D



Other nonholonomic constraints

Holonomic

$\mathbf{v} \cdot \nabla \times \mathbf{v} = 0$

Nonholonomic

$P\,dx + Q\,dy + R\,dz = 0$

Holonomic

$\nabla \times \mathbf{v} = 0$

Extensions: 1. Multiple Constraints

$$\mathrm{d}x_2 - x_3\,\mathrm{d}x_1 = 0$$

and

$$\mathrm{d}x_3 - x_1\,\mathrm{d}x_2 = 0$$

Are the constraint equations non holonomic?

Individually: YES!

Together:

$$\mathrm{d}x_3 - x_1\,\mathrm{d}x_2 = \mathrm{d}x_3 - x_1\,(x_3\,\mathrm{d}x_1) = 0 \qquad x_3 = ke^{\frac{x_1^2}{2}}, \qquad x_2 = \int ke^{\frac{x_1^2}{2}}dx_1 + c$$

Extensions 2: Constraints in > 3 generalized speeds

- n dimensional configuration space
- m independent constraints (i=1,..., m)

$$\sum_{j=1}^{n} a_{ij}\,dx_j - b_i\,dt = 0$$

or

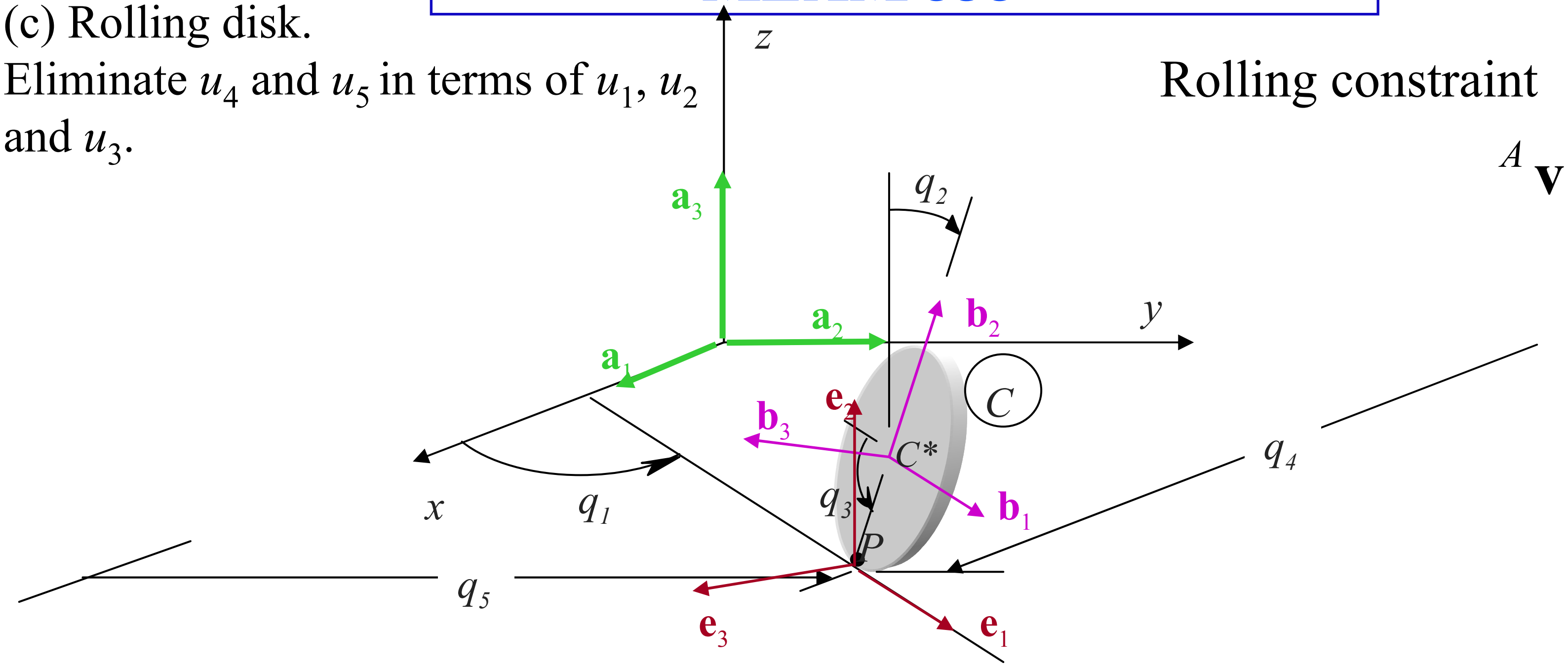
$$\sum_{j=1}^{n} a_{ij}u_j - b_i = 0$$

(c) Rolling disk.

Eliminate u_4 and u_5 in terms of u_1, u_2 and u_3.

Rolling constraint

$$ {}^A\mathbf{v}^P = 0 $$



$$ {}^A\mathbf{v}^P = {}^A\mathbf{v}^{C^*} + {}^A\omega^C \times \overrightarrow{C^*P} $$

$$ = {}^A\mathbf{v}^{C^*} + \left(u_1\mathbf{b}_1 + u_2\mathbf{b}_2 + u_3\mathbf{b}_3\right)\times\left(-R\mathbf{b}_2\right) $$

$$ = 0 $$

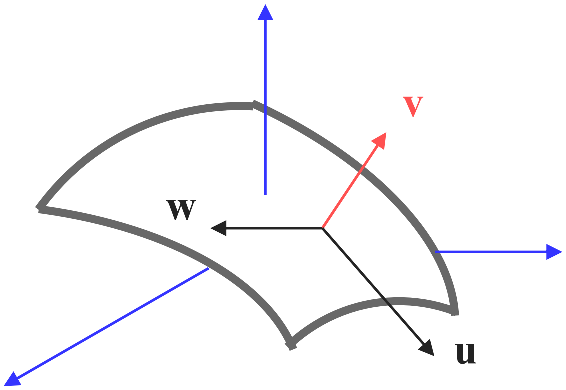
$$ u_4 \cos q_1 + u_5 \sin q_1 - R u_2 \tan q_2 + R u_3 = 0 $$

$$ -u_4 \sin q_1 \sin q_2 + u_5 \cos q_1 \sin q_2 = 0 $$

m=2 constraints

n=5 generalized coordinates

Frobenius Theorem: Generalization to *n* dimensons



n dimensional configuration space

m independent constraints (*i*=1,..., *m*)

$$\sum_{j=1}^{n} a_{ij} dx_j = 0$$

The necessary and sufficient condition for the existence of *m* independent equations of the form:

$$f_i(x_1, x_2, ..., x_n) = 0, \qquad i=1,..., m.$$

is that the following equations be satisfied:

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\left(\frac{\partial a_{il}}{\partial x_k} - \frac{\partial a_{ik}}{\partial x_l}\right) u_k w_l = 0$$

where u_k and w_l are components of any two *n* vectors that lie in the null space of the *m*x*n* coefficient matrix $\mathbf{A} = [a_{ij}]$:

$$\sum_{j=1}^{n} a_{ij} u_j = 0, \quad \sum_{j=1}^{n} a_{ij} w_j = 0,$$

Generalized Coordinates and Speeds

Holonomic Systems

Number of degrees of freedom of a system in any reference frame

- the minimum number of variables to completely specify the position of every particle in the system in the chosen reference

The variables are called generalized coordinates

There can be no holonomic constraint equations that restrict the values the generalized coordinates can have.

q_1, q_2, …, q_n denote the generalized coordinates for a system with n degrees of freedom in a reference frame A.

n generalized coordinates specify the position (configuration of the system)

The number of independent speeds is also equal to n

In a system with n degrees of freedom in a reference frame A, there are n scalar quantities, u_1, u_2, …, u_n (for that reference frame) called generalized speeds. They that are related to the derivatives of the generalized coordinates by :

$$u_i = \sum_{j=1}^{n} Y_{ij}(q_1, q_2, \ldots, t)\dot{q}_j + Z_i(q_1, q_2, \ldots, t)$$

where the nxn matrix $\mathbf{Y} = [Y_{ij}]$ is non singular and $\mathbf{Z}$ is a nx1 vector.

Example 1

Generalized Coordinates

$q_1\,, q_2\,, q_3\,, q_4\,, q_5$

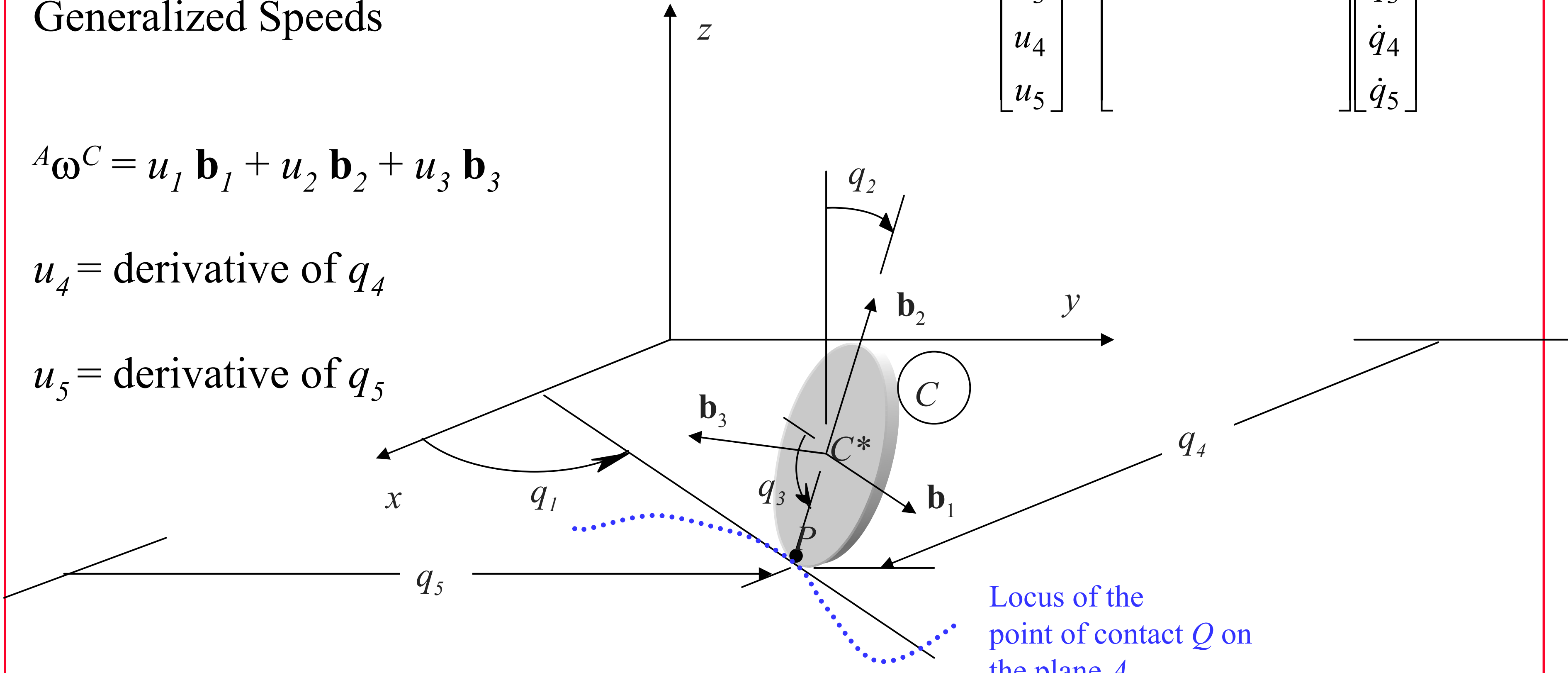
Generalized Speeds

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} & & & & \\ & & & & \\ & & \mathbf{Y} & & \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \end{bmatrix}$$

$^A\omega^C = u_1\,\mathbf{b}_1 + u_2\,\mathbf{b}_2 + u_3\,\mathbf{b}_3$

$u_4 =$ derivative of q_4

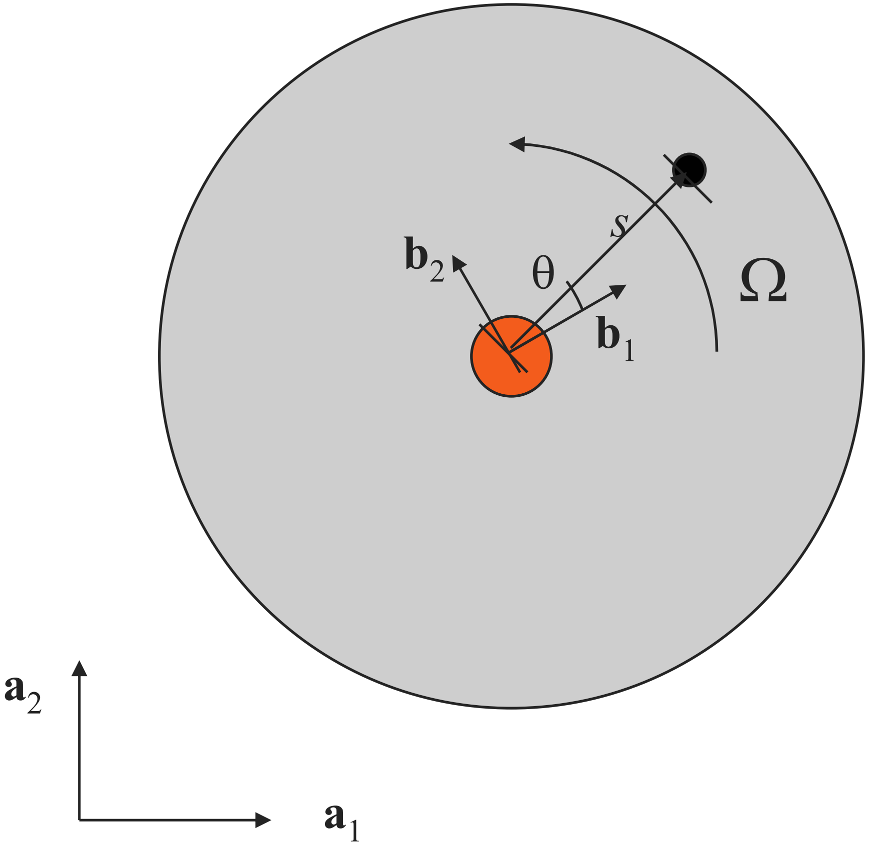
$u_5 =$ derivative of q_5



Locus of the point of contact *Q* on the plane *A*

Example 2

Bug on the turntable



Generalized coordinates in *A*

- s, θ

Generalized speeds

$$u_1 = \frac{dx}{dt}$$

$$u_2 = \frac{dy}{dt}$$

Generalized speeds and derivatives of generalized coordinates

$$u_i = \sum_{j=1}^{n} Y_{ij}(q_1, q_2, \ldots, t)\dot{q}_j + \boxed{Z_i(q_1, q_2, \ldots, t)}$$

Appears in
rheonomic constraints

Nonholonomic Constraints are Written in Terms of Speeds

m constraints in *n* speeds

$$\sum_{j=1}^{n} C_{ij}\left(q_1, q_2, \ldots, t\right) u_j + D_i\left(q_1, q_2, \ldots, t\right) = 0$$

m speeds are written in terms of the *n*-*m* (*p*) independent speeds

$$u_i = \sum_{k=1}^{p} A_{ik}\left(q_1, q_2, \ldots, t\right) u_k + B_i\left(q_1, q_2, \ldots, t\right) = 0$$

Define the *number of degrees of freedom for a nonholonomic system* in a reference frame *A* as *p*, the number of independent speeds that are required to completely specify the velocity of any particle belonging to the system, in the reference frame *A*.

Example 3

Number of degrees of freedom

- $n - m = 2$ degrees of freedom

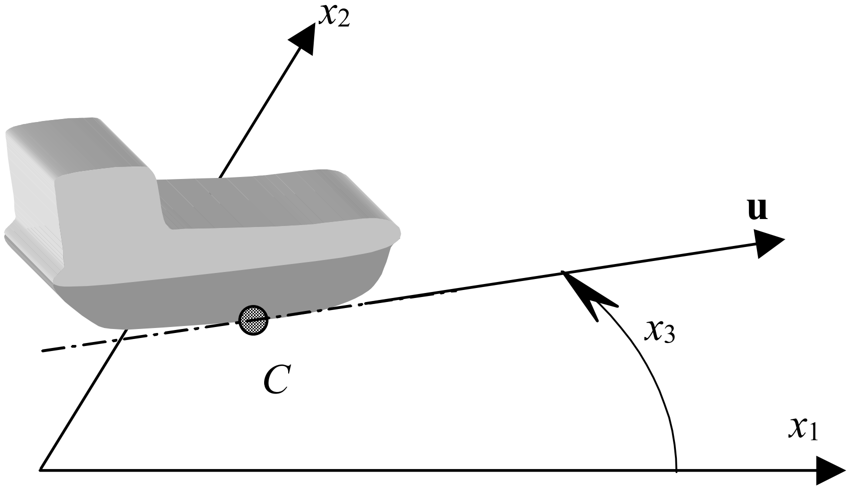
Generalized coordinates

- (x_1, x_2, x_3)

Speeds

- forward velocity along the axis of the skate, v_f
- the speed of rotation about the vertical axis, ω
- and the lateral (skid) velocity in the transverse direction, v_l

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} v_f \\ \omega \\ v_l \end{bmatrix}$$



$$\mathbf{u} = \mathbf{Y}\dot{\mathbf{q}} + \mathbf{Z}$$

$$\mathbf{Y} = \begin{bmatrix} \cos x_3 & \sin x_3 & 0 \\ 0 & 0 & 1 \\ -\sin x_3 & \cos x_3 & 0 \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$